Taking Away the Punch Bowl: 
Monetary Policy and Financial Instability*

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First draft: 19th December 2017 
This version: 4th December 2018

Abstract

In the last decade, the problem of financial instability has surged to the forefront of economists’ and policymakers’ attention. This paper presents a theory of financial crises due to excessively loose monetary policy. Under political pressure to choose a monetary policy that is popular with the average person in the economy, the central bank sets low interest rates, which most of the time delivers rapid asset-price inflation and a build-up of debt, but also occasional financial crises where asset prices collapse and deleveraging occurs. Pareto-inefficient financial instability occurs even though the central bank maximizes average welfare because too many individuals would lose if the central bank were to ‘take away the punch bowl’. To avoid financial crises, the policy implication is that central banks need to become ‘conservative’ in the sense that monetary policy should be set in the interests of savers rather than borrowers.

JEL classifications: E32; E44; E51; E52; G01.

Keywords: financial crises; monetary policy; house prices; heterogeneous agents; political economy; risk premiums; rare events; portfolio choice.

*I thank seminar participants at the Bank of England, U. Bonn, CASS-RCUK conference on “Managing Financial Risk and Promoting Stable Economic Growth”, the Einaudi Institute of Economics and Finance, the London School of Economics, the 2018 Fall Midwest Macro Meeting, and U. York for helpful comments. An earlier draft of this paper was circulated under the title “A Tale of Two Inflation Rates: House-Price Inflation and Monetary Policy”.

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The Federal Reserve, after the recent increase in the discount rate, is in the position of the chaperone who has ordered the punch bowl removed just when the party was really warming up.

William McChesney Martin, Chairman of the Federal Reserve (1951–1970)

1 Introduction

Since the financial crisis of 2007–2008, financial instability has been back at the forefront of the attention of economists and policymakers. There has been much debate about the causes of financial crises, which policy responses are appropriate, and whether there are reforms to the financial system or regulatory changes that might reduce the likelihood of crises occurring in the future.

One hypothesis is that monetary policy bears some responsibility for crises. By setting interest rates too low, central banks create asset-price booms and encourage an unsustainable build-up of debt. Once asset prices eventually drop, the presence of large amounts of debt leads to a financial crisis. This view has been articulated by Taylor (2009) in the context of the U.S. monetary policy prior to the 2007–2008 financial crisis, and more generally as the 'BIS view' (Borio and Lowe, 2002, Borio, 2012). However, in spite of some economists attributing blame to central banks and monetary policy, most remain unconvinced, in part because of the lack of a clear theoretical mechanism through which monetary policy can have such dramatic real effects on financial instability. If central banks cannot influence long-term real interest rates, how can they be responsible for booms and busts in asset prices and debt?

The contribution of this paper is to present a theory of financial crises caused by pressure on central banks to adopt excessively loose monetary policies, in other words, the difficulty of 'taking away the punch bowl'. In this theory, monetary policy takes centre stage in explaining financial crises because it has long-lasting real effects on borrowing costs and the volatility of asset prices. By studying the political economy of monetary policy, the theory also explains why the central bank will find itself under pressure to adopt policies that create the risk of financial crises.

In the equilibrium of the model, monetary policy keeps real borrowing costs low for as long as possible. The consequence of this policy is that most of the time there is sustained inflation in house prices and a build-up of debt. However, the policy also results in occasional financial crises: rare events where house prices collapse and significant deleveraging occurs. The occurrence of financial crises and the booms in asset prices and credit that precede them imply a highly inefficient allocation of resources in the economy. Monetary policy is at fault in the sense that it would be possible for the central bank to achieve an efficient allocation of resources by setting higher interest rates to prevent house-price booms and busts — but tight monetary policy will not be popular with the average person in the economy.

There are two key ingredients of the theory. First, an incomplete markets friction whereby borrowers are restricted to nominal debt contracts. Households who want to borrow to buy houses only have access to standard mortgages. They cannot raise finance by selling equity shares in their houses, or hedge their exposure to house prices by buying or selling derivatives. Markets
are also incomplete in that households cannot decouple their enjoyment of housing services from homeownership by making use of a perfect rental market.

