Contagion and the Emergence of Convention in Small Worlds

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October 2004

Abstract

We model a simple dynamic process in which boundedly rational agents learn through their interactions with others. Of interest is to study the process of contagion where by one action `spreads throughout the population' and becomes conventional. We vary the network of player interaction between a regular lattice and a random network allowing us to model contagion in small world networks. Through simulation results we highlight the importance of network structure on both the possibility of contagion and the rate of contagion.

JEL Classification: C70, C72, C62

Keywords: Best reply, risk dominant, contagion, small world.

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1 Introduction

A large literature has addressed the issue of myopic learning in coordination games (see Young 1998 and Fudenberg and Levine 1998). Of principal interest has been to address the issue of equilibrium selection. It has been widely demonstrated that in two strategy coordination games the risk dominant strategy emerges as the 'long run equilibrium'. Crucial, however, to interpreting this conclusion is to have some understanding of how long the long run may be. Ellison (1993), Blume (1995) and Cartwright (2004), amongst others, demonstrate that convergence rates depend on the matching network of player interaction and further, depending on this network, convergence times can range from economically 'realistic' to 'unrealistic'. The prior literature has, however, focussed for the most part on a particular form of network - namely lattice networks. The primary motivation for the current paper is to consider a more general class of matching networks and, in particular, to evaluate the stability of equilibrium in small world networks.

To illustrate the issues a simple example may be useful: Play evolves other a number of discrete periods. At the start of each period, simultaneously, each of n computer users must decide whether to use software 'good' (G) or software 'not so good' (B). During the period, computer users interact through a series of pairwise matchings. Whenever two computer users who have chosen G interact they each receive a payoff of 2. If two computer users who have chosen B interact they each receive a payoff of 1. Finally, if two users interact who have chosen different software they each get a payoff of 0. 'Everybody choose G' and 'everybody choose B' are Nash equilibrium of this game. Intuition would suggest that strategy G - the risk dominant strategy - should emerge as a long run convention. Suppose, however, that every player is currently playing B. If we consider it unlikely that a player would choose software G when everybody else is choosing B it is clearly questionable whether a transition from B to G is plausible.

Crucial is the 'stability' of the equilibrium to play B. More specifically, suppose that each player behaves myopically in choosing the software that would have maximized his payoff in the previous period. Let the *stability* of the equilibrium to play B be represented by the number of players that must play G in the initial period in order for software G to be selected by others and ultimately become, over time, conventional. If this number is small then we consider the equilibrium to play B to be unstable - a transition from a state in which all players play B to one where all play G could be seen as 'realistic' if we allow some player 'experimentation'. Conversely, if the number is large then we consider the equilibrium to play B to be stable -

even if we allow player experimentation, the transition from a state in which all players play B to one where all play G could be seen as 'unrealistic'.

The previous literature (e.g. Ellison 1993, Blume 1995, Chapter VI of Young 1998 and Morris 2000) has demonstrated that the stability of B as a convention will depend on the matching network - in the context of the example this equates to which computer users interact with each other. With the notable exception of Morris (2000) the literature has focussed on a particular form of matching network - namely lattice networks. This has led to results in respect of network size with the distinction drawn between local and global networks. Lattice networks have, however, a very particular structure and do not appear particular representative of observed social networks. It thus appears crucial to model more general interaction structures and, in particular, to model interaction structures that more closely resemble those observed in social interaction.

The approach taken in this paper is motivated by the literature on small world networks (e.g. Watts 2000). This literature essentially builds on two principles. First, many observed social networks have so called small world characteristics; more formally they have two properties: (a) a high amount of clustering or, equivalently, a large overlap in the interaction sets of agents and yet (b) the path length between any two individuals is small. Second, small world networks can be seen as intermediate between lattice networks (which have property (a) but not (b)) and random networks (which have property (b) but not (a)).

As already discussed, lattice networks have been the subject of a large literature. The broad conclusion of that literature is that the risk dominated equilibrium is unstable in lattice networks if each player interacts with a relatively small number of players. In Cartwright (2004) we consider random networks and find that, even if player interact with only a small number of other players, equilibria are stable. In this paper will make use of a framework for modelling interaction structures intermediate between that of lattice and random networks allowing us to consider equilibrium stability in small world matching networks.

Our main conclusion will be that the risk dominated equilibrium is unstable in small world networks provided that players are not 'nearly indifferent' between strategies. Thus, stability in small world networks is comparable to that of local interaction on a regular 1 lattice. One particularly interesting aspect of this result is that it demonstrates the potential instability of risk dominated equilibria in networks where the path length between players is relatively short. Previous results demonstrating instability of the risk dominated equilibrium (see, in particular, Morris (2000)) assume a low neighborhood growth property implying long path lengths. Our research would suggest that, while being sufficient, a low neighborhood growth property is not necessary for equilibrium instability. What does appear necessary is a high amount of clustering. Note that this distinction can only be made by considering small world networks precisely because these networks do not satisfy the 'normal' relationship between clustering and path length characteristics.

That the risk dominant strategy will realistically emerge as conventional will depend not only on the stability of the risk dominated equilibrium but also on the *rate of contagion* of the risk dominant strategy once it is embedded within the population. This latter effect is likely to be of secondary influence relative to the former but may still be important. Previous results (see in particular Blume 1995) in treating interaction on a regular lattice have shown that the rate of contagion is relatively slow in local networks and fast in global networks. This would suggest a trade off between instability of the risk dominated equilibrium and a slow rate of contagion. Intuitively, however, it seems possible that the rate of contagion will, in general, be related to the average path length between players. This opens up an interesting possibility that in small world networks we could observe both instability of the risk dominated equilibrium and a fast rate of convergence to the risk dominant equilibrium. In fact, we find this to typically *not* be the case, with the rate of contagion in small world networks being relatively slow and similar to that of local interaction on regular 1 lattice. Thus, the rate of contagion does not appear closely linked to the path length between players. Interestingly, however, we do observe for *particular types* of small world network the hypothesized relationship. Thus, in certain networks we observe both instability of the risk dominated equilibrium and a fast rate of convergence to the risk dominant equilibrium.

We proceed as follows: Section 2 introduces the dynamic and additional notation. In Section 3 we discuss different types of network. In Section 4 we set our approach before providing results in Sections 5 and 6. Section 7 concludes.

2 The Learning Dynamic

There exists a finite population of players $N = \{1, ..., n\}$. Members of the population are linked by an (undirected) *matching network* represented by matrix $R = [r_{ij}]$. If $r_{ij} = 1$ there is an *edge* between players i and j and if $r_{ij} = 0$ there is no edge between players i and j. If there is an edge between two players i and j then we say, interchangeably, that i and j are *neighbors* or that i and j *interact with each other*. We denote by (i, j) an edge linking players i and j. For each player $i \in N$ we denote by $R(i) \subset N$ the set of neighbors of i.

Given any two distinct players $i, j \in N$ we denote by $\tau(i, j)$ the *distance* between i and j in the matching network. Distance is defined by the minimum number of edges that need to be traversed to go between i and j. Thus, for example, $\tau(i, j) = 1$ if and only if i and j are neighbors. For each player $i \in N$ we set $\tau(i, i) = 0$.

Play proceeds over an indefinite number of discrete time periods, indexed $t = 0, 1, 2, ...$ In each period every member of the population chooses one of two *strategies* G or B. A *strategy vector* is given by a vector $s \in \{G, B\}^N$ where s_i is the strategy of player i. Let S denote the set of strategy vectors. We denote by $s(t)$ the strategy vector in period t. Thus, $s(0)$ is the initial strategy vector.

Given strategy vector $s \in S$ we denote by $g(s)$ the number of players playing strategy G. We say that strategy G is *conventional* if $g(s) > \frac{n}{2}$; that is, if half of the population are playing it. Let $\overrightarrow{B}$ denote the strategy vector where every player plays B and $\overrightarrow{G}$ the strategy vector where every player plays G. Thus, $g\left(\overrightarrow{B}\right) = 0$ and $g\left(\overrightarrow{G}\right) = n$.

There exists a *threshold value* q. In each period $t > 0$ every player $i \in N$ is assumed to choose strategy G if and only if proportion q or more of those players with whom he interacts played G in period $t - 1$.[1] Note that every player revises his choice of strategy in every period. It will be assumed throughout that $0 < q < 0.5$ implying a bias in favor of strategy G.

The above behavior gives rise to a deterministic dynamic that we refer to as a *best reply dynamic*. We denote the dynamic by $\mathcal{D}(s, R, q)$ where s is the initial state, R the reference network and q the threshold value. Given any initial state $s \in S$ the long run dynamics of the best reply dynamic $\mathcal{D}(s, R, q)$ can be traced out with certainty. Note that both $\overrightarrow{G}$ and $\overrightarrow{B}$ are absorbing states.

3 Networks

We wish to contrast networks that have differing structures. Two common measures of network structure are characteristic path length and clustering

[1] The reference network R and threshold value q are assumed to be constant over time.

coefficient.[2] After defining the characteristic path length and clustering coefficient we introduce the type of networks we will model in this paper.

The *characteristic path length* L of matching network R is a measure of the typical distance between players in the network.[3] Formally, define L_i as follows,

$$L_i = \frac{1}{n-1} \sum_{\substack{j \in N \\ j \neq i}} \tau(i,j)$$

The characteristic path length L is defined as the median value of L_i, taken over all players $i \in N$. Thus, if a network has a characteristic path length of x this suggests that on average the shortest path length between two players is x.

The *clustering coefficient* C is a measure of the cliquishness of a network. Formally, define C_i as follows,

$$C_i = \frac{1}{\binom{|R(i)|}{2}} \sum_{j \in R(i)} \sum_{\substack{l \in R(i) \\ l \neq j}} r_{jl}$$

Note that $\binom{|R(i)|}{2}$ is the total number of edges that there could be between the neighbors of player i while $\sum_{j,l} r_{jl}$ is the actual number of edges. The clustering coefficient C of network R is defined as the mean value of C_i. If a network has a clustering coefficient of C this would suggest that, for any three players $i, j, k \in N$, if $j, k \in R(i)$ then $j \in R(k)$ with probability C.

3.1 Regular 1 lattice

A *regular* 1 *lattice of degree* k has the property that each player $i \in N$ is neighbors with the nearest $\frac{k}{2}$ players to each side. Thus, player i interacts with players $i \pm d \pmod n$ where $1 \leq d \leq \frac{k}{2}$. Each player has, therefore, k neighbors. Ellison (1993) primarily treats interaction on a regular 1 lattice while Blume (1993, 1995) treats interaction on a 2 lattice.

3.2 Random regular networks

A model of *random networks* consists of a set of networks M and a probability distribution over that set p. Thus, for example, M may be the set of

[2] A third characteristic is network size - the total number of edges. This is something we will keep constant.

[3] Our definitions of the charactersitc path length and clustering coefficient are standard - see, for example Watts (2000).

all networks that have m edges and p selects amongst them equally. This particular model is not very useful for our purposes; for example, there will, almost surely, be players who interact with no player. An alternative, as taken by Cartwright (2004) is to consider random regular networks. Let $\mathcal{G}(n, k)$ denote the set of all networks that have n vertices and where each vertex has k edges. We refer to a network as a *random regular network of degree k* if it was randomly selected from the set $\mathcal{G}(n, k)$.[4] [Models of random networks and random regular networks are covered in detail by Ballobas 2001].

3.3 Small worlds

Between the extremes of a regular 1 lattice and a regular random network we can conceive of networks with 'varying degrees of randomness'. To model this we make use of the concept of a *β-graph* as defined by Watts (2000). [The material covered in this section is treated in more detail by Watts (2000)] A β-graph, where $\beta \in [0, 1]$ is constructed using the following algorithm:

1. begin with a perfect regular 1-lattice of degree k.

2. taking each player $i \in N$ in turn consider the edge between player i and his nearest neighbor in a clockwise sense $i + 1$. With probability β the edge $(i, i + 1)$ is *rewired*. If the edge is rewired then it is deleted from the network and player i is randomly linked to a player j for which there is no existing link to form an edge (i, j).[5] With probability $1 - \beta$ edge $(i, i + 1)$ is not rewired in which case the edge remains and the algorithm continues.

3. taking each player $i \in N$ in turn consider, the edge between i and its second nearest neighbor in a clockwise sense $(i, i + 2)$ for rewiring, and

[4] In the simulations, reported below, we did not consider random regular networks. Instead, as we explain immediately below, we considered a form of random network where the number of neighbours of a player can very between $\frac{k}{2}$ and (theoretically) $n - 1$. The mean number of neighbours of a player is k. To the best of our knowledge there are no results on the charactersitics of networks of this form. Our strong intuition, however, is that these types of network will have almost identical properties to those of random regular networks. We base this presumption on the fact that each player must interact with at least $\frac{k}{2}$ players. Note that this immediately overcomes the principal problem (for our purposes) of other models of random networks where some players almost surely have no neighbours.

[5] Note that edge $(i, i + 1)$ could be deleted and then immediately reinstated.

so on. In all there are $k/2$ rounds of the rewiring process as outlined in stage 2 of the algorithm.

We note that in any β graph each player has at least $\frac{k}{2}$ neighbors and the average number of neighbors is k. If $\beta = 0$ then the resulting network is a regular 1 lattice - no edges are rewired. If $\beta = 1$ then the network will resemble a random regular network - all edges are rewired. In varying β between these two extremes we can generate a range of different networks.

Figure 1 [all figures follow the Conclusion] illustrates how the clustering coefficient and characteristic path length vary with β (when $n = 1000$ and $k = 10$). Note that L varies between 3.3 and 46.65 while C varies between 0.009 and 0.666. The difference in structure between a regular 1 lattice and random network are apparent and important. Note also the possibility of small world networks. Formally, we say that a network *is a small world network* if it has a relatively *small characteristic path length* and relatively *large clustering coefficient*. We observe small world networks for a value of β around 0.01 to 0.1.

4 Matching networks and stability

In this section we set out our approach and summarize results concerning the two extreme cases of a regular 1 lattice and random regular network.

4.1 Clusters

We say that a *strategy vector s has a cluster of size D centered on player i* if and only if there exists integer $\widetilde{\tau}$ such that,

1. $g(s) = D$,
2. if $\tau(i, j) < \widetilde{\tau}$ then $s_j = G$,
3. if $\tau(i, j) > \widetilde{\tau}$ then $s_j = B$,

The first condition states that D players are playing strategy G. The second and third conditions state that all those sufficiently close to i are playing G and all those sufficiently far away are playing B. This leaves those players at distance $\widetilde{\tau}$ from i who may be playing either G or B. We refer to $\widetilde{\tau}$ as the *radius of the cluster*.

4.2 Simulation approach

Given a network R, let $s(0)$ be an initial state that has a cluster of size D centered on some player i and consider the best reply dynamic $\mathcal{D}(s(0), R, q)$. The evolution of the dynamic can be mapped out and the number of players playing strategy G recorded. Thus, given values for β, n, k, q and D we simulate the best reply dynamic by randomly generating a β-graph, randomly picking a player i, randomly choosing an initial state that has a cluster of size D centered on player i and then running the best reply dynamic. Data recorded included whether or not strategy G became conventional, that is, whether there existed some $t > 0$ such that $g(t) > \frac{n}{2}$, the characteristic path length and clustering coefficient of the network, and whether $g(t)$ exceeded values other than $\frac{n}{2}$ and if so the number of periods it took to do so. We discuss this data in more detail as we proceed.

4.3 Stable and unstable

Our principal objective is to consider the stability of state $\overrightarrow{B}$. Under the best reply dynamic $\overrightarrow{B}$ is an absorbing state and so its stability is not under question. Suppose, however, we consider a scenario in which players may switch to alternative G even if this is not a best reply. For example, a player i may be able to 'persuade' those players 'near to him' in the network to 'experiment with G'. [Note that at no point will we formally model this possibility]. How many people does i need to persuade to experiment in order for the best reply dynamic to converge on states where G is conventional? If this number is small or, more specifically, independent of n then we might think of $\overrightarrow{B}$ as unstable. Conversely, if the number is large and increasing in n then we might think of $\overrightarrow{B}$ as stable.

The issue of stability can be highlighted by considering the extreme cases of a random regular network and a regular 1 lattice. The following two propositions (somewhat informally stated) are proved in Cartwright (2004). (Proposition 1 is easily derived from Ellison (1993) or Blume (1993)).

Proposition 1: If R is a regular 1 lattice of degree k then given any initial state with a cluster of size $k + 1$ the best reply dynamic converges on state $\overrightarrow{G}$.

Proposition 2: Let $qk > 1$.[6] Given any integer D and any real number $\varepsilon > 0$, as $n \to \infty$ for almost every network in $\mathcal{G}(n, k)$ and any initial state

[6] That is, a typical player plays G only if at least two of his neighbors played G in the previous period.

with cluster of size D the probability that the best reply dynamic converges to state $\overrightarrow{B}$ is at least $1-\varepsilon$.

Thus, in a regular 1 lattice the state $\overrightarrow{B}$ is relatively unstable and in a random regular network relatively stable.

5 The data on stability

To question the stability of $\overrightarrow{B}$ for intermediate values of β we look to measure (given β, n, k, q) what values of D are consistent with strategy G becoming conventional for some $t > 0$. The procedure we used can be detailed as follows: a β-graph was generated. Fixing a value of D ten simulations of the best reply dynamic were run (with a randomly generated initial state that had a cluster of size D). By incrementally increasing D we recorded: (i) the minimum D such that G became conventional in at least one of ten simulations (denote this DL) and (ii) the minimum D such that G became conventional in all ten simulations (denote this DU). The reported values of DU and DL are the mean average values from repeating the above procedure in generating many networks for the given value of β.[7]

We treat DU and DL as measures of the stability of $\overrightarrow{B}$. The larger is DL and DU then the more stable it would appear is $\overrightarrow{B}$. Clearly, the above procedure is somewhat arbitrary but we feel is as good as any other in providing bounds on the cluster size for which G can be seen to become conventional.[8] Certainly, the data (Figures 2-4) appear to give a reliable picture and experimentation with all aspects of the procedure did not change the 'look of this data'.[9]

We present our results. Figures 2, 3 and 4 (and Tables A1 - 7 in the Appendix) give the recorded estimates of DL and DU. Figure 2 presents the data for the case where $n = 1000$ and $k = 10$; i.e. there are 1000 players and on average a player has 10 neighbors. The data confirm that $\overrightarrow{B}$ is relatively unstable when $\beta \approx 0$ and stable when $\beta \approx 1$. There is also clear evidence

[7] Typically, 'many' equated to 20 different networks. There was, however, vary little observed variance in the values of DL and DU between the different networks (for the fixed β).

[8] Note that given any β and D there is some probability that G will become conventional and that probability will (realistically) be intermediate between 0 and 1. It will thus inevitably be a matter of interpretation as to whether we judge G as likely or not to become conventional.

[9] The particular procedure used had the benefits of saving on computer time.

of an increase in the stability of $\overrightarrow{B}$ as β increases. Indeed, the data would suggest the following conjecture,

Conjecture 1: Given values of N, k and q there exists a critical value β^* such that if $\beta < \beta^*$, almost surely, for any β-graph R and any initial state s that has a cluster of size $k+1$ the best reply dynamic $\mathcal{D}(s, R, q)$ will converge to a state where G is conventional. Further, β^* is an increasing function of q.[10]

Thus, for sufficiently small β the stability of $\overrightarrow{B}$ is similar to that of a regular 1 lattice. Figure 4 provides results for a range of values of n and k.[11] It is worth noting that, even when $\beta = 1$, many players have 10 neighbors; so, for example if $q = 0.35$ this implies that a typical player will choose G if and only if she has 4 or more neighbors who played G in the previous period. This implies that the crucial variable is not so much q as $\frac{1}{k}\lceil qk \rceil$ where $\lceil \cdot \rceil$ denotes the nearest integer greater than or equal to $\cdot$. For example, having $q = 0.475$ and $k = 20$ would appear the analogue to having $q = 0.45$ and $k = 10$ in that in both scenarios the typical player will only play G if half or more of his neighbors played G in the previous period. With reference to Figure 4 we suggest the following.

Conjecture 2: The critical value β^* is invariant to changes in n, q and k if $\frac{1}{k}\lceil qk \rceil$ is held constant.

Given that we are interested in modelling large populations, crucial is to observe whether the values of DU and DL are independent of n. Propositions 1 and 2 suggest that they should be when $\beta = 0$ but not when $\beta = 1$. Consider the doubling of n from 1000 to 2000 while k remains constant at 10. We appear to observe that for $\beta < \beta^*$ both DU and DL are invariant to the change in n; this is to be expected given Conjecture 1 and Proposition 1. If, however $\beta > \beta^*$ we observe a doubling of both DU and DL as n doubles. Thus, it appears the DU and DL change proportionally to changes in n. Changing qk while keeping n fixed appears to have little effect on DU and DL. The following appears consistent with the data.

[10] Note that an alternative would be to think in terms of a critical value of $\frac{1}{k}\lceil qk \rceil$ (see below) for each value of β. We prefer using a critical value of β on the basis that β is continuous while $\frac{1}{k}\lceil qk \rceil$ is discrete and, for small k, can take relatively few values.

[11] Additional values of n and k were considered and the picture remained as consistent as Figure 3 suggests.

Conjecture 3: For $\beta > \beta^*$, fixed k and q and any D the probability that G becomes conventional (for randomly constructed β-graphs and random initial states with cluster of size D) is decreasing in the size of the population.

Conjecture 3 would imply that in large populations $\overrightarrow{B}$ is stable when $\beta > \beta^*$. It would also imply a very distinct change is observed between $\beta < \beta^*$ and $\beta > \beta^*$. One important issue is what level the critical value β^* takes as a function of n, k and q. Of particular interest is to see on what side of the critical value we find small world networks. Let β^s denote a value of β consistent with β-graphs that have small world characteristics. Figures 2 and 4 would suggest that $\beta^s < \beta^*$ when $\frac{1}{k}\lceil qk \rceil \leq 0.4$ but that $\beta^s > \beta^*$ when $\frac{1}{k}\lceil qk \rceil = 0.5$. Figure 4 presents results where $n = 3000$ and $k = 40$; the larger value of k permits us to look at a range of q values between 0.4 and 0.5. The data would certainly suggest the following,

Conjecture 4: If $\frac{1}{k}\lceil qk \rceil < 0.4$ then $\beta^s < \beta^*$.

More generally it appears that strategy vector $\overrightarrow{B}$ will typically be unstable in small world networks unless the critical value is close to 0.5; that is, unless players only marginally favor strategy A. This is an important result. First, and most importantly, it suggests that in small world networks, even if a risk dominated strategy is a current convention, the experimentation of relatively few can tip the dynamics in favour of the risk dominant strategy. Thus, it would not be unreasonable to talk of the risk dominant strategy as being likely in the medium to long run. Second, it answers an open question in the literature as to when the risk dominated equilibrium is unstable in this way. Previous results (Ellison 1993, Blume 1993, 1995, Young 2001) suggest that whenever the clustering coefficient is high the risk dominated equilibrium will be unstable. Morris (2000), however, shows that a low neighborhood growth property is a sufficient condition for instability of the risk dominated equilibrium. In the terminology of the current paper this low neighborhood growth property is essentially equivalent to a high characteristic path length. Given that 'in most networks' a high clustering coefficient and a high characteristic path length go hand in hand it is difficult to distinguish what is really causing the instability of the risk dominated equilibrium. In considering small world networks we can answer this question. Our results suggest that the clustering in the network is the principal determinant of the stability of the risk dominated equilibrium. More conclusively we find that a high characteristic path length is not a necessary condition for the risk dominated equilibrium to be unstable.

6 Other issues

While our central motivation was to consider the stability of $\overrightarrow{B}$ we briefly consider in this section two other questions of interest. First, how quickly strategy G is adopted by players if G does become conventional. Second, the possibility of long run outcomes in which both strategies G and B coexist.

6.1 The rate of contagion

One motivation for considering the stability of $\overrightarrow{B}$ was to gain some understanding of whether the weighting times for strategy G to become conventional could be seen as economically plausible or not. We argued that the stability of $\overrightarrow{B}$ is crucial to addressing this issue as it reflects the likelihood that experimentation could 'tip the balance' in favor of strategy G. A second determinant of the expected weighting time for strategy G to be conventional will be the time it takes for players to take up strategy G *once* the balance has been tipped in its favor. As discussed by Blume (1995) the overall weighting time for strategy G to become conventional will be primarily determined by the first of these factors but the second factor is also clearly of interest.

To obtain some data on the speed of convergence simulations were run as follows: a β-graph is generated and the best reply dynamic simulated with an initial cluster size of D^* (see below); recorded (for the case of $n = 1000$) are the number of periods before $500, 600, 700$ and 800 players are playing strategy G.[12] The choice of D^* is clearly important and we choose $D^* \approx DU$ where DU is the value noted earlier in Section 4.1. We highlight that the value of DU is specific to the value of β and as a consequence relatively little can be gained by comparing across β the number of periods it takes before 500 players are playing strategy G. The choice of D^* reflected two considerations (1) to guarantee that strategy G become conventional in the majority of cases and (2) to have as few players as possible playing strategy G in the initial period in order that the contagion process is not overly driven by the initial state.[13]

Figure 5 (see also table A8) provides results for the case where $n = 1000$ and $k = 10$. Figures 6, 7 and 8 (see also tables A9 - A11) provide analogous

[12] One data point (for fixed β and D) was the average of 500 simulations as given by 50 different β-graphs and 10 different initial strategy vectors with a cluster of size D for each β-graph.

[13] Having experimented with alternative values of D^* our results seem robust to the choice of initial strategy. Also, in the majority of cases $D^* < k + 1$.

results for differing values of n and k. One notable feature of the data is the apparent proportionality between the number of periods elapsed and the number of players playing G. Let Δ denote the time taken for the number of players playing G to increase by 100. It appears that Δ is independent of the number of players playing G. Comparing Figures 5 and 7 (and 6 and 8) we observe that Δ appears highly dependent on n. This would imply long weighting times in large populations. Also, comparing Figures 5 and 6 (and 7 and 8), we observe that a ceteris paribus increase in k decreases Δ except when q is near to 0.5.

A key issue is how Δ varies with β. One clear observation is that Δ appears to be a decreasing function of β. That is, as we move to more random networks the rate of contagion is quicker. Generally, however, we observe that in small world networks the value of Δ is relatively high and the same as that in a regular 1 lattice.[14] It may have been conjectured that the rate of contagion would be related to the characteristic path length (as will be the case when $q = 0.01$). Our results clearly suggest the contrary.

One aspect of the data is, however, worth highlighting: While the rate of contagion does not appear to be fast for small world networks *in general* for *particular* values of β we do obtain both a small world network and relatively fast rates of convergence. In particular, we observe this for values of β around 0.1.[15] Table 1 provides results of simulations in which we directly compared the cases $\beta = 0, 0.1$ and 1.

β	L	C	$q = 0.05$		$q = 0.15$		$q = 0.25$		$q = 0.35$		$q = 0.45$	
			NC	TC	NC	TC	NC	TC	NC	TC	NC	TC
0	50.45	0.667	1	30	1	37	1	50	1	75	1	150
0.1	4.42	0.49	1	1	1	2	1	11	0.07	66	0	–
1	3.26	0.009	1	1	0.02	1	0	–	0	–	0	–

Table 1: Comparing properties of the best reply dynamic with $n = 1000$ and $k = 10$. Recorded are the characteristic path length (L), the clustering

[14] Two exceptions are when $q = 0.01$ and $q = 0.45$. When $q = 0.01$ the rate of contagion will be directly related to the characteristic path length - hence, the observed results (see Figure 1). Interpreting the results for $q = 0.45$ is difficult given the neccessarily large value of D^* needed to guarantee convergence. How much can be read into the results is thus open to question but this does seem an issue worth pursuing further.

[15] Note that this cannot be due to the choice of D^* as D^* was typically less than $k + 1$ for β around 0.1.

coefficient (C) and for differing values of q the proportion of simulations out of 500 (NC) where strategy G became conventional given a cluster size of $k+1=11$ and, if it did so, the time taken (TC) for the number of players playing G to increase from 500 to 800.

The results in Table 1 suggest the interesting possibility that in certain networks we have that both (1) $\overrightarrow{B}$ is relatively unstable and (2) the weighting time for G to become conventional once 'the dynamics are in its favor' is relatively short. These two factors clearly combine to suggest a relatively short weighting time for strategy G to become conventional. As can be seen in Table 1, for $qk \leq 3$ we do observe, when $\beta = 0.1$ both relative instability of $\overrightarrow{B}$ and a relatively fast rate of convergence.

6.2 The coexistence of multiple strategies

Typically we observed that the best reply dynamic converged to either $\overrightarrow{B}$ or $\overrightarrow{G}$. There is, however, the potential for long run dynamics in which both strategies B and G coexist. For results and discussion on the possible long run existence of multiple strategies see Young (1998) or Morris (2000). To obtain some data on the likelihood that multiple strategies coexist, in the simulations discussed in the previous section 6.1, we recorded the number of players playing strategy G in the absorbing state to which the dynamic converged.[16]

The case of $q < 0.45$ was uninteresting with state $\overrightarrow{G}$ emerging in over 99% of simulations. The case of $\lceil qk \rceil = \frac{k}{2}$ proves more interesting with the results summarized in Figure 9. When β is close to 0 or 1 we rarely see the long run coexistence of strategies. In small world networks, however, we find a dramatic increase in the possibility of coexistence of strategies. Indeed the coexistence of strategies is both likely and significant: significant in the sense that a significant number of players may be playing strategy B even if strategy G is conventional. For example, when β is around 0.01 we observe that in over one half of the simulations in which strategy G becomes conventional more than one fifth of the population typically continue to play strategy B in the long run.[17]

[16] There is the potential for non-singleton communication classes - or equilibrium cycles. These appeared at most very few times and if they did occur the number playing G after 4000 periods is recorded.

[17] This cannot be due to insufficient time - it is a simple matter to check when an absorbing state has indeed been reached.

What are we to make of this result? The first thing to note is that we only observe the likely coexistence of strategies when qk is near $\frac{k}{2}$ and are thus in the range where $\overrightarrow{B}$ is relatively stable in small world networks. It would seem therefore that in small world networks, if qk is near $\frac{k}{2}$, strategy G can 'survive' in the population without necessarily 'spreading' throughout the population. This is in contrast to random regular networks where strategy G can only survive in the population by spreading throughout that population.

7 Conclusion

We have consider a simple model of learning in which players interact through a network. Of key interest was to consider the process whereby a strategy can become conventional. It has long been known that the structure of the interaction network can influence the likelihood of contagion (Ellison 1993, Blume 1995 and Morris 2000). In a companion paper (Cartwright 2004) we compare a regular lattice to random networks. The main contribution of the current paper has been to consider interaction networks that lie between the extremes of randomness and a regular lattice.

Of particular interest has been to consider interaction in small world networks. Our interest in this is motivated partly by the fact that many economic and social networks appear to have small world characteristics (Watts 2000). Also, as small world networks lie somewhere between the extremes of randomness and regularity we wished to question whether the likelihood of contagion is more akin to that in random or in regular networks. The general conclusion we make is that contagion in the small world networks that we model is comparable to that of a regular lattice. One implication is that in small world networks a risk dominated equilibrium is relatively unstable while the rate of contagion is relatively slow. We do find, however, that for certain small world networks that the rate of contagion can be relatively fast and the risk dominated equilibrium unstable. Our results certainly suggest that in treating small world networks it is not unreasonable to talk of the risk dominant equilibrium as being the expected medium to long run convention (of 'a best reply dynamic with experimentation') even if the risk dominated equilibrium is the starting convention.

There seem two clear avenues for future research. First, to consider more general models of small world networks. Here we consider a particular form of small world network constructed using the algorithm of Watts (2000). Our conjecture is that the risk dominated equilibrium is generally speaking unstable in small world networks - stability being related to the clustering in the network. Our results suggest, however, that the rate of contagion can differ depending on the specific form of the network. In particular, there may be a class of networks that produce both instability of the risk dominated equilibrium and a fast rate of contagion. One problem with pursuing this avenue of research is a lack of algorithms for constructing general small world networks. A second possible line for future research is to consider heterogenous populations. That is a population in which the critical value q differs across the population. Again, our conjecture would be that the stability of the risk dominated equilibrium would not be altered by this change but that the rate of contagion may be. In running preliminary simulations with heterogenous populations we, in fact, found results similar to those reported in this paper. A problem, however, is how to introduce heterogeneity - in our simulations results were highly sensitive to the way heterogeneity was introduced.[18] Introducing heterogeneity, while being an interesting possibility, is thus one we leave for future research.

[18] We considered the following, a player's q is uniformly drawn from some interval $\left[\underline{q}, \overline{q}\right]$. Qualitatively we obtained the same results as reported in the paper. A slight change in $\underline{q}$ and $\overline{q}$ could, however, lead to a case where strategy G became conventional for $D = k$ irrespective of β to one where it never became conventional for $D = k$ irrespective of β.

8 Appendix

Table A1: Values of DU and DL for $n = 1000$ and $k = 10$.

	$q = 0.15$		$q = 0.25$		$q = 0.35$		$q = 0.45$	
β	DU	DL	DU	DL	DU	DL	DU	DL
0.0001	5	5	5	5	5.9	5.3	9.3	7.6
0.000373	5	5	5	5	5.8	5.2	35.1	9.6
0.0014	5	5	5	5	5.9	5.1	179.8	29.3
0.0052	5	5	5	5	6	5.2	257	111.9
0.0193	5	5	5	5	6.5	5.3	358.8	261
0.072	5	5	5.2	5	43.8	8.1	454.9	396.5
0.2683	5.1	5	15.9	5.8	208	176	467.4	440.2
1	19.5	14.6	104	95.1	268.6	259.6	436	427.2

Table A2: Values of DU and DL for $n = 2000$ and $k = 10$.

	$q = 0.15$		$q = 0.25$		$q = 0.35$		$q = 0.45$	
β	DU	DL	DU	DL	DU	DL	DU	DL
0.0001	5	5	5	5	5.8	5.3	28	7.4
0.000373	5	5	5	5	5.9	5.1	203.8	29.8
0.0014	5	5	5	5	6	5.2	381.5	97.3
0.0052	5	5	5	5	5.9	5.3	467	257.5
0.0193	5	5	5	5	6.2	5.5	674.8	556.2
0.072	5	5	5.1	5.1	62.3	14.5	877.5	799.3
0.2683	5.1	5	28.8	6.1	385.2	329.1	900.5	860.6
1	35.3	25.4	194.6	179.2	532	515.4	879	872.9

Table A3: Values of DU and DL for $n = 3000$ and $k = 10$.

	$q = 0.15$		$q = 0.25$		$q = 0.35$		$q = 0.45$	
β	DU	DL	DU	DL	DU	DL	DU	DL
0.0001	5	5	5	5	5.9	5.3	13.6	7.4
0.000373	5	5	5	5	6	5.1	400.2	38.5
0.0014	5	5	5	5	6	5.4	557.7	144.9
0.0052	5	5	5	5	6	5.2	688.1	429.9
0.0193	5	5	5	5	6.3	5.7	1005.2	823.9
0.072	5	5	5	5	76	29.4	1323.2	1203.2
0.2683	5.1	5	36	7.9	602.3	501	1348.4	1286.2
1	51.4	39	283.6	270.4	799	787.6	1317	1292.2

Table A4: Values of DU and DL for $n = 1000$ and $k = 20$.

	$q = 0.15$		$q = 0.25$		$q = 0.35$		$q = 0.45$	
β	DU	DL	DU	DL	DU	DL	DU	DL
0.0001	5	5	7.4	7	11.3	10.2	27.4	13.8
0.000373	5	5	7.5	6.9	11.3	10.2	143	26.2
0.0014	5	5	7.4	7	11.3	10.1	265.3	92.7
0.0052	5	5	7.4	7.1	11.3	10.2	371	255.8
0.0193	5.1	5	8.1	7.2	12.1	10.4	475.8	396.2
0.072	5.6	5.1	10.1	8	14	11.4	477.7	426.7
0.2683	8.9	5.9	17	11.6	306.9	214	503.4	491.6
1	79.1	73.6	204.1	199.1	341.5	331	489.3	473.2

Table A5: Values of DU and DL for $n = 2000$ and $k = 20$.

	$q = 0.15$		$q = 0.25$		$q = 0.35$		$q = 0.45$	
β	DU	DL	DU	DL	DU	DL	DU	DL
0.0001	5	5	7.6	7	11.1	10.3	158.2	14.2
0.000373	5	5	7.4	7	11.2	10.2	391.5	88.2
0.0014	5	5	7.5	7	11.1	10.3	515.4	240.8
0.0052	5	5	7.7	7.1	11.4	10.5	696.7	549.2
0.0193	5.1	5	7.9	7	11.9	10.2	969.7	853.4
0.072	5.5	5.1	9.6	7.8	17.1	11.8	1038.5	972.3
0.2683	9	5.9	17.8	12.7	585	523.3	969.4	942.4
1	157.6	149.5	410	399	669.6	661.6	967	967

Table A6: Values of DU and DL for $n = 3000$ and $k = 20$.

	$q = 0.15$		$q = 0.25$		$q = 0.35$		$q = 0.45$	
β	DU	DL	DU	DL	DU	DL	DU	DL
0.0001	5	5	7.3	6.7	11.2	10.3	103.4	15.2
0.000373	5	5	7.6	7.1	11.2	9.9	589.6	124.9
0.0014	5	5	7.5	6.9	11.3	10.1	770.3	360.7
0.0052	5.1	5	7.3	6.8	11.4	10.1	1054.8	839.9
0.0193	5.1	5	8.2	7	12	11	1381.2	1245.3
0.072	5.7	5	9.6	7.6	14.1	11.4	1518	1421.2
0.2683	9.5	5.7	18	12.3	830.4	772.7	1441.2	1414
1	237.1	223	599	599	1014	995.2	1414	1414

Table A7: Values of DU and DL for $n = 3000$ and $k = 40$.

	$q = 0.4125$		$q = 0.4375$		$q = 0.4625$		$q = 0.4875$	
β	DU	DL	DU	DL	DU	DL	DU	DL
0.0001	23.2	21.5	25.2	23.2	27.6	25.5	483.8	80.8
0.000373	23.3	21.5	25	23	27.6	25.2	801.9	383.2
0.0014	23.2	21.8	25.4	23.7	27.8	25.2	1066.2	827.7
0.0052	23.6	21.6	25.3	23.8	27.9	25.5	1507	1222.5
0.0193	24.8	22.6	27.2	24.4	659.2	183.6	1615.4	1357.2
0.072	29.6	26	1005.2	277.6	1341.8	1262.8	1503.6	1428
0.2683	1203.2	1025	1348	1298.6	1482	1461.6	1576.8	1554
1	1226	1226	1286	1286	1414	1414	1482	1482

Table A8: The number of periods taken for $g(s)$ to increase from 500 to 800 (or in the case of $q = 0.45$ from 600 to 800) when $n = 1000$ and $k = 10$.

β	$q = 0.01$	$q = 0.15$	$q = 0.25$	$q = 0.35$	$q = 0.45$
0.0001	27	38	50	75	114
0.000373	19	38	50	75	142
0.0014	10	38	50	75	100
0.0052	3	37	50	74	68
0.0193	1	19	49	73	51
0.072	1	3	28	62	28
0.2683	0	1	2	6	12
1	0	1	1	1	1

Table A9: The number of periods taken for $g(s)$ to increase from 500 to 800 (or in the case of $q = 0.475$ from 600 to 800) when $n = 1000$ and $k = 20$.

β	$q = 0.01$	$q = 0.175$	$q = 0.275$	$q = 0.375$	$q = 0.475$
0.0001	12	22	30	50	119
0.000373	8	22	30	50	141
0.0014	3	22	30	50	83
0.0052	1	21	29	50	56
0.0193	1	21	29	48	36
0.072	1	11	27	45	23
0.2683	0	1	3	9	8
1	1	0	1	1	1

Table A10: The number of periods taken for $g(s)$ to increase from 1000 to 1600 (or in the case of $q = 0.45$ from 1200 to 1600) when $n = 2000$ and $k = 10$.

β	$q = 0.01$	$q = 0.15$	$q = 0.25$	$q = 0.35$	$q = 0.45$
0.0001	50	75	100	150	232
0.000373	29	75	100	150	258
0.0014	10	74	100	150	164
0.0052	3	72	99	149	110
0.0193	1	25	97	147	68
0.072	1	3	37	101	41
0.2683	0	1	2	7	14
1	1	1	1	1	2

Table A11: The number of periods taken for $g(s)$ to increase from 1000 to 1600 (or in the case of $q = 0.475$ from 1200 to 1600) when $n = 2000$ and $k = 20$.

β	$q = 0.01$	$q = 0.175$	$q = 0.275$	$q = 0.375$	$q = 0.475$
0.0001	19	43	60	100	257
0.000373	9	43	60	100	188
0.0014	3	43	60	100	134
0.0052	1	43	60	99	89
0.0193	1	42	58	97	33
0.072	0	14	54	89	41
0.2683	0	1	4	10	10
1	0	0	1	1	1

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