On the valuation and capital cost of project flexibility within sequential investment

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Abstract

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Valuing sequential investment requires choosing uncertainty processes. Having committed to a diffusion and its differential equation, solutions and options are customized to terminal boundary conditions but generalizations for other processes or flexibility sequences has not been possible.

By placing investment within a mathematical graph (network), we separate diffusion choices from the flexibility sequencing. To facilitate smooth pasting, standardized discount functions are placed within an investment matrix.

This allows the investment costs and values for multiple flexibilities to be determined as a function of their triggers for any diffusion. Insights are offered concerning valuation and capital costs of flexibility.

Keywords: value matching, smooth pasting, flexibility discounting, real options, investment sequence and bi-partite directed graph.
1 Introduction

Great strides have been made in valuing financial flexibility through the development of dynamic asset pricing. In particular, delta hedging and risk neutral valuation\(^1\) have for a range of underlying stochastic processes allowed options on stocks, bonds, rates and operational assets to be priced. In the last asset class, if timing flexibility is present and investment in a new project must justify capital costs, then fundamental asset valuation can benefit from the study of operational flexibility.\(^2\)

At the heart of dynamic asset valuation is the idea that projects having operational flexibility can be valued using similar means to those with financial optionality. Given a diffusion specification for a random variable and the costs of control, this frequently reduces to a question of determining optimal trigger points for a policy. Whilst it may be appropriate to question the applicability of risk neutral valuation to corporate or operational situations (if the underlying risks are not traded) this assumption has facilitated theoretical progress and many papers, tailored to energy and other tradeable assets, have adopted such techniques to solve investment and switching problems.\(^3\)

However the complexity of the operations that have been modelled to date is limited, often to a few sequential decisions after which the project becomes inflexible. Even when the number of embedded option functions that require matching at boundaries is limited, presenting trigger thresholds as a function of investment quantities results in a non-linear system of equations, which cannot be solved analytically. Consequently for numerical solutions that depend on a particular choice of parameters, it is often difficult to say if the results would also apply to other parameter or diffusion choices or to find useful heuristics and generalizations.

It is also unrealistic to assume a simple sequence, especially one that ends with no flexibility. Most projects do not become passive and inflexible after early decision making has been exercised and managers must continually examine compound and value improving strategies. It is also unrealistic to assume that the uncertainty process remains the same before and after each

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\(^1\) Black and Scholes (1973) [3], Merton (1973) [15] and Cox, Ross and Rubinstein (1976) [6].

\(^2\) The term real options dates from Myers (1977) [16], but also see Brennan and Schwartz (85) [4], McDonald and Siegel (85) [14] and the texts of Dixit and Pindyck (1994) [8] and Trigeorgis (1996) [20].

\(^3\) Also including costly reversibility (Abel and Eberly 96, [2]), Q theory, marginal cost of capital (Hayashi 82 [12], Abel et al. 96 [1]) and research and development (Childs and Triantis 99 [5] and Liu and Wong [13] 11).
investment so it would be useful to be able to vary diffusions across the stages of a project’s life and flexibility.

In this article we formalize a solution technique for flexibility that encompasses a wide range of multistage problems, arbitrary numbers of thresholds and different diffusion choices. This tractability is attained by presenting investment quantities and option values as functions of trigger thresholds, rather than the other way round. Initially assuming thresholds whilst ensuring these are consistent with optimal values produces a set of equations for investment quantities and flexibility values that are linear in other factors. This system of equations is capable of solution by matrix methods.

At each decision threshold, we examine what flexibility is consumed and what is generated net of required investment. The careful identification and separation of flexibility before and after investment allows the use of discounting and matrix techniques to untangle and optimize problems of any scale.

Discount factors work for flexibility because, unlike the project’s current cash revenue and cost rates, the option to change flows in the future has no immediate flow or dividend. Flexibility is valued only as a discount instrument whose payoff depends on future changes in revenues and costs commensurate with its use. Understanding that flexibility generated today will be consumed in the future, allows linking dependence together using present values techniques.

We use discount functions\(^4\) in matrices to form flexibility chains in a valuation system that captures sequential dependence. The passage of decision taking is marked on a mathematical graph (network) by transitions from one operational and flexibility state to another. Especially for those systems that contain recursive or circular investment cycles (that cannot be worked back from a final end condition) this facilitates solution of simultaneous linear equations. The particular choice of matrices is also driven by the need to encompass first order optimality conditions in the same framework. The formal solution method derived applies to frictional, perfectly reversible and terminal situations across as many states as desired, also allowing different stochastic processes to be embedded at different stages.

This development is important for several reasons. Much of the investment literature produces project values for one particular diffusion and parameter set so results depend heavily on that choice. If market price dynamics differ pre and post a firm’s investment, having one diffusion throughout a

project life is unsatisfactory.

Secondly, fixing on one diffusion can make empirical work and estimation difficult; a valuation technique that puts the project’s flexibility structure ahead of its diffusion dynamics will be more robust and amenable when testing different uncertainty specifications against empirical data.

This paper proceeds as follows, in Section 2 we examine the classical two stage cycle captured by hysteresis, or costly investment and divestment (Dixit (89) [7] etc. discuss systems with two states and decision points). This allows some notation and the main principles of the paper to be laid out for a problem that has been well studied.

In Section 3 we discuss the discount factors that are at the heart of this solution method, indicating how they can be adapted for stochastic processes other than Geometric Brownian Motion (GBM in Section 2). This allows restatement of the same problem as in Section 2 but with general notation and solutions for use in Sections 3 to 6. During these we study investment sequences with more states, decisions and alternate diffusion processes.

In Section 4 we represent two way flexibility utilizing discount factors and use these in four stage investment ladder including two way switching (an Appendix shows a worked solution to this case). The solution method proposed here can accommodate entanglement between many flexibility values.

In Section 5 we employ a serial hysteresis sequence with four stages and levels of action over two cycles. This allows hysteresis to be modelled with different processes at different stages. This is achieved either by using different functional forms to represent different stochastic processes by flexibility state or by using common discount factors but with differing parameters.

Notation is developed progressively but the objective is to develop a common solution framework. Since the majority of equations in this paper fall into one of three types, we use consistent prefixes; Vm for value matching, Sp for smooth pasting and Di for discounting equations (Gr for growth and other less common types are spelt out). Since each Section contains these equations, but with contents specific to its application, their subsection number is appended as a suffix, e.g. Vm2.1 in Section 2.1. After the main idea of the paper has been laid out in Section 2, Section 3 moves to more compact vector and matrix quantities (both in bold font).

Whilst matrix and vector representations are identical across Sections, it must be emphasized that their contents differ. This is because the number and structure of the flexibility components and matrixes differ by Section with the particulars of each flexibility application.

Each Section’s application can be seen visually in its own graph (within
a Table also labelled by Section). These display the numerous flexibilities and their interplay. It is particularly useful to separate their identification using two different colors.

Finally before an Appendix showing a worked example from Section 4, conclusions are offered in Section 6.

2 Classical hysteresis under GBM

Before working on a more complex project, to illustrate the techniques of this paper it is useful to apply them to a well known problem with only two decision points. Investment hysteresis concerns a project with dynamic stochastic underlying value $P$ that can be controlled in only two ways; either being opened or closed with optimal timing when project $P$ coincides with known thresholds $P_2$ and $P_1$ respectively. The former state variable $P$ is random across time whilst the latter levels $P_{2,1}$, at which the former is controlled, are subject to choice.

A project can be launched by incurring a lump sum cost $X_2$ (an exercise price equal for example to a risk free capitalized constant perpetual rate $x/r$ plus a switching on cost $K_2 > 0$) when the project value at that time $P_2$ is gained. Although the launch flexibility is consumed, in addition to the cash flow’s present value on opening $P_2$, a further option to close is acquired.

This inherited flexibility is the next option to close or suspend the diminished and unprofitable project at threshold $P_1$ in return for recouping a lower divestment sum $X_1$ (for example the capitalized cost saving $x/r$ less a switch off cost $K_1 > 0$). When divestment savings and project losses are lower than at investment ($X_1 < X_2$ and $P_1 \ll P_2$) economic hysteresis is said to occur. (Dixit (89) [7] etc. Note that net payoffs at thresholds $P_2 - X_2$ and $X_1 - P_1$ are both positive and thus in the money.)

Identifying idle $V_i$ and full on $V_f$ states and the value of both at each of the two thresholds in particular allows possible switching to be captured by two value matching equations (Vm2.0).

$$
V_i(P_2) = V_f(P_2) + P_2 - X_2 \quad (Vm2.0)
$$

$$
V_f(P_1) = V_i(P_1) + X_1 - P_1
$$

At the upper threshold on the first line of Vm2.0, idle flex is converted into full state flex plus the full project value less the present value of costs. At the lower threshold on the second line, the converse is true.

It is easiest to illustrate this with a price process for $P$ that follows geometric Brownian motion (GBM – but later Sections include other processes).
Flexibility values then depend on positive constants $A, B$ and powers $a > 1, b < 0$ that satisfy a quadratic condition. In this case the option solutions (functions of current project value $P$) that correspond to opening and closing flexibilities are constant multiples of power functions $V_i(P) = AP^a$, $V_f(P) = BP^b$ of $P$ valued at two project thresholds $P_{1,2}$.

For time homogeneous problems in this paper, by identifying conditions on either side of the investment boundary ($V_{m2.0}$) we interpret constants $A, B$ through flexibility values.

### 2.1 Discounting between $V_i(P_1), V_i(P_2)$ and $V_f(P_2), V_f(P_1)$

Measuring idle and full flex at two thresholds each, means that the constants $A, B$ can be eliminated in favour of working with four individual values. Furthermore elimination yields a natural discount factor relationship between the flexibility values at the beginning and end of each state. For the idle or call flex the initial value is a function of its end value $V_i(P_1) = (P_1/P_2)^a V_i(P_2)$, so too for the full put state where the initial value (at $P_2$) is a function of its value at $P_1$ (the lower threshold) $V_f(P_2) = (P_2/P_1)^b V_f(P_1)$.

Although we have doubled the number of constants from two to four, a discounting condition Di2.1 provides the two extra conditions to ensure that the system can still be solved by relating option values to their discounted later selves.

$$D_{f21} = \left(\frac{P_2}{P_1}\right)^b V_f(P_2) = D_{f21}V_f(P_1) \quad (\text{Di2.1})$$

$$D_{i12} = \left(\frac{P_1}{P_2}\right)^a V_i(P_1) = D_{i12}V_i(P_2)$$

Between thresholds and decision times no action is taken and no benefits are realized from flexibility because investment changes are not made or payoffs gained. Therefore the flexibility values $V_{i,f}(P)$ we seek are pure discount instruments (zero coupon but random maturity) that reflect option time value alone. Having identified Di2.1 as discounting equations subject to payoffs, $D_{f21}, D_{i12}$ will solve a Bellman equation without cashflow (subject

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5For a risk free rate of $r$ and GBM for $P$ with $\delta$ dividend yield and $\sigma$ volatility

$$dP = P(r - \delta)dt + P\sigma dW \quad a, b = \frac{1}{2} - \frac{r - \delta}{\sigma^2} \pm \sqrt{\left(\frac{r - \delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}.$$
to capital growth alone) subject to boundary conditions $D_{f11} = 1, D_{i22} = 1$ (and no bubble conditions).

Although it seems as if moving from two constants $A, B$ to four flex values $V_{i,f}(P_{1,2})$ makes the problem harder, firstly it brings greater economic meaning to the interpretation of flexibility values. Rather than work with constants for each state we show that it is more tractable to work with values at the beginning and at the end of the diffusion state, i.e. after a recent investment transition and just before the next.

Secondly we have a natural means to link these functionally, one that captures the discounting dynamics of the diffusion process within that state. These discounting equations $Di2.1$ give extra two conditions that compensate for the inclusion of flexibility values at both the beginning and end of each state. They intuitively account for the economics of flexibility value which has no current cashflow benefit or cost but which grows or shrinks like a discount bond with its proximity to a threshold.

Isolating the functional role of discounting is important when replacing one stochastic process with another and the discount factor representation is generalized allowing this separation method to solve flexibilities for other diffusions. This means that flexibility modelling can occur graphically before the need to chose stochastic processes.

### 2.2 Rate of return on flexibility

Having captured the functional form of the flexibility in discount factors and separated it using two particular values of interest, we can look at the relative beta or elasticity of flexibility values. This allows us to interpret the third required condition, that of smooth pasting. Local changes in $P$ generate changes in either of $V_i(P)$ or $V_f(P)$ that depend on powers $a, b$ identified earlier. These also play the role of betas or option elasticities; for GBM these claims have constant betas over their life between $P_2, P_1$ and within Beta2.2 the call and put are isoelastic.

\[
\beta_i = \frac{P_1}{D_{i12}} \frac{\partial D_{i12}}{\partial P_1} = a \quad \text{(Beta2.2)}
\]

\[
\beta_f = \frac{P_2}{D_{f21}} \frac{\partial D_{f21}}{\partial P_2} = b
\]

In the idle state the call is active and since $a > 1$, this flex will have a relative beta higher than the project itself, whilst in the full state the put is active with negative beta ($b < 0$) offering insurance at a cost.
2.3 Smooth pasting as a beta condition

For the decision making at $P_2, P_1$ to be optimal, not only must the total values match either side of $V_{m2.0}$ but the slopes of value with respect to the threshold must match too. This is to say that $\frac{\partial V_i (P_2)}{\partial P_2} = \frac{\partial V_f (P_2)}{\partial P_2} + 1$ at $P_2$ and $\frac{\partial V_f (P_1)}{\partial P_1} = \frac{\partial V_i (P_1)}{\partial P_1} - 1$, which has a rate of return interpretation (see Sødal (1998) [18] and Shackleton and Sødal (2005) [17]) and in this context a natural beta interpretation too. Since the GBM claims representing the flex values are isoelastic, the elasticities in Beta2.2 for the call are the same at both thresholds and for the put too, the same at both. This means that smooth pasting has an easy interpretation as weighted average cost of capital matching (not across debt and equity but across asset and option).

$$aV_i (P_2) = bV_f (P_2) + P_2 \quad \text{(Sp2.3)}$$

$$bV_f (P_1) = aV_i (P_1) - P_1$$

These are derived from smooth pasting and Beta2.2 and they determine the relative rate of return on each of the flexibility components $V_i, V_f$ and the investment gain or divestment loss $\pm P$. Their components are weighted by the values of flexibility and payoffs present in $V_{m2.0}$.

Smooth pasting equation Sp2.3 says that at the optimal thresholds, the project and its flexibility pre and post investment have weighted average costs of capital or local expected returns that balance. When the project value is between $P_2$ and $P_1$, value matching does not hold and no action is taken.

This third set of conditions Sp2.3 are as important as the previous two; together with the two value matchings and two discountings, $V_{m2.1}$ and Di2.2, they provide six equations in total. These are enough linear constraints to solve for $V_i (P_1), V_f (P_1), V_i (P_2), V_f (P_2), X_1, X_2$ as functions of $P_1, P_2$, the discount factors $(P_1/P_2)^a, (P_2/P_1)^b$ that depend on these two and their elasticities or betas $a, b$.

Most typically the project’s physical attributes (cost rates and frictions) would dictate investment and divestment quantities $X_2, X_1$. Then a solution would require finding values for the options $V_i (P_1), V_f (P_1), V_i (P_2), V_f (P_2)$ and thresholds $P_1, P_2$ that solve six equations. This would be a non-linear system incapable of solution in $P_1, P_2$ because these thresholds cannot be expressed as a linear set of functions of quantities $X_2, X_1$ etc.

This is one reason why we advocated choosing $P_2, P_1$ first along with the identification of flex values at the beginning and end of states. This
means that it is possible to solve for constants and investment quantities $V_i(P_1), V_f(P_1), V_i(P_2), V_f(P_2), X_1, X_2$ as a function of thresholds $P_1, P_2$ since these can be expressed as a set of linear functions of thresholds $P_{1,2}$ and discount factors. Numerical methods are still necessary to search across $P_1, P_2$ if a specific pair of $X_2, X_1$ are required to match physical attributes.

Thus the way to solve such flexibility problems is to start with given project thresholds that are assumed optimal and then calculate corresponding investment and divestment costs. This solution can be done using ad hoc means but simultaneous equations are best resolved using matrices. In this setting the structure we develop is that of a graphical discount matrix and its associated partial derivatives operating on flexibility values within one of two vectors.

Lastly the separation within Di2.1 allows the diffusion dynamics to be summarized within the discount function alone. This separation facilitates valuation because the only relevant dynamics from the chosen stochastic process are the discount factor itself (conditional on thresholds) and its sensitivity to those assumed thresholds. Using discount factors and their derivatives (related to betas) these can be manipulated separately.

### 2.4 Hysteresis discount matrix solution

The value matching equations Vm2.0 can be stacked into compact vector form, each row being evaluated at one of the thresholds $P_2$ or $P_1$.

$$\begin{bmatrix} V_i(P_2) \\ V_f(P_1) \end{bmatrix} = \begin{bmatrix} V_f(P_2) \\ V_i(P_1) \end{bmatrix} + \begin{bmatrix} P_2 \\ -P_1 \end{bmatrix} - \begin{bmatrix} X_2 \\ -X_1 \end{bmatrix} \quad \text{(Vm2.4)}$$

It makes sense to think of the flexibility values in each of the two vectors differently. In Vm2.4 the vector on the left of the equality sign $[V_i(P_2), V_f(P_1)]^\top$ values flexibility at the end of both states whilst the vector to the right of the equality sign $[V_f(P_2), V_i(P_1)]^\top$ values flex at the beginning of the next states just after the investment decision and project payoffs have been achieved.

The other two elements on the right comprise the net payoff associated with value matching at a transition; the benefits and costs depend on project value at that threshold and investment cost or divestment benefit. These are grouped with $P$ and $X$ into a project payoff vector $[P_2, -P_1]^\top$ and an investment cost vector $[X_2, -X_1]^\top$ both of which track the change in (net) investment through their embedded signs. At $P_2$ the project value $P_2$ is

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$^6\top$ represents the transpose of a vector.
gained at a cost of $X_2$ and at $P_1$ the divestment $X_1$ gain is made net of foregoing project value $P_1$.

Furthermore with the discounted interpretation of flexibility we have another relationship between values at the beginning and the end of a state, that of the beginning values being discounted versions of their end values. This relationship (Di2.4 is a compact version of Di2.1) between the two involves first a discount matrix with off diagonal elements associated with functions from the GBM options.

\[
\begin{bmatrix}
V_f(P_2) \\
V_i(P_1)
\end{bmatrix} =
\begin{bmatrix}
0 & D_{f21} \\
D_{i12} & 0
\end{bmatrix}
\begin{bmatrix}
V_i(P_2) \\
V_f(P_1)
\end{bmatrix}
\]  
(Di2.4)

\[
\begin{bmatrix}
V_i(P_2) \\
V_f(P_1)
\end{bmatrix} =
\begin{bmatrix}
0 & D_{i12}^{-1} \\
D_{f21}^{-1} & 0
\end{bmatrix}
\begin{bmatrix}
V_f(P_2) \\
V_i(P_1)
\end{bmatrix}
\]  
(Gr2.4)

The matrix in Di2.4 has an inverse which is shown in Gr2.4. Finally, the smooth pasting or WACC condition in matrix form is

\[
\begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix}
\begin{bmatrix}
V_i(P_2) \\
V_f(P_1)
\end{bmatrix} =
\begin{bmatrix}
b & 0 \\
0 & a
\end{bmatrix}
\begin{bmatrix}
V_f(P_2) \\
V_i(P_1)
\end{bmatrix} +
\begin{bmatrix}
P_2 \\
P_1
\end{bmatrix}.
\]  
(Sp2.4)

From Sp2.4, Di2.4 one of the flexibility vectors (end values) can be eliminated to solve for the other (in Flex2.4).

\[
\begin{bmatrix}
V_i(P_2) \\
V_f(P_1)
\end{bmatrix} =
\begin{bmatrix}
a & -bD_{f21} \\
-aD_{i12} & b
\end{bmatrix}^{-1}
\begin{bmatrix}
P_2 \\
P_1
\end{bmatrix}
\]  
(Flex2.4)

Finally investment quantities $X_{1,2}$ are derived using Vm2.4. Although this explicit solution to the simple hysteresis case is not yet intuitive, we have shown that flexibility valuation is possible via a system of equations linear in functions of $P_{1,2}$. Especially for larger systems, the key ingredient used so far was the identification and separation of flexibility elements both at the beginning and end of their lives. In order to better understand the intuition of this flexibility valuation method, in the next Section vector and matrix notation for these solution elements is developed further. Section 3 also moves away from GBM and generalizes the standard discount factors to allow for other diffusions. The next Section formalizes this result and interprets the solution matrices.

### 3 General discount factors and solution

The discount factor approach rests on a future fixed amount paid at a random time associated with a diffusion achieving a prespecified threshold. This
approach has been studied in Dixit, Pindyck, Sødal, etc. (99) [9], (06) [19].

Due to time value and uncertainty, the present value of a future payoff involves discounting and risk; not over the quantity of the payoff (say a dollar), only over its timing.

Flexibility states exist between the time taken for the diffusion to travel from beginning to end thresholds $P_{beg}$ to $P_{end}$. In hysteresis the idle or call state existed from $P_{beg} = P_1$ to $P_{end} = P_2$ but in the full state where the put was in play the beginning was $P_2$ and end $P_1$. We are therefore concerned with the expected discounted time taken for a diffusion to travel from $P_{beg}$ to $P_{end}$ and in particular the present value discount and growth factors that should be applied between the two. Here we use risk neutral methods for valuation to form the expectation of discounted time to payoff between thresholds consistent with diffusion and option within $s$.

From the perspective (and time) of initial $P_{beg}$, the discounting must take into account a continuous risk free $r$ rate applied to the risk neutral expectation of a random stopping time $\tau$ defined by the coincidence of the underlying value process first touching the second (higher) threshold, $P_\tau = P_{end}$

$$D_{s,beg,end} = E_{P_{beg}}^Q \left[ e^{-r\tau} \mid P_\tau = P_{end} \right]. \quad (D\text{factor}3.0)$$

This discount function solves a second order partial differential equation; there are two solutions which distinguish put and call flexibilities; setting the subscript $s = i, f$ along with $beg, end$ for the two threshold levels of concern yields $D_{i12}$ or $D_{f21}$.

Both discount factors are positive but not greater than one $1 \geq D_{s,beg,end} > 0$ and depend on the diffusion of $P$ as well as parameters of the process $s$.

Many processes have solutions for their discounts functions; some require numerical methods but the function and its differential can still be evaluated. However in this paper the emphasis is centred on the means to combine solutions rather than list specific processes and solutions themselves. However in Section 5 we present some other processes.

The discount function $D_{s,beg,end}$ has an elasticity $\beta_{s,beg/end}$ or relative beta which generally can be different at the beginning compared to the end of a state. The discount factor is a zero yield claim, supporting itself only via capital gains. Under physical (not risk neutral) probabilities its rate of return is linked to its local beta.

\footnote{This is to say that $D_{i12}$ and $D_{f21}$ both solve the same Bellman asset pricing equation but with different boundary conditions. Second order equations typically have two solutions; one of which depends on a positive and one a negative coefficient; these naturally associate with call (idle $s = i$) and put (full $s = f$) states.}
action, threshold idle $V_i$ investment payoff fully active $V_f$

| open, $P_2$ | $V_i(P_2)$ | $P_2 - X_2$ | $V_f(P_2)$ |
| close, $P_1$ | $V_i(P_1)$ | $X_1 - P_1$ | $V_f(P_1)$ |

Table 3: Hysteresis flexibility values $V_{i,f}$ red before and blue after transitions (horiz arrows) occurring at $P_{2,1}$ with payoffs net of PV costs $X_{2,1}$ (vertical arrows are diffusions).

### 3.1 Value matching in vectors

For the same hysteresis problem as in Section 2, Table 3 shows an investment graph with the four flex values colored differently according to whether they are at the beginning (blue) or end (red) of a diffusion state. Those at the end of a diffusion and flexibility state occur just before an investment transition (i.e. on the left hand side of a value matching equation) whilst those at the beginning of a diffusion state occur just after an investment transition (and on the right of a value matching equation).

As in Section 2 it is most convenient to define beginning and end flex values separately as vectors (also colored) $U = [V_f(P_2), V_i(P_1)]^T$, $W = [V_i(P_2), V_f(P_1)]^T$, along with another vector that captures the investment quantities $X = [X_2, -X_1]^T$.

The six quantities within $U$, $W$, $X$ are the outputs for which we seek a solution. They will depend upon four matrices and also the project payoff vector $\Omega = [P_2, -P_1]^T$ all of which are a function of the optimal project thresholds $[P_2, P_1]^T$. These thresholds may be treated as optimal, in order to recover appropriate $X_{2,1}$ (within $X$). If other desired $X_{2,1}$ are required, then $P_{2,1}$ should be treated as candidate values and adjusted to achieve required $X_{2,1}$.

$$W = U + \Omega - X$$  \hspace{1cm} \text{(Vm3.1)}

Having placed the flexibility values into two vectors $U$ and $W$, the value matching equation can be expressed succinctly $W = U + \Omega - X$ using a vector that also captures the project payoffs from cashflows $\Omega$ along with the required investment quantities $X$. Expression Vm3.1 says that the end of state flexibility $W$ transforms into beginning values $U$ net of a payoff
defined by the project payoff vector $\Omega$ net of flexibility exercise costs $X$ (investment quantities or call and put exercise prices).

### 3.2 Discount and growth matrices

For general discount factors, a matrix that captures the discount relationship between $U$ flex values at the beginning of states and all those at the end of states $W$ can be formed. We label this discount matrix $D$ and it relates beginning $U$ to end values $W$ so

$$U = D W$$  \hspace{1cm} \text{(Di3.2)}

$$W = G U$$  \hspace{1cm} \text{(Gr3.2)}

Equations Di3.2 and Gr3.2 along with Vm3.1 represent two of the three sets of independent equations required to solve for $W, U, X$. The last condition comes from the smooth pasting equations where in addition to $D, G$ differentials of discount factors must be captured in matrix form.

### 3.3 Smooth pasting matrices

As in Section 2, the final stage in smooth pasting in matrix form is to re-express Vm3.1 so that the matrices and not the flex values themselves (which we wish to treat as constant) carry the line by line differentiation. The key is to use growth (Gr3.2 $W = GU$) and discounting (Di3.2 $U = DW$) projections to produce an equivalent, Vm3.3, which is amenable to row by row differentiation.

$$G U = D W + \Omega - X$$  \hspace{1cm} \text{(Vm3.3)}

Within the expression Vm3.3, backward in time (on the left) and forward (on the right) translation has occurred using growth and discount matrices. This separates the diffusion dynamics (in matrices) from the problem constants (in vectors). The matrix products from the top row depend on
threshold $P_2$ only through $D^{-1}_{12}$ and $D_{21}$ whilst in the second row they depend on threshold $P_1$ through $D^{-1}_{21}$ and $D_{12}$.

Because $D, G$ have no leading diagonal elements, when the matrix products $[GU]$ or $[DW]$ are differentiated with respect to the first and second rows, the result is that the time value function alone carries the effect of the row by row differentiation.

In order to conform with a beta interpretation (equation Beta3.0), following differentiation each row must be multiplied by its threshold value. This is captured using a partial elasticity differential operator\(^8\) and notation $[D']$, $[G']$ to indicate row wise differentiation of the elements within $[D]$ or $[G]$ followed by threshold multiplication. The row by row scaled partial elasticities with respect to the thresholds $P_2, P_1$ required for smooth pasting simplify using two new matrices $D', G'$ given by

\[
D' = \begin{bmatrix}
P_2 \frac{\partial}{\partial P_2} & 0 \\
0 & P_1 \frac{\partial}{\partial P_1}
\end{bmatrix}
D = \begin{bmatrix}
0 & P_2 \frac{\partial D_{21}}{\partial P_2} \\
P_1 \frac{\partial D_{12}}{\partial P_1} & 0
\end{bmatrix}
(D'3.3)
\]

\[
G' = \begin{bmatrix}
P_2 \frac{\partial}{\partial P_2} & 0 \\
0 & P_1 \frac{\partial}{\partial P_1}
\end{bmatrix}
G = \begin{bmatrix}
0 & P_2 \frac{\partial D^{-1}_{12}}{\partial P_2} \\
P_1 \frac{\partial D^{-1}_{21}}{\partial P_1} & 0
\end{bmatrix}.
(G'3.3)
\]

Now the third required condition, smooth pasting, is achieved by performing this row differentiation on $V m 3.3$ giving $[GU]' = [DW]' + [\Omega - X]'$. This yields Sp3.3

\[
\begin{bmatrix}
0 & P_2 \frac{\partial D^{-1}_{12}}{\partial P_2} \\
P_1 \frac{\partial D^{-1}_{21}}{\partial P_1} & 0
\end{bmatrix}
\begin{bmatrix}
V_f(P_2) \\
V_i(P_1)
\end{bmatrix}
= \begin{bmatrix}
P_2 \frac{\partial D_{21}}{\partial P_2} \\
P_1 \frac{\partial D_{12}}{\partial P_1}
\end{bmatrix}
\begin{bmatrix}
V_i(P_2) \\
V_f(P_1)
\end{bmatrix}
+ \begin{bmatrix}
P_2 \\
-P_1
\end{bmatrix}
\]

The result of row differentiation and multiplication of the payoff vector $\Omega$ with respect to $P_2, P_1$ is represented by $\Omega' = [P_2, -P_1]^\top$. The final terms on the right of Sp3.3 are the scaled payoff’s sensitivity to each threshold. These do not depend on fixed $X$ ($X' = 0$). The sensitivity of the net payoff with respect to the threshold is 1 on investment and $-1$ on divestment because the net payoff increases 1:1 at $P_2$ as a function of $P_2$ and conversely decreases 1:1 with $P_1$ at $P_1$. When scaled by the project’s gross value at those points

\(^{8}\)i.e. the matrix elasticity operator pre-multiplies its contents, so $[D]' = \begin{bmatrix}
P_2 \frac{\partial}{\partial P_2} & 0 \\
0 & P_1 \frac{\partial}{\partial P_1}
\end{bmatrix}[D]$ and $[DW]' = \begin{bmatrix}
P_2 \frac{\partial}{\partial P_2} & 0 \\
0 & P_1 \frac{\partial}{\partial P_1}
\end{bmatrix}[DW] = D'W$. This is because the dependency of elements in rows of $DW$ are carried in $D$ alone.
\([P_2, -P_1]^T\) results (this is the same as \(\Omega\) here but in Section 4 this is not the case).

In Sp3.3, the first term on the right \(D'W\) is the sensitivity of the beginning flex values \(U\) to changes in thresholds, however it is most easily expressed through the partials within the discount matrix \(D'\) applied to the flex values at the end of states \(W\).

The one on the left of Sp3.3 \(G'U\) corresponds to changes in end flex values \(W\) i.e. before a transition is effected, again expressed as a matrix (\(G'\)) function of \(U\). If the transition is optimal, the sum of the two former components (on the right) equals the latter.

Putting the smooth pasting conditions into matrix form required row by row differentiation of \(D, G\) with respect to each threshold and scaling. To reiterate, the top row in the last equation was differentiated with respect to \(P_2\) and the second \(P_1\) (this is because the flex values in the top row of the vectors are optimized with respect to the former whilst those in the second, the latter) and both were rescaled by \(P_2, P_1\).

Since the partial derivatives of discount functions also carry the diffusion dynamics, this means that the smooth pasting equations have also been reduced to linear algebra involving solution quantities \(V_i(P_1), V_i(P_2), V_f(P_1), V_f(P_2)\) for any \(D_{i2, f21}\). This is to say that the flexibility’s functional form rests on factors \(D_{i,f} D_{i,f}^{-1}\) for diffusions and their scaled partials whilst the boundary flex values at thresholds can be treated as problem constants.

### 3.4 Beta matrix interpretation

As discussed in Section 2, it is easier to understand the smooth pasting conditions and solution for flex values if we relate them to weighted average cost of capital and beta concepts. This is possible because the scaled elasticity if the present value discount factor is its relative beta (Beta3.0).

From value matching \(W = U + \Omega - X\), we seek to convert Sp3.3 into a new form similar to Sp2.3 as a weighted average cost of capital (WACC). This will be done using new matrices \(\beta_W, \beta_U, \beta_\Omega\) that operate on \(W, U, \Omega\) respectively.

With solutions for \(W\) and \(U\) already secure, the question is what matrix expressions for \(\beta_W, \beta_U, \beta_\Omega\) allow a WACC to hold? Since flexibilities in \(W, U\) are intertwined, expressions for \(\beta_W, \beta_U\) will include off diagonal elements, \(\beta_\Omega\) which applies to payoffs however will be diagonal.

The means to derive an expression to define these betas\(^9\) with matrices

---

\(^9\)Note that the constant elasticity under GBMs means that within either the idle (\(a\)) or full states (\(b\)), beginning and end betas are the same. The relative elasticities under
is to start with the smooth pasting expression $G'U = D'W + \Omega'$.
Substituting from $D_i3.2$ $U = DW$ and $Gr3.2$ $W = GU$ converts it to
$G'DW = D'GU + \Omega'$ which is of the desired WACC form $Sp3.4$

$$\beta_W W = \beta_U U + \beta_{\Omega} \Omega.$$ (Sp3.4)

Since $\beta_{\Omega} \Omega = [P_2, -P_1]^T = \Omega'$ for hysteresis the beta payoff $\beta_{\Omega} = I$ is the
identity matrix (payoffs are linear and $\Omega = \Omega'$ carry sign changes).

The beta matrix for the beginning flexibilities $\beta_U$ is

$$D'G = \begin{bmatrix}
0 & P_2 \frac{\partial D_f}{\partial F_2} \\
P_1 \frac{\partial D_{i2}}{\partial f_1} & 0
\end{bmatrix}
\begin{bmatrix}
0 & D_{i2}^{-1} \\
D_{f21}^{-1} & 0
\end{bmatrix}$$ (BetaU3.4)

This expression contains two betas. The first $\beta_{f21} = \frac{P_2}{D_{f21}} \frac{\partial D_{f21}}{\partial F_2}$ is the sen-
tivity with respect to $P_2$ of the put scaled up by the threshold value and
down by its discount factor itself. This relative cost of capital or beta is
applied to $V_f(P_2)$ the value of the put at this threshold ($b$ for the GBM
case).

The second $\beta_{i12} = \frac{P_1}{D_{i12}} \frac{\partial D_{i12}}{\partial f_1}$ is applied to $V_i(P_1)$ and is the relative cost
of capital of the call in the idle state, evaluated at $P_1$. It is derived from the
sensitivity of call’s idle discount factor, appropriately scaled ($a$ in the GBM
case). The beta matrix $\beta_W$ for the end flexibilities $W$ is

$$G'D = \begin{bmatrix}
0 & P_2 \frac{\partial D_{i2}}{\partial F_2} \\
P_1 \frac{\partial D_{f21}}{\partial f_1} & 0
\end{bmatrix}
\begin{bmatrix}
0 & D_{f21} \\
D_{i12} & 0
\end{bmatrix}$$ (BetaW3.4)

which also contains two betas. The first $\beta_{i21} = \frac{P_1}{D_{i12}} \frac{\partial D_{i12}}{\partial F_1}$ multiplies $V_i(P_2)$
at $P_2$, it is the beta sensitivity of the idle (call) growth factor at $P_2$ looking

GBM in $Sp2.4$ are

$$\beta_W = G'D = \begin{bmatrix}
a & 0 \\
0 & b
\end{bmatrix} \quad \beta_U = D'G = \begin{bmatrix}
b & 0 \\
0 & a
\end{bmatrix}.$$
back to the threshold $P_1$, again scaled by the threshold $P_2$ and its own growth factor ($a$ for GBM).

The second is $\beta_{12} = \frac{P_1}{P_{21}} \frac{\partial P_{21}}{\partial P_1}$ which is the beta sensitivity of the put at $P_1$ again derived from a scaled growth factor sensitivity, from $P_1$ looking back to $P_2$ ($b$ for GBM).

Writing out Sp3.4 in full would on the first line of the beta matrices show the idle (call) option value weighted by its beta $\beta_{21}$ $V_i(P_2)$ equating the full (put) option value times its beta $\beta_{21} V_f(P_2)$ plus the payoff $P_2$ (times its unit beta). The sunk cost $X_2$ (also $X_1$) has zero beta so does not contribute to this equation or the excess cost of capital above the risk free rate.

In the second line, the beta weighted full (put) option at its exercise threshold $\beta_{12} V_f(P_1)$ equates to the idling (opening) option weighted by its beta $\beta_{i2}$ $V_i(P_1)$ less the unit beta times the loss of project value $-P_1$. This representation of smooth pasting with betas is intuitive and allows a WACC to be derived for general diffusions (see last Section – in particular when beginning and end flex betas are not the same as with GBM).

### 3.5 Simultaneous matching, pasting and discounting

Now we have three matrix equations, Vm3.1, Di3.3, and Sp3.4 for the vectors $[V_i(P_2), V_f(P_1)]^T$, $[V_f(P_2), V_i(P_1)]^T$, $[X_2, -X_1]^T$. These equation contain matrices $(D, G, D', G')$ along with the payoff vector $\Omega$, and its sensitivity $\Omega' = \beta_{1} \Omega$ all of which depend on $P_2, P_1$. The sensitivity matrices were converted into beta matrices $\beta_W = G'D$ and $\beta_U = D'G$.

The three sets of equations can be stacked with rows representing value matching (Vm3.1), discounting (Di3.3) and smooth pasting (Sp3.4) simultaneously (I is the identity matrix and 0 one of zeros).

$$
\begin{bmatrix}
I & -I & I \\
D & -I & 0 \\
\beta_W & -\beta_U & 0
\end{bmatrix}
\begin{bmatrix}
W \\
U \\
X
\end{bmatrix} =
\begin{bmatrix}
\Omega \\
0 \\
\beta_{1} \Omega
\end{bmatrix}
$$

Vm

$$
\begin{bmatrix}
I & -I & I \\
D & -I & 0 \\
\beta_W & -\beta_U & 0
\end{bmatrix}
\begin{bmatrix}
W \\
U \\
X
\end{bmatrix} =
\begin{bmatrix}
\Omega \\
0 \\
\beta_{1} \Omega
\end{bmatrix}
$$

Di

$$
\begin{bmatrix}
I & -I & I \\
D & -I & 0 \\
\beta_W & -\beta_U & 0
\end{bmatrix}
\begin{bmatrix}
W \\
U \\
X
\end{bmatrix} =
\begin{bmatrix}
\Omega \\
0 \\
\beta_{1} \Omega
\end{bmatrix}
$$

Sp

(System3.5)

As can be seen the result has a block in the lower left hand corner which allows solution for $U, W$ first. If we define a new system matrix $S$ and a new joint vector of values $V$, the joint values $V$ can be solved

$$
S = \begin{bmatrix}
D & -I \\
\beta_W & -\beta_U
\end{bmatrix},
V = \begin{bmatrix}
W \\
U
\end{bmatrix} = S^{-1} \begin{bmatrix}
\Omega \\
0
\end{bmatrix}
$$

since

$$
SV = \begin{bmatrix}
0 \\
\beta_{1} \Omega
\end{bmatrix}.
$$
3.6 Flexibility returns at investment

The value matching equations link flexibility values on either side of an investment decision. The beta matching equations link the local rates of return at every investment. The WACC within Sp3.4 shows that prior to investment, the local rate of return and elasticity of flex is driven by $\beta W$. but after the transition is effected (and $X$ is sunk/recouped) it is driven by $\beta U$, that of the new flex, plus that of the payoff $\beta \Omega$. This is an intuitive condition on the interlinkage between, and constraints on, flexibility values.

Furthermore from this equalization and the discounting relationship, the end flex value can be isolated easily. From $\beta W = \beta U + \beta \Omega$ substitute for $U$ and collect terms $[\beta W - \beta U]W = \beta \Omega$ to solve

$$W = [\beta W - \beta U]^{-1} \beta \Omega.$$  (W3.6)

The two equations in the prior text represent dollar rates of return or net beta equalizations, whilst $W3.10$ represents present values as a ratio of $\beta \Omega$ premultiplied by an (inverse) factor $[\beta W - \beta U]^{-1}$.

This last element $\beta W - \beta U$ is easy to interpret, it is the immediate beta $\beta W$ of $W$ less that derived from $U$ which itself is expressed as a discounted product. The amount by which the gross beta needs to be decreased, $-\beta U$ is the beta on $U$ discounted by the expected time factors until these betas are applied.

Furthermore, for a given risk free rate and risk premium, a beta equation maps into an excess return equation. This is true of the calculation involving $[\beta W - \beta U]^{-1}$ meaning that its inverse in the third line can be treated as a perpetuity factor.

Thus the implication for flexibility values $W$ is that they depend on aperiodic perpetual payoffs proportional to $\beta \Omega$ with the relative timing of these events carried though the perpetuity factor $[\beta W - \beta U]^{-1}$. Cycles will be of different lengths for each component but the matrices carry the magnitude and timing of each through their embedded discount functions.

A similar grown beta argument applies to $U$ and $[\beta W G - \beta U]^{-1}$ as can be seen in $U3.6$

$$U = [\beta W G - \beta U]^{-1} \beta \Omega.$$  (U3.6)

To summarize, in a system where flexibility is sequential and its use begets another form, optimal timing is dictated by a net discounted condition applied to their betas. When optimally used, the maximum flexibility values (elements $U$ and $W$) satisfy relative beta or net sensitivity conditions on their interlinkage. A discounted net beta operator $[\beta W G - \beta U]$ or
\[ \beta_W - \beta_U \mathbf{D} \] applied to their values \( \mathbf{U} \) and \( \mathbf{W} \) smooths the sensitivities of their payoffs. This is to say the beta of each flexibility net of its future use depends on its current and a discounted (grown) beta. Note that these beta conditions are matrices which can carry complex interactions between each of the flexibility components (the betas are interlinked in a similar manner to the flexibilities themselves).

Since they have zero elasticity, the value of \( X_n \) that correspond to the assumed threshold \( P_n \) are not determined through betas. Instead \( X \) is derived through value matching alone \( X = U + \Omega - W \).

Although these equations were demonstrated for simple hysteresis, they are much more general and the next Sections show how to expand the matrix contents for different investment problems and different stochastic processes. This solution method is robust to many different setups. Although the components of discount, growth matrices, payoff vector \( \mathbf{D}, \mathbf{G}, \Omega \) and scaled elasticities, \( \mathbf{D}', \mathbf{G}', \Omega' \) or more intuitively beta matrices \( \beta_W, \beta_U, \beta_{\Omega} \) may change, the same solution technique and formulae can be used.

4 Two way flexibility

In this Section we illustrate a more complex investment scenario involving a stage with both the chance to further invest if things go well, but simultaneously the chance to recoup if not. First we need to understand discount factors that can accommodate two different diffusion outcomes.

4.1 Two way discount factors

Two way discount factors which yield a unit payment at one threshold but a zero value at another are also possible. These are linear combinations of a pair of one way factors. For example starting from \( P_3 \) there might be two thresholds of interest \( P_4 > P_3 \) and \( P_1 < P_3 \) where at \( P_4 \) a flexibility payoff is attained and at \( P_1 \) this flexibility becomes worthless (because use of the other flex causes this one to be killed).

Two way factors are defined with the first threshold in the argument \( \text{beg} \) as the starting level, the second \( \text{end} \) the target level and the third \( \text{out} \) e.g. the level at which the flexibility dies or is knocked out e.g. \( D_{n,\text{beg,end,out}} \) firstly for the “up and in, down and out” and secondly the converse “down
and in, up and out”.

\[
D_{s341} = E_{P_3}^Q \left[ e^{-rt} \big| P_\tau = P_4, \min P > P_1 \right] \quad \text{(Dfactors4.1)}
\]

\[
D_{s314} = E_{P_3}^Q \left[ e^{-rt} \big| P_\tau = P_1, \max P < P_4 \right]
\]

\[
D_{s441} = 1 \quad D_{s114} = 1 \quad D_{s141} = 0 \quad D_{s414} = 0
\]

The two are complimentary in the sense that if one pays off the other dies and vice versa; this can be seen in their boundary conditions (applied at \( P_3 = P_4 \) or \( P_3 = P_1 \)). They are each constructed from a linear combination of one way discount factors, such that these conditions are met.

\[
D_{s341} = cD_{s34} + dD_{s31} = \frac{D_{s34} - D_{s31}D_{s41}}{1 - D_{s41}D_{s41}}
\]

because \( 1 = c + dD_{s41} \) and \( 0 = cD_{s41} + d \)

similarly \( D_{s314} = \frac{D_{s31} - D_{s34}D_{s41}}{1 - D_{s31}D_{s41}} \)

Although for GBM the component discount factors \( D_{s34}, D_{s31} \) have constant local betas, when mixed in this fashion \( D_{s314} \) has a dynamic beta because the weights on each change. Within state \( s \) and region \( P_1, P_4 \) the process for \( P \) could follow any diffusion of choice so long as the discount factors can be found analytically or numerically. Furthermore, because the flexibility and discount factors pertain to state \( s \) and region \( P_1, P_4 \) alone, in a subsequent states and regions different processes can be used.

### 4.2 Investment ladder

We consider another system with the flexibility to ratchet up or down a value ladder of three operational states; idle \( i \), power \( p \) and full \( f \) including an intermediate power \( \gamma \) (elasticity) of an underlying flow\(^{10} \). We aim to use the same algebra but to vary the matrix and vector contents alone in order to examine this new investment setup.

Operational cashflows and value are again nil in the idle state, but now vary with \( P^\gamma \) in a new power state and full (power 1) in the third state. This leads to non-flex values varying with level according to \( 0, P^\gamma \) and \( P \). Starting with the transition at the highest threshold, from power to full, this occurs at threshold \( P_4 \) with a payoff of \( P_4 - P_4^\gamma - X_4 \) net of transaction cost. That is to say that on going to full at the top threshold, \( P_4 \) is gained but \( P_4^\gamma \) is lost with the incremental investment \( X_4 \).

\(^{10}0 < \gamma < 1 \) is generally a sufficient convergence condition for the PV of the power of a GBM diffusion \( P_0^\gamma = E_{P_0}^Q \left[ \int_0^\infty p(t)^\gamma e^{-rt} dt \right] \).
from/to thr. idle $V_i$ p.off  

<table>
<thead>
<tr>
<th>from</th>
<th>to</th>
<th>power</th>
<th>V</th>
<th>V</th>
<th>full</th>
<th>p.off</th>
<th>full $V_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>thr</td>
<td>idle</td>
<td>$V_p$</td>
<td>(P4)</td>
<td>$V_f$ (P4)</td>
<td>$P_4 - P_4^\gamma - X_4$</td>
<td>$V_p$</td>
<td>$P_4 - P_4^\gamma - X_4$</td>
</tr>
<tr>
<td>thr</td>
<td>idle</td>
<td>$V_p$</td>
<td>(P3)</td>
<td>$V_f$ (P3)</td>
<td>$P_3^\gamma - P_3 + X_3$</td>
<td>$V_p$</td>
<td>$P_3^\gamma - P_3 + X_3$</td>
</tr>
<tr>
<td>thr</td>
<td>idle</td>
<td>$V_p$</td>
<td>(P2)</td>
<td>$V_f$ (P2)</td>
<td>$P_2^\gamma - X_2$</td>
<td>$V_p$</td>
<td>$P_2^\gamma - X_2$</td>
</tr>
<tr>
<td>thr</td>
<td>idle</td>
<td>$V_p$</td>
<td>(P1)</td>
<td>$V_f$ (P1)</td>
<td>$X_1 - P_1^\gamma$</td>
<td>$V_p$</td>
<td>$X_1 - P_1^\gamma$</td>
</tr>
</tbody>
</table>

Table 4: Elasticity ladder with flexibility states Idle, Power and Full flow $V_{i,p,f}$; red before and blue after transitions (horiz arrows) occurring at $P_{1,2,3,4}$ with payoffs net of investment/divestment costs $X_{1,2,3,4}$ (vertical arrows are diffusions).

Reversion can occur at $P_3$ yielding the reverse $P_3^\gamma - P_3 + X_3$ including a partial return of fixed investment cost $X_3 < X_4$. Similar transitions occur at $P_2$ into the power state from the idle and $P_1$ back to the idle from the power. These last two have investment exercise prices of $X_2$ and $X_1$ ($X_1 < X_2$).

The value matching conditions within this system have different contents to those in the previous Sections and are shown in Vm4.2 $W = U + \Omega - X$.

$$
\begin{bmatrix}
V_p (P_3) \\
V_f (P_3) \\
V_i (P_2) \\
V_p (P_1)
\end{bmatrix} =
\begin{bmatrix}
V_f (P_4) \\
V_p (P_3) \\
V_p (P_2) \\
V_i (P_1)
\end{bmatrix} +
\begin{bmatrix}
P_4 - P_4^\gamma \\
P_3^\gamma - P_3 \\
P_2^\gamma - P_1^\gamma \\
X_4 \\
-X_3 \\
X_2 \\
-X_1
\end{bmatrix} -
\begin{bmatrix}
X_4 \\
-X_3 \\
X_2 \\
-X_1
\end{bmatrix}.
$$

(Vm4.2)

For the three states, four thresholds $P_{1-4}$ and switching costs $X_{1-4}$ the new investment/divestment graph is given in Table 4. It is another bipartite and directed graph but one that requires two way discount factors within the power state. Note that the extra state notation $V_p$ required for flexibility in the power state along with two new possible outcomes of entering the power state, either from the idle or full conditions (rows two and three).

From the power state, since reversion to the idle state is possible (at $P_1$) as well as elevation to the full state (at $P_4$), the discount matrix is populated with six elements, and in particular some rows now contain complementary

---

11Bipartite in the sense that investment nodes are either beginning (blue) or end (red) of diffusions and from one you can only move to the other type. Directed in the sense that each link between nodes is only forward and does not offer the opportunity for immediate reversal (see Wilson 1985 [21]).
discount factors with mutually exclusivity i.e. conditional upon each other not paying off. For example in row three (within the power state) the element \( D_{p341} \) represents from the point of view of threshold \( P_3 \), the discounted chance of paying a dollar at \( P_4 \) knowing that if \( P_1 \) is reached the opportunity dies (zero value).

The second \( D_{p314} \) in row two is complementary and pays off at 1 assuming 4 is not hit. Other discount factors are interpreted similarly but % \( D_{f43} \), \( D_{i12} \) are one way factors in full and idle states respectively. The detailed discounting equations \( U = DW \) are thus given in Di4.2 (also different contents to prior Di3.2).

\[
\begin{bmatrix}
V_f (P_4) \\
V_p (P_3) \\
V_p (P_2) \\
V_i (P_1)
\end{bmatrix}
= 
\begin{bmatrix}
0 & D_{f43} & 0 & 0 \\
D_{p341} & 0 & 0 & D_{p314} \\
D_{p241} & 0 & 0 & D_{p214} \\
0 & 0 & D_{i12} & 0
\end{bmatrix}
\begin{bmatrix}
V_f (P_4) \\
V_p (P_3) \\
V_p (P_2) \\
V_i (P_1)
\end{bmatrix}
\]

(Di4.2)

The inverse discount matrix \( D^{-1} = G \) exists and is shown in Gr4.2 but it is harder to interpret since it has negative elements. These occur because within the power state, hitting one boundary and achieving a payoff is conditional on not hitting the other, thus not all growth paths are assured.

\[
G = 
\begin{bmatrix}
0 & \frac{D_{p214}}{D_{f43}} & \frac{D_{p314}}{D_{p214}D_{p341}-D_{p214}D_{p314}} & 0 \\
0 & 0 & 0 & 0 \\
0 & \frac{D_{p341}}{D_{p214}D_{p341}-D_{p214}D_{p314}} & \frac{D_{p241}}{D_{p341}} & 0 \\
0 & 0 & \frac{1}{D_{i12}} & 0
\end{bmatrix}
\]

(Gr4.2)

Finally along with the line by line derivatives of \( D, G \) (with respect to \( P_{4,3,2,1} \)) the scaled sensitivity of the payoff vector with respect to the thresholds \( \Omega' = [P_4 - \gamma P_4', \gamma P_3', -P_3, \gamma P_2', -\gamma P_1']^T \) is required\(^{12}\) in order to construct the solution from smooth pasting \( U = [\beta_W G - \beta_U]^{-1} \beta_U \Omega \).

Numerical results for this system under GBM are shown in the Appendix for \( P_{4,3,2,1} = 4, 3, 2, 1 \) with \( a = 2, b = -1 \) and \( \gamma = 0.5 \). Figure 1 shows graphically the relationship between the flexibility values (on the vertical) and project value (on the horizontal). The Appendix also shows values for certain limiting cases.

The advantage of this solution system is apparent. It is modular in the sense that cashflow components and their intertwined flexibility features can be placed within a system governed by a common framework and solution

\(^{12}\) All beta matrices still hold but note that here since \( \Omega' \neq \Omega, \beta_U \neq I \).
Figure 1: Investment ladder example for GBM with $a, b = 2, -1$ and $\gamma = 0.5$ in the power state. Table 5 shows the investment graph whilst this plot shows the optimal flex values and investment quantities for idle, power and full states with potential two way switching.
method. This is a particularly important aspect of sequential investment (see Gamba and Fusari (2009) [11]). With two way discount factors different investment paths can be modelled (i.e. 1,2,4,3 or 1,2,1,2 etc.).

4.3 Flexibilities and their use as assets

There are several different types of restriction on $P_{1-4}$. Firstly for GBM (and several other diffusions excluding arithmetic Brownian motion) thresholds must be positive. Secondly in order for the discount function to make sense, each pair involved above, must satisfy inequalities such as $P_2 > P_1$. This immediately constrains the flex values $U, W$ and the commensurate outputs $X_{1-4}$. We would also like flexibility to be an asset (not a liability, negative) and therefore restrict ourselves to solutions where $W > 0, U > 0$ (vector of zeros 0).

However within a complex system, it is not always obvious that simply satisfying these explicit constraints will generate combinations of $X_{1-4}$ that (given the investment graph) represent project maxima with clear economic meaning. Consequently we must consider a further economic set of constraints on and therefore subsets of $P_{1-4}, \Omega_{1-4}$ and matching $V_{i,p,f}(P_{1-4}), X_{1-4}$.

Those systems that are easiest to interpret and the ones we focus on here, have values of $X_{1-4}$ that are positive for positive elements $\Omega_{1-4}$, and negative for negative project value changes within $\Omega$. These are systems of sequential investment where consuming cash (positive $X_{1-4}$) produces a payoff (positive $\Omega_{1-4}$) and where divestment (diminishing the present value of cashflows i.e. negative $\Omega_{1-4}$) liberates cash (negative $X_{1-4}$). Furthermore with investment and divestment flexibility, we normally require a positive net payoff at a trigger point where flex is consumed

$$W - U = \Omega - X > 0.$$  \hfill (Restrict3.7)

When this condition holds for a system it corresponds to harvesting of flexibility (in the money) at each of the thresholds. (Rather than always harvesting flexibility, other cases which involve investing in flexibility at one or more thresholds $W - U = \Omega - X < 0$ for one threshold $P$ may also prove interesting but we leave these.)

Any solution should also be confirmed as a maximum and neither a minimum nor saddle point in value. To test this, for fixed (not solved) $X$ from the proposed maximal solution, a numerical investigation can be conducted varying each of the thresholds $P_{it}$ in turn. Bearing in mind that there are
multiple levels and multiple thresholds, we must take care to identify which values we would expect to have zero and non zero slopes.

Smooth pasting holds for values (in pairs one each in $\mathbf{W}, \mathbf{U}$) at their one joint threshold, so their slope with respect to this level will not be zero; rather one value will increase as the other net of payoff, decreases. At a turning point, it is the sensitivity of a value with respect to a threshold other than its own (smooth pasted) one that should be zero and with negative curvature for a maximum. So for the solution to be a maxima, the second order condition on these values with respect to thresholds other than those smooth pasted, is that their perturbation should lead to a value decrease. Since we do not wish to construct more matrix differentials than necessary, here we recommend a numerical investigation to establish maximization.

Finally with an analytic method for mapping from a restricted set within $P_{1-4}$ to optimal $X_{1-4}$ in $\mathbf{X}$, the system can be used numerically (via search) to find the $P_{1-4}$ given economically reasonable values $X_{1-4}$.

The next Section shows how to incorporate different stochastic processes.

5 Serial hysteresis and changing dynamics

In this Section we employ a cyclical system for tracking switching decisions again at four different thresholds and random times. This similar to Ekern (1993) [10] who evaluates operational flexibility in a sequential investment/divestment situation without cyclicity. His firm has limited capacity to open or close and therefore can switch between idle and full status a finite number of times before losing all remaining flexibility. Whilst such situations can be solved in reverse from a terminal condition, one aim here is to solve cyclical systems without the ability to refer to terminal conditions. We also show how to accommodate different stochastic processes at different investment stages.

5.1 Flexibility timeline and investment graph

What flexibility value applies depends on the number of potentially limited switching opportunities. Here four sequential and cyclical decisions at independent thresholds $P_{1-4}$ are labelled by their level (not their sequence –

\[ \text{For fixed (candidate optimal) } \mathbf{X}, \text{ in turn perturb each threshold } P_n \text{ and check if all values } V_s(P_{m\neq n}) \text{ other than those evaluated at } P_n \text{ decrease. Those flexes present at } P_n \text{, say } V_s(P_n), V_t(P_n) \text{ in states } s, t \text{ will not be maximal; one will increase the other decrease as } P_n \text{ is changed. This is because of smooth pasting in } P_n. \text{ However these flexes } V_s(P_n), V_t(P_n) \text{ will be optimized with respect to all other thresholds } P_{m\neq n}. \]
Table 5: Serial hysteresis flexibility values $V_{i,f,I,F}$ red before and blue after transitions (horiz arrows) occuring at $P_{2,3,4,1}$ with payoffs net of PV costs $X_{4,2,3,1}$ (vertical arrows are diffusions).

In time order these are encountered in the sequence 4, 2, 3, 1). Using colors again facilitates identification of different flexibility age within state (blue young and red old). Table 4 reflects the simple time line and usage of sequential flexibility.

Transitions (opening or closing rows) at project thresholds are labelled in the left hand column. States (in columns) can either be idle or full on and the flexibility value in this region is again labelled in its subscript, e.g. $V_i$ (idle), $V_f$ (full). However idle and full conditions that occur later may have different transition costs or result from different diffusion dynamics. This means they must have different labels $V_i, V_f$ and different values. At the two thresholds $P_3, P_4$ opening occurs whilst at $P_1, P_2$ closing occurs. Note that although a full sequence is implied here, on occasion this investment and divestment pattern may get stuck, either open at a very high price, or closed with a low one (this is indicated by a column having an open top or bottom).

By allowing for a difference between $X_1$ and $X_3$ (or $X_4$ and $X_2$) this also generalizes single (Dixit (1989) [7]) to serial hysteresis allowing for potential different cost rates with each cycle of operation. For example if costs rates and required capital in full on state $V_f$ are higher than those in state $V_f$, then $X_4 > X_3$. Similarly the present value of spared operational costs and recovered capital at the closing thresholds may be different $X_1 <> X_2$ but the savings on closure at each point (of depressed project worth) will be the
positive quantities \(X_1 - P_1\) and \(X_2 - P_2\).

\[
W = \begin{bmatrix}
W_1 (P_4) \\
W_2 (P_3) \\
W_3 (P_2) \\
W_4 (P_1)
\end{bmatrix}
\begin{bmatrix}
U^+ \\
V^+ \\
V^- \\
U^-
\end{bmatrix}
\begin{bmatrix}
\Omega^- \\
\Omega^+ \\
\Omega^+ \\
\Omega^-
\end{bmatrix}
\begin{bmatrix}
P_4 \\
P_3 \\
P_2 \\
P_1
\end{bmatrix}
\begin{bmatrix}
X_1 = \frac{x_f}{r} + K_{i-f} \\
X_2 = \frac{x_f}{r} + K_{i-F} \\
-\frac{x_f}{r} - X_2 = K_{i-I} - \frac{x_f}{r} \\
-\frac{x_f}{r} = K_{i-F} - \frac{x_f}{r}
\end{bmatrix}
\] (Vm5.1)

where \(x_f, x_F\) are the operational cost rates associated with the project in the different phases and \(r\) the risk free rate so that the PV of perpetual costs \(x/r\). Opening and closing frictions \(K\) are incurred at the end of idle and full states, i.e. on opening the PV cost rate must be incurred plus an additional amount whilst on closing, the saving is less than the PV operational cost. They can be customized to the nature of each transition.

5.2 Sequential processes for investment impact

Dixit, Pindyck, Sodak (99) [9] and others detail discount factors for different processes such as the mean reverting Hypergeometric process, which along with GBM, Arithmetic Brownian motion (ABM), their fundamental quadratics \(q(\beta)\) = 0 and discount factors are shown.

Proc. diffusion \(dP = (r - \delta) P \, dt + \sigma P \, dW\) \(q(\beta) = 0\) \(D_{s12}^{G,A,H}\) (params)
GBM \(= \beta^2 \beta (\beta - 1) + \frac{\sigma^2}{\beta - r} = 0 \quad \left(\frac{P_t}{P_0}\right)^{\beta + \frac{\sigma^2}{\beta - r}} = e^{\beta(1 - r)}\)
ABM \(= \frac{1}{2} \beta^2 \beta^2 + \alpha \beta - r = 0 \quad e^{\beta(P_1 - P_2)}, e^{-\beta(P_1 - P_2)}\)
HYP \(= \eta (\bar{P} - P) \, P \, dt + \sigma P \, dW\) \(q(\beta) = 0\) \(\left(\frac{P_t}{P_2}\right)^{\beta + \frac{\sigma^2}{\beta - r}} = H\left(e^{\beta(P_1 - P_2)}, e^{-\beta(P_1 - P_2)}\right)\)

These explicit functions and their differentials can be combined at will inside a customized matrix \(D\). For instance taking this serial hysteresis case, a different diffusion can be placed into the scheme at each investment stage. This is especially important when projects are large and their market impact needs to be reflected through different dynamics in various stages of its investment process.

Using \(D^{A}(\alpha, \sigma_A, r)\) and \(D^{H}(\eta, \bar{P}, \sigma_H, r)\) for the arithmetic and hypergeometric respectively along with two GBMs e.g. \(D^{G}(\delta_i, \sigma_i, r)\), \(D^{G}(\delta_I, \sigma_I, r)\) with different convenience (dividend) yield \(\delta_i, \delta_I\) and volatility \(\sigma_i, \sigma_I\) parameters highly, a highly customized version of Df6.2 can be constructed.
\( G = D^{-1} \) is also required,

\[
D = \begin{bmatrix}
0 & 0 & \mathcal{D}_{f42}(\alpha, \sigma_A, r) & 0 \\
0 & 0 & 0 & \mathcal{D}_{F31}(\eta, \overline{P}, \sigma_H, r) \\
D^G_{i14}(\delta_i, \sigma_i, r) & 0 & 0 & 0
\end{bmatrix}
\]

both are amenable to row wise differentiation and \( P \)-scaling to facilitate smooth pasting. The required matrices are

\[
D' = \begin{bmatrix}
0 & 0 & 0 & \partial \mathcal{D}_{f42} / \partial P_4 \\
0 & 0 & 0 & 0 \\
P_1 \partial \mathcal{D}_{i14} / \partial P_1 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
G' = \begin{bmatrix}
0 & 0 & 0 & P_4 \partial \mathcal{D}_{i14}^{-1} / \partial P_4 \\
0 & 0 & 0 & 0 \\
P_1 \partial \mathcal{D}_{i14}^{-1} / \partial P_1 & 0 & 0 & 0
\end{bmatrix}
\]

from which the solution follows

\[
U = \begin{bmatrix}
-P_4 \partial \mathcal{D}_{f42} / \partial P_4 & 0 & 0 & P_4 \partial \mathcal{D}_{i14}^{-1} / \partial P_4 \\
0 & -P_3 \partial \mathcal{D}_{F31} / \partial P_3 & 0 & 0 \\
P_2 \partial \mathcal{D}_{i14}^{-1} / \partial P_2 & 0 & -P_2 \partial \mathcal{D}_{i14}^{-1} / \partial P_2 & 0 \\
0 & P_1 \partial \mathcal{D}_{i14}^{-1} / \partial P_1 & 0 & -P_1 \partial \mathcal{D}_{i14}^{-1} / \partial P_1
\end{bmatrix}^{-1}
\begin{bmatrix}
P_4 \\
P_3 \\
-P_2 \\
-P_1
\end{bmatrix}
\]

No matter how numerous the sequential layers, if the matrix that represents its investment graph can be inverted, it can be solved via construction of flexibility value vectors \( U, W \) with values at the thresholds identified.

Traditionally in the literature, four inputs of cost rates (within \( X \)) would have been provided with sufficient conditions (eight) to pin down four option constants and four output thresholds. Here we advocated assuming four optimal thresholds, splitting the flexibilities into two per state (eight in total) and then using twelve conditions (in three matrix expressions) to retrieve their eight values along with the four investment and divestment costs. This works because we also get an extra set of discounting conditions.

If the output values in \( X \) do not match the physical properties of the project, then it remains a numerical task to iterate on the values for all \( P \) until target \( X \) are reached. Whilst this search may remain a computational task, considerable tractability has been generated with this solution.

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After the design of the flexibility investment graph, diffusions can be tested and combined at will in a modular fashion. Although each of the functions in \( D \) may be required, the only values that affect decision making and flexibility are the discounts and derivatives at the thresholds. The flexibility determines where the entries occur within \( D \) but the processes in play in each state determine what those entries capture functionally and economically through the discount timing (present values) and discount slopes (returns).

6 Conclusion

Flexibility to time the launch, closure or another transformation of a project are all discount claims on an underlying process for uncertainty. Their values can be identified on a mathematical graph or network which lends itself to modular analysis through separation and discounting.

Discounting has been used before (Dixit, Pindyck, Sødal etc. (1999, 2006) [9], [19]) and the factor that is appropriate during the life of flexibility state depends on the diffusion dynamics and its interlinkage with other options. We captured these features in a discount matrix.

The rate of return on each flexibility value depends on an elasticity that carries through from its discount factors. Not only must discounting hold for flex values, but for optimal smooth pasting, their rates of return at exercise must match too. This can be captured using beta matrices derived from, but complementary to, the discount matrices.

For a given set of threshold separations, optimal flexibility values at the beginning and end of each state are thus driven by two considerations; discounting and smooth pasting or beta matching. The third and most natural condition of value matching only contributes to the solution of fixed investment costs.

For thresholds to match real project characteristics however, determining optimal solutions would involve matching a known set of fixed costs. These must still be solved numerically, but the system of equations presented here greatly facilitates this task. They also shed light on the theoretical valuation of and rate of return on project flexibility.

References


7 Appendix: Investment ladder values

7.1 Two way GBM example

To illustrate Section 4 numerically, in this appendix we show worked examples for a GBM with \( a,b = 2,-1 \) (one of \( r,\delta,\sigma \) still free) and power elasticity \( \gamma = 0.5 \). Table 4 shows the graph for the investment problem. For ease of exposition we set \( P_4, P_3, P_2, P_1 = 4,3,2,1 \) and then calculate each of \( X, W, U \). In this case \( D_{i4.2} \) and \( G_{4.2} \) become (to three decimal places)

\[
D = \begin{bmatrix}
0 & 0.750 & 0 & 0 \\
0.550 & 0 & 0 & 0.196 \\
0.222 & 0 & 0 & 0.444 \\
0 & 0 & 0.250 & 0 \\
\end{bmatrix}
\quad G = \begin{bmatrix}
0 & 2.211 & -0.974 & 0 \\
1.333 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 \\
0 & -1.105 & 2.737 & 0 \\
\end{bmatrix}
\]

The first scaled differentials with respect to thresholds \( P_{4,3,2,1} \) are

\[
D' = \begin{bmatrix}
0 & -0.750 & 0 & 0 \\
1.164 & 0 & 0 & -0.624 \\
0.540 & 0 & 0 & -0.635 \\
0 & 0 & 0.5 & 0 \\
\end{bmatrix}
\quad G' = \begin{bmatrix}
0 & 5.368 & -4.079 & 0 \\
-1.333 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 \\
0 & 1.579 & -3.053 & 0 \\
\end{bmatrix}
\]
and the beta matrices are calculated from \( \beta_W = G'D \) and \( \beta_U = D'G \) are

\[
\beta_U = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 3.263 & -2.842 & 0 \\
0 & 1.895 & -2.263 & 0 \\
0 & 0 & 0 & 2
\end{bmatrix}
= \begin{bmatrix}
0.48 & 0 & 0 & -0.762 \\
0 & -1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0.190 & 0 & 0 & -1.048
\end{bmatrix}
\]

The payoff vector \( \Omega \), it scaled elasticity \( \Omega' \) and beta matrix \( \beta\Omega \) are

\[
\begin{bmatrix}
P_4 - P_4^* \\
P_3^* - P_3 \\
P_2^* - P_2 \\
- P_1^*
\end{bmatrix}
= \begin{bmatrix}
2.000 \\
-1.268 \\
1.414 \\
-1.000
\end{bmatrix}
\begin{bmatrix}
P_4 - \gamma P_4^* \\
\gamma P_3^* - P_3 \\
\gamma P_2^* - P_2 \\
- \gamma P_1^*
\end{bmatrix}
= \begin{bmatrix}
3.000 \\
-2.134 \\
0.707 \\
-0.500
\end{bmatrix}
\begin{bmatrix}
1.5 & 0 & 0 & 0 \\
0 & 1.683 & 0 & 0 \\
0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0.5
\end{bmatrix}
\]

From these \( U, W, X \) can be solved, with \( \Omega \) they satisfy value matching

\[
W = \begin{bmatrix}
V_p (P_4) : 1.303 \\
V_f (P_4) : 0.672 \\
V_i (P_2) : 0.564 \\
V_p (P_1) : 0.445
\end{bmatrix}
= \begin{bmatrix}
0 & 0.750 & 0 & 0 \\
0.550 & 0 & 0 & 0.196 \\
0.222 & 0 & 0 & 0.444 \\
0 & 0.250 & 0 & 0.445
\end{bmatrix}
\begin{bmatrix}
1.303 \\
0.895 \\
0.487 \\
0.141
\end{bmatrix}
\]

and discounting \( U = DW \)

These numerical results can be seen in the Section 4 Figure, which represents these values plotted with underlying (full) project value on the horizontal and component values on the vertical. Idle (zero cashflow), power \( (P^*) \) and full (states) have values zero, \( P^{0.5} \) (square root \( P \) in purple) and \( P \) (45 degree red line) respectively, however their switching options \( V_i, V_p \) and \( V_f \) must be added. These are displayed in the vertical separation between the non-flex and flexible values \( (P + V_f \) is in blue, \( P^{0.5} + V_p \) in green and \( V_i \) in turquoise). The four quantities in \( X \) are represented by the vertical black lines at each of the four thresholds 4, 3, 2, 1.

In the idle state, as potential project cashflow picks up, so does the call option \( V_i \) and at an increasing rate until at \( P_2 = 2 \) it is worth sinking fixed PV operating costs of \( X_2 = 1.338 \) in order to launch the power mode which then has value \( \Omega_2 = 1.414 \). This is because although the launch option of
$V_i(P_2) = 0.564$ was sacrificed, $V_p(P_2) = 0.487$ was gained. This last option is a discounted combination allowing either switching to full (at $P_4 = 4$) or reverting to idle (at $P_1 = 1$) but not both.

If project cashflows continue to increase, at $P_4 = 4$, the switching option $V_p(P_4) = 1.303$ is sacrificed along with the power flow (now $\Omega_2 = 4 - 2 = 2$) but in return the full flow 4 is gained. This costs $X_4 = 1.369$ but compensation is made by the receipt of a closure, put, option $V_f(P_4) = 0.672$.

On the way down as cashflows fall, the closure put option increases in value and then pays off at $P_3 = 3$, at this point it is worth sacrificing its higher value $V_f(P_3) = 0.895$ along with the full flow value 3 in return for the power flow 1.732 (i.e. $\Omega_3 = -1.268$) and cost rate savings with PV of 1.359 (i.e. $X_3 = -1.359$).

Due to two way switching in the power state, this sequence of transitions can occur in a different order to that just presented $P_2, P_4, P_3, P_1$; e.g. $P_2, P_4, P_3, P_4$ is also possible as is $P_2, P_1, P_2, P_1$ etc.

Since it determines local rate of return matching, smooth pasting is best represented with the beta matrices

$$
\beta_W W = \begin{bmatrix}
2.048 & 0 & 0 & -0.762 \\
0 & -1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0.190 & 0 & 0 & -1.048
\end{bmatrix} \begin{bmatrix}
1.303 \\
0.895 \\
0.564 \\
0.445
\end{bmatrix} = \beta_U U + \beta_\Omega \Omega
$$

which shows the relative beta calculation of each flexibility element at each point in the investment graph.

The matrix for $\beta_W$ contains the elasticities for the components in $W$ evaluated at each of the thresholds $P_n$. When premultiplying it on $W$, the aggregate beta of flex consumed is calculated, thus on the first line (top $P_4 = 4$) $V_p(P_4) = 1.303$ has a beta itself of 2.048 (at this threshold the put has zero value). However the put has a non zero derivative and contributes to the slope here with a beta of -0.762 expressed as a function of its value at $P_1$ which is $V_p(P_1) = 0.445$. Thus the composite beta of flex foregone is
2.048 \times 1.303 - 0.762 \times 0.445 = 2.329. This matrix also contains simple one way, put and call betas (−1, 2) at $P_3$ and $P_2$ respectively.

7.2 Special cases

We can also look at the consequences of either matching two thresholds or sending one threshold to an extreme level. Using the formalization of multiple smooth pasting and value matching, the purpose is to be able to model either perfect reversibility (with continuous switching) or irreversibility (by making the attainment of one remote). The former allows cap and floor like flows to be modelled whilst the latter terminates flexibility at one point within a larger system.

The first and second cases are when an exit threshold becomes unobtainably low (allowing entry only) or a re-entry threshold becomes unobtainably high (to model exit). The third case is when two thresholds coalesce. All are tackled by first using the matrix algebra for finite threshold and differences and then evaluating the solved flexibility values in a limit. Subject to a certain level of numerical precision, the matrix algebra can handle these special situations as limiting cases increasing the modelling scope.

7.3 One time entry

For the hysteresis example in Sections 2 and 3, consider the limiting case as $P_1 \to 0$. Although the discount and growth matrices are degenerate when $P_1 = 0$, they still have non zero determinants for finite $P_1$. Evaluating the two components of end flex within $W$, it is possible to show that a finite flex value remains for entry $V_i(P_2)$ in the limit but not for exit $V_f(P_1)$

$$
\lim_{P_1 \to 0} \begin{bmatrix}
V_i(P_2) \\
V_f(P_1)
\end{bmatrix} = \begin{bmatrix}
\frac{-1}{a}P_2 \\
0
\end{bmatrix}.
$$

Since exit $P_1$ is no longer attainable, entry at $P_2$ yields no further flex ($V_f(P_2) = 0$) so the classic relationship between one time entry value and trigger level $P_2$ can be recovered from the payoff and cost alone

$$
V_i(P_2) = P_2 - X_2 = \frac{1}{a}P_2 \quad P_2 = \frac{a}{a-1}X_2.
$$

Whilst not specifying how the system would have got into a potential launch position without having first been shut (since $P_1 \to 0$), a one time entry option value can be calculated from this payoff for any intermediate ($0 < P_{1.5} < P_2$) insertion level $P_{1.5}$ via $V_i(P_{1.5}) = D_{1.5.2}V_i(P_2)$.
Although this last example shows how to recover classic “one shot” real option values, we can use this feature of the matrix algebra to solve first and take limits later for the investment ladder in Table 4. Suppose in Section 4 we wished to model “one time entry” into the power state, with subsequent opportunity to move to the full state and back again, but without the opportunity to suspend back to the idle condition. This would be important when valuing an initial option to invest in such a flexible project from a clean sheet as oppose to the option to invest having been in a prior idle state.

Analytically the system for Di4.2 can be inverted (Gr4.2), differentiated, scaled \((D', G')\) and used to solve for all finite values of \(P_{4,3,2,1}\). Again under GBM (other processes such as ABM will have other natural limits, e.g. \(P_1 \rightarrow -\infty\)) letting the threshold \(P_1\) become as close to zero as can be tolerated numerically, will achieve the goal of separating power from idle states at \(P_1\). Other flexibility values will adjust accordingly. Having effectively set \(P_1 = 0\), any one time option from a non-zero insertion point can be calculated. Calculations below use parameters common from Section 4 to shows values for this limiting case.

The limits on the numerical calculations as \(P_1 \rightarrow 0\) will be the precision and conditioning within the matrices; for example the ability to calculate the necessary determinants.

7.4 One time exit

Looking at the classical GBM hysteresis from Sections 2 and 3 one last time, we can model one time exit by allowing \(P_2 \rightarrow \infty\). This means that once in the idle state, the full threshold \(P_2\) is not achievable, i.e. that any exit is final and “one time”. Again solving for \(W\) before taking limits shows

\[
\lim_{P_2 \rightarrow \infty} \begin{bmatrix} V_i(P_2) \\ V_f(P_1) \end{bmatrix} = \begin{bmatrix} \frac{1}{b}P_2 \\ -\frac{1}{b}P_1 \end{bmatrix} \rightarrow \begin{bmatrix} \infty \\ -\frac{1}{b}P_1 \end{bmatrix}.
\]

Since the payoff when launching at an arbitrarily high level \((P_2)\) is unbounded, so is the option value \(\frac{1}{b}P_2\) at that threshold. However due to the effect of discounting, the option flex value at \(P_1\) is bounded (and since \(b < 0\), positive) at \(-\frac{1}{b}P_1\). Its exercise, which begets \(X_1 - P_1\) only (no further flex), is consistent with an optimal put threshold of \(P_1 = \frac{b}{b-1}X_1\) which is natural given our understanding of the classical GBM put.

Applying this idea to the Investment Ladder in Section 4, if \(P_4 \rightarrow \infty\) then the full project can switch to the power but not the reverse. Thus if the project were found in the full state (maybe having switched to full for the
last available opportunity – see next section), we can model its one time exit from there into the flexible power state. Values for flex values and options at all other thresholds are also shown below, which shows how to evaluate this one time exit from the full state at \( P_{3.5} = 3.5 \) say.

Both this and the one time entry case can also be represented in a simple sequential model where switching can occur a finite number of times as in Ekern (1993) [10]. This occurs in Section 6 for sequential hysteresis includes terminal limiting cases (\( P_{\text{open}} \to \infty, P_{\text{close}} \to 0 \)) to model finite switching ending in closed or open, before moving on to show how different stochastic processes can be incorporated at different stages of a sequence. Before moving to that case, one other limiting situation is handled.

7.5 Perfect reversibility

Consider one reversible boundary separating only two regions and a policy of continuous switching at a common point. This form of collapsed hysteresis can be modelled as \( P_2 \to P_1 \) in Sections 2 and 3 (for GBM). This would be an appropriate policy in a situation where two frictions \( K_{2,1} \) incurred at \( P_{2,1} \) (contained in \( X \) in Vars3.3) tend to zero so that hysteresis disappears and action thresholds match.

We would expect the output costs \( X \) to align as well (since \( K_{2,1} \to 0 \)) but the matrix inversion is not strictly possible if \( P_2 = P_1 \). This is because as thresholds merge, discounting disappears since the time to the next switch becomes zero. Furthermore the key matrix (\( S \) from Section 3) that requires inversion has a determinant that tends to zero

\[
\lim_{P_2 \to P_1} D, G \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \det [\beta_W G - \beta_U] \to 0.
\]

However, analytical progress can be made with the solution before the limit is taken. With the GBM system in Section 3 for example, it is possible to show using L’Hôpital’s rule on the quotients that finite values of \( V_i(P_1) \), \( V_f(P_2) \) result, i.e. for \( U \)

\[
\lim_{P_2 \to P_1} \begin{bmatrix} V_i(P_2) \\ V_f(P_1) \end{bmatrix} = \begin{bmatrix} P_1 \frac{a - 1}{b - ab} \\ P_1 \frac{1 - b}{a' - ab} \end{bmatrix}.
\]

The common value \( X \) of \( X_2 = -X_1 = \frac{\delta}{r} P_1 \) corresponds to a flow condition \( \delta P_1 \geq r X \) which says that the perfectly reversible option to switch should be based on the opportunity cost rates of the current project flow \( \delta P_1 \) and operational costs \( r X \).
Whilst L'Hôpital's rule works analytically, the same result can be retrieved numerically with arbitrary precision by using a small but finite difference between the two unique thresholds. In the next subsection $10^{-6}$ (consistent with the numerical precision available in the computational package) is used for the Investment Ladder threshold differences $P_2 - P_1$ and $P_4 - P_3$.

Safe in the knowledge that the limit is secure, a system with a truly reversible threshold can be modelled numerically to any desired level of accuracy as having an arbitrarily small hysteretic band. Thus within a larger system, reversibility can be built in as a feature at any pair of thresholds that are to act as one (knowing that $X_4 = -X_3$ etc. will be recovered in Section 4). Along with the one time options above, this is a useful modular feature of the solution method of this paper.

### 7.6 Limiting cases

The calculations tolerate limiting special cases. The first is as $P_1 \to 0$ when exit from the power state is no longer allowable, and the only entry possible is one time. For the same values of $P_{4,3,2}$ but with $P_1 = 10^{-6}$, the solved flex values and investment costs are (limited only by the precision available within the matrix inversion, 0.000 indicates values less than $10^{-4}$)

$$
\begin{bmatrix}
P_n \\
4 \\
3 \\
2 \\
10^{-6}
\end{bmatrix}
\begin{bmatrix}
W \\
1.210 \\
0.772 \\
0.656 \\
0.000
\end{bmatrix}
\begin{bmatrix}
U \\
0.579 \\
0.681 \\
0.303 \\
0.000
\end{bmatrix}
\begin{bmatrix}
X \\
1.369 \\
-1.359 \\
1.061 \\
0.000
\end{bmatrix}
\begin{array}{c}
\end{array}
$$

One time entry say from a jumping or insertion point of $P_{1.5} = 1.5$ is therefore worth $\left(\frac{1.5}{10^{-6}}\right)^2 \times 0.303 = 0.170$.

Modelling a one time switch from full is also possible. Sending $P_4 \to \infty$ prohibits the transition from power to full but not full to power. Although setting $P_1 = 10^6$ yields two extreme values and investment quantities, at other thresholds these get discounted to zero due to their remoteness

$$
\begin{bmatrix}
P_n \\
10^6 \\
3 \\
2 \\
1
\end{bmatrix}
\begin{bmatrix}
W \\
5.000 \times 10^5 \\
2.257 \\
0.261 \\
0.369
\end{bmatrix}
\begin{bmatrix}
U \\
0.000 \\
0.123 \\
0.185 \\
0.065
\end{bmatrix}
\begin{bmatrix}
X \\
5.000 \times 10^5 \\
-3.402 \\
1.338 \\
-1.304
\end{bmatrix}
\begin{array}{c}
\end{array}
$$

Although it could not have come from $P_4$ (unless perhaps for a final time – see Section 6) were the project to be inserted in the full state say at
$P_{3.5} = 3.5$, it would be worth $3.5 + \left(\frac{3.5}{3.0}\right)^{-1} \times 2.257 = 5.435$. The moderate flex premium to its project value of 3.5 reflects the timing option at $P_3$ to got to power flow with a project payoff net of cost savings $(P_3^2 - P_3 + X_3)$ of $1.732 - 3 + 3.402 = 2.134$ equal to the flex value use $2.257 - 0.123$.

Finally, to demonstrate prefect reversibility at each of the two thresholds $P_3 = 3$ and $P_2 = 2$ (and two way passage between them), this final example has $P_4 = P_3 + 10^{-6}$ and $P_1 = P_2 - 10^{-6}$

\[
\begin{bmatrix}
    P_n \\
    3 + 10^{-6} \\
    3 \\
    2 \\
    2 - 10^{-6}
\end{bmatrix}
\begin{bmatrix}
    W \\
    1.019 \\
    0.803 \\
    0.525 \\
    0.702
\end{bmatrix}
\begin{bmatrix}
    U \\
    0.803 \\
    1.109 \\
    0.702 \\
    0.525
\end{bmatrix}
\begin{bmatrix}
    X \\
    1.051 \\
    -1.051 \\
    1.591 \\
    -1.591
\end{bmatrix}
\]

Firstly note that as expected the investment divestment quantities match $X_4 = -X_3$ and $X_2 = -X_1$ (zero frictions $K$). Secondly since payoffs are also matched at $P_{4,3}$ (and also $P_{2,1}$) cross pair option elements within $W, U$ are also equal. The matrix algebra can only be inverted if all Sp conditions are unique, whereas in the limit both two equations at a common limit have the same WACC. This system can be used to search for specific $X$, say $X_4 = -X_3 = 1, X_2 = -X_1 = 1.5$ in which case the solution is near $P_{4,3} = 2.924$ and $P_{2,1} = 1.778$ (and $W, U$ also differ).

Also note that Section 5 can be made equivalent to Ekern (1993) [10], bringing an end to flex by setting a final suspension threshold very close to zero (e.g. $P_1 \rightarrow 0$ to limit flex after the transition at $P_3$ and remain in full mode) or a final activation threshold very high (or $P_3 \rightarrow \infty$ to limit flex after the transition at $P_2$ and remain idle).