The second key ingredient is political economy. The central bank cannot ignore pressure to choose a monetary policy that is in the interests of the average household in the economy. The danger this poses to monetary policy is that whenever the central bank uses a criterion based on households’ utilities to guide policy, it introduces distributional considerations that can sway policy from a focus purely on promoting economic efficiency. The tension between distribution and efficiency stems from the central bank having limited instruments: interest rates affect both the distribution of wealth and the efficiency of the allocation of resources in the economy. The central bank does not have access to individual-specific lump-sum taxes and transfers to sterilize the distributional consequences of changes to interest rates.

The theory is built on a mechanism through which the central bank can have a long-lasting impact on the real cost of borrowing. This is the effect of monetary policy on risk premiums, for which there is a straightforward logic. Even without introducing any frictions, monetary policy can always affect nominal asset prices and thus influence the probability distribution of unexpected nominal changes in asset prices. Comparing these uncertain nominal returns to the perfectly predictable nominal return on nominal bonds, standard asset-pricing theory predicts that monetary policy will be able to affect the risk premium over nominal bonds of real assets such as housing or shares, even though it cannot change the relative returns on two real assets. Combined with the incomplete markets friction whereby borrowers need to obtain credit in the form of nominal debt, monetary policy is therefore able to vary the real cost of borrowing through its impact on the relative returns on nominal and real assets.

The mechanism behind this effect of monetary policy on real borrowing costs is not restricted to a ‘short run’ defined in terms of sluggish price adjustment or imperfect information. Consequently, monetary policy operating through the risk premium channel can have long-lasting effects on real asset prices and credit. The greater the risk premium, the larger the gap between mortgage rates and the expected capital gains on housing. A lower real cost of borrowing stimulates a build-up of mortgage debt and pushes up real house prices. The impact is magnified by the financial accelerator where increases in house prices improve balance sheets and stimulate further mortgage lending. Note that the ability of monetary policy to affect real borrowing costs through the mechanism here does not mean it can raise or lower real returns simultaneously for all assets: higher real returns on housing are associated with lower real returns for bondholders.

Importantly, the central bank cannot change real interest rates through the risk premium channel without this having consequences for financial stability. The logic behind the risk premium means there must be in equilibrium a negative relationship between the interest rate on bonds and the volatility of asset prices. Lower interest rates can then only be achieved with a concomitant rise in financial risk. The boom in real asset prices and credit resulting from loose monetary policy is not one that can be sustained forever in equilibrium. All else equal, the larger the boom and the longer on average it is sustained, the larger must be the fall in asset prices when the boom unexpectedly comes to an end.
The theory is able to explain the occurrence of financial crises if the central bank pursues loose monetary policy for long periods of time. In this case, there must be some rare occasions when there is a large fall in nominal house prices. Since the crisis has been preceded by a build-up of nominal debt, the financial accelerator mechanism goes into reverse. New lending is reduced as balance sheets deteriorate, and the fall in nominal assets prices largely ends up reflecting a reduction in the relative price of housing, with goods prices and nominal incomes responding much less during the crisis.

Even if monetary policy could create a risk of financial crises because of the logic above, it is less obvious why policy would be conducted in this way, especially if the central bank acts to maximize the average utility of households in the economy (monetary policy is the solution of a Ramsey problem). The explanation lies in the distributional effects of monetary policy. Savers lose from low real interest rates, while borrowers gain. Since low real interest rates generate a credit boom in normal times, borrowers who have taken on debt to buy houses also gain from faster house-price inflation that raises the expected real return on their leveraged investment in housing. All else equal, leveraged homeowners dislike financial instability because they are exposed to house-price risk, but all else is not equal because of the negative equilibrium relationship between the interest rate and the volatility of house prices. Borrowers might then like low interest rates overall even if these have to be accompanied by a small risk of a financial crisis. In the model presented in this paper, low interest rates together with greater financial instability always constitute a net gain to borrowers. All that is needed then for the central bank to run a loose monetary policy creating the risk of financial crises is for the interests of borrowers to outweigh the interests of savers in the calculation of average utility. In the model, this always occurs unless the population is sharply declining.

The theory of financial crises in this paper does not depend on agents being irrational, or forming expectations with systematic errors. Furthermore, policymakers are benevolent in the sense of maximizing the utility of the average person in the economy. In the model, everyone knows that loose monetary policy creates the conditions for financial crises, even if the timing of crises cannot be predicted. Nonetheless, there is sufficient political pressure for low interest rates in spite of the negative consequences. To avoid financial crises, the policy implication is that central banks need to become ‘conservative’ by giving more weight to the interests of savers rather than borrowers.

In explaining why there are low-probability events with large falls in house prices, the theory endogenously generates ‘rare events’ of the kind that have been emphasized in the literature on asset pricing (see, for example, Barro, 2006). As in that literature, rare events have a large impact on risk premiums. Here, the difference is that the rare events are the consequence of monetary policy trying to generate a housing boom for as long as possible, with the transmission mechanism of monetary policy working through risk premiums. The importance of risk premiums in understanding house price developments is stressed by Favilukis, Ludvigson and Van Nieuwerburgh (2017) in a richer quantitative model, but that paper does not study the consequences of monetary policy for risk premiums.

The paper differs radically from the common approach in models of financial frictions where a linearized model is subject to shocks that generate small fluctuations in the neighbourhood of
a steady state (Bernanke, Gertler and Gilchrist, 1999, Kiyotaki and Moore, 1997). In trying to understand financial crises as large discrete changes that hit an economy, the paper’s goals have some similarity to Brunnermeier and Sannikov (2014). However, the mechanism for generating financial crises is completely different from that paper. There, the key assumption is that capital can be used most effectively by financially constrained ‘experts’. By finding the full non-linear solution for the model’s equilibrium, it is shown how an accumulation of small shocks can trigger a crisis episode. Here, financial crises occur in the housing market and are caused by the political economy of monetary policy. These are novel features in a formal model of financial crises.

While the model in this paper is purely theoretical, its predictions of a credit cycle with credit booms preceding financial crises is consistent with the empirical evidence in Schularick and Taylor (2012). There is also some evidence supporting the role of monetary policy in generating credit booms (Jiménez, Ongena, Peydró and Saurina, 2014). The normative implications of the paper also build on a earlier literature that has explored the implications of incomplete financial markets for monetary policy (Koenig, 2013, Sheedy, 2014).

The modelling strategy in the paper is to present a highly stylized model that is stripped down to the essentials. Everything can be solved analytically without the need for any approximations or numerical methods, including the constrained optimization of social welfare used to determine the central bank’s monetary policy. The essential features of the model are heterogeneous agents, a housing market, nominal debt contracts, aggregate risk, and monetary policy set endogenously to solve a Ramsey problem. The simplest model that allows an exposition of the theory has overlapping generations of individuals with stylized hump-shaped profiles of non-financial income and the marginal utility from housing over their three-period lives, a stochastic endowment of consumption goods, and an exogenous supply of housing. This simple model exogenously assumes incomplete markets, but it is shown the main conclusions are robust to an extension where there are also markets subject to frictions for renting houses and trading equity shares in houses.

The plan of the paper is as follows. Section 2 sets up a model of a credit economy with housing and incomplete markets. Section 3 analyses the political economy of monetary policy and derives the equilibrium with financial crises when the central bank maximizes the sum of household utilities. Policy implications are studied in section 4, where it is shown that monetary policy can implement a first-best allocation of resources by making financial stability its goal. Section 5 endogenizes incomplete markets by extending the analysis to include additional markets subject to frictions. Finally, section 6 draws some conclusions.

2 A credit economy with housing and incomplete markets

The economy has overlapping generations of individuals who have deterministic lives spanning three discrete time periods. Individuals of different generations are referred to as the ‘young’, ‘middle-aged’, and ‘old’, indexed by $\tau \in \{y,m,o\}$. The number of people in generation $\tau$ at date $t$ is denoted by $N_{\tau,t}$, and the total population is $N_t \equiv N_{y,t} + N_{m,t} + N_{o,t}$. The birth rate per middle-aged individual is $\gamma$ (with $0 < \gamma < \infty$), that is, $N_{y,t} = \gamma N_{m,t}$. Since $N_{m,t} = N_{y,t-1}$ and $N_{o,t} = N_{m,t-1}$, the
age distribution and the dynamics of the total population are:

\[
\frac{N_{y,t}}{N_t} = \frac{\gamma^2}{1 + \gamma + \gamma^2}, \quad \frac{N_{m,t}}{N_t} = \frac{\gamma}{1 + \gamma + \gamma^2}, \quad \frac{N_{o,t}}{N_t} = \frac{1}{1 + \gamma + \gamma^2}, \quad \text{and} \quad N_t = \gamma N_{t-1}, \tag{2.1}
\]

which means the growth rate of the population is equal to \(\gamma - 1\).

Individuals born at date \(t\) have the following lifetime expected utility function:

\[
\mathcal{U}_t = \log C_{y,t} + \delta E_t [\log C_{m,t+1} + \Theta(H_{m,t+1})] + \delta^2 E_t \log C_{o,t+2}, \tag{2.2}
\]

where \(C_{\tau,t}\) denotes per-person consumption of a composite good by individuals of age \(\tau\) at date \(t\) (with \(C_{\tau,t} \geq 0\)). Utility is logarithmic in consumption of goods.\(^1\) The subjective discount factor is \(\delta\) (satisfying \(0 < \delta < \infty\)), and \(E_t\) denotes expectations conditional on all date-\(t\) information, where expectations are formed rationally. There is no altruism across generations.

The utility function (2.2) assumes a stylized life-cycle pattern of housing demand that is concentrated in middle age. Making this assumption provides a simple reason for houses to be traded between the generations. Formally, utility from housing services \(H_{m,t}\) is received only while middle-aged, with \(H_{y,t}\) and \(H_{o,t}\) not appearing in the utility function.\(^2\) Housing services \(H_{m,t}\) is a continuous variable (with \(H_{m,t} \geq 0\)), and the housing utility function \(\Theta(H)\) is strictly increasing, strictly concave, and satisfies the Inada conditions (\(\Theta'(H) > 0, \Theta''(H) < 0, \lim_{H \to 0} \Theta'(H) = \infty\), and \(\lim_{H \to \infty} \Theta'(H) = 0\)).

Individuals receive an exogenous real non-financial income, where ‘real’ refers to units of the composite consumption good. Non-financial income is assumed to have a stylized life-cycle pattern that is concentrated in middle age. Making this assumption provides a reason for some individuals to save, and for others to borrow both for consumption and to buy houses. Formally, each middle-aged individual at date \(t\) receives a stochastic real non-financial income \(y_{m,t} > 0\), while the young and the old receive nothing, so \(y_{y,t} = 0\) and \(y_{o,t} = 0\). Non-financial income can be interpreted as labour income, with labour supplied inelastically only by the middle aged, and where \(y_{m,t}\) is proportional to labour productivity. All middle-aged individuals at date \(t\) receive the same realization of \(y_{m,t}\), so there is aggregate risk, but no idiosyncratic risk.

The economy is an endowment economy where \(Y_t\) is the exogenous supply of goods available for consumption. Consumption goods are not storable. The aggregate endowment is equal to the total non-financial income of all individuals:

\[
Y_t = N_{m,t} y_{m,t}. \tag{2.3}
\]

The supply of houses \(H_t\) is assumed to be inelastic. The housing stock tracks the number of young

\(^1\)Conditional on additive separability between the utilities from consumption of goods and housing services, logarithmic utility in consumption is required for the existence of a balanced growth path. See footnote 6 below.

\(^2\)Equivalently, it could be assumed that individuals require some minimum amount of housing services at all points in life, and are endowed with this amount of housing at birth. The variable \(H_{m,t}\) then denotes the additional demand for housing services above the minimum while middle aged, for example, the desire for a larger family house.
Individuals:

$$H_t = N_{y,t},$$ [2.4]

which means it grows in line with the population at rate $\gamma - 1$. There are no construction or maintenance costs.

The only exogenous stochastic variables are the economy’s real GDP growth rate $g_t$, and a ‘sunspot’ variable $\psi_t$ independent of the fundamentals of the economy. Real GDP growth is:

$$g_t = \frac{Y_t - Y_{t-1}}{Y_{t-1}}, \quad \text{where} \quad g_t \sim [g, \bar{g}] \text{ with } -1 < g < \bar{g} < \infty,$$[2.5]

Real GDP growth can have any bounded probability distribution. At least one of the probability distributions of $g_t$ and $\psi_t$ is non-degenerate so there is something individuals are uncertain about.

The economy has markets for goods, houses, and financial assets. All markets are perfectly competitive and all prices are fully flexible. Money is used as a unit of account, but the economy is ‘cash-less’ in the sense of having no physical monetary units. The nominal price of a unit of goods is denoted by $P_t$ and the nominal price of unit of housing by $V_t$.

Financial markets are incomplete. Apart from housing, the only asset or liability is a one-period nominal bond. Let $B_{\tau,t}$ denote the quantity of nominal bonds purchased (or issued, if negative) by households of age $\tau$ at the end of time period $t$. Each nominal bond is a riskless claim to one monetary unit of account at time $t + 1$ (the face value is one unit of money). There is no option of voluntary default, and there must be no involuntary default in any state of the world. At time $t$, the nominal price of a bond is $Q_t$. A key missing market is for securities where payments depend on the realization of house prices. Here, house purchases cannot be financed by issuing such securities, but the presence of such a market subject to frictions is considered later in section 5.

Utility from housing services $H_{m,t}$ can only be obtained through homeownership. Houses must be purchased and held between $t-1$ and $t$ to enjoy utility flows at time $t$. Any change in the housing stock accrues proportionately to existing homeowners. Another missing market is a perfect rental market that would allow consumption of housing services to be separated from homeownership. The presence of a frictional rental market is considered later in section 5.\(^3\)

Individuals are born with no initial assets or liabilities and leave no bequests. The budget identities of the young, middle-aged, and old are respectively:

$$C_{y,t} + \frac{V_t H_{m,t+1}}{P_t} + \frac{Q_t B_{y,t}}{P_t} = 0;$$ [2.6a]

$$C_{m,t} + \frac{Q_t B_{m,t}}{P_t} = y_{m,t} + \frac{\gamma V_t H_{m,t}}{P_t} + \frac{B_{y,t-1}}{P_t};$$ [2.6b]

and $C_{o,t} = \frac{B_{m,t-1}}{P_t}.$ [2.6c]

\(^3\)Even in the absence of a rental market, it is conceivable that individuals might hold housing purely as an investment with the only returns coming from capital gains or losses. This possibility is ruled out for now ($H_{o,t} = 0$), but it will be considered later as a special case of the analysis in section 5.
The non-negativity constraints on consumption and housing services require that the borrowers in the economy are the young \((B_{y,t} \leq 0)\) and the savers are the middle-aged \((B_{m,t} \geq 0)\). To satisfy the non-negativity constraints and rule out involuntary defaults requires:

\[
y_{m,t} + \frac{\gamma V_t H_{m,t}}{P_t} + \frac{B_{y,t-1}}{P_t} \geq 0, \quad [2.7]
\]

which must hold with probability one.

Lifetime utility (2.2) is a strictly concave function and the constraints in (2.6) are linear in the choice variables, so first-order conditions are necessary and sufficient for a global maximum. Maximizing expected utility (2.2) with respect to house purchases \(H_{m,t+1}\) subject to the budget identities (2.6a) and (2.6b) implies an Euler equation for the house purchases of the young:

\[
\frac{V_t}{P_tC_{y,t}} = \delta E_t \left[ \Theta'(H_{m,t+1}) + \frac{\gamma V_{t+1}}{P_{t+1}C_{m,t+1}} \right]. \quad [2.8]
\]

Maximizing expected utility with respect to bond holdings \(B_{\tau,t}\) subject to the budget identities in (2.6) implies the following Euler equations for the young and the middle-aged:

\[
\frac{Q_t}{P_tC_{y,t}} = \delta E_t \left[ \frac{1}{P_{t+1}C_{m,t+1}} \right], \quad \text{and} \quad \frac{Q_t}{P_tC_{m,t}} = \delta E_t \left[ \frac{1}{P_{t+1}C_{o,t+1}} \right]. \quad [2.9]
\]

The no-default condition (2.7) is not imposed as an additional constraint because it is equivalent to the requirement of non-negative consumption, which will always hold because logarithmic utility satisfies the Inada conditions.\(^4\)

There is a central bank that sets the risk-free nominal interest rate \(i_t\), which determines the nominal bond price:\(^5\)

\[
Q_t = \frac{1}{1 + i_t}. \quad [2.10]
\]

The nominal interest rate \(i_t\) is not subject to the zero lower bound because there is no physical cash. Monetary policy is conducted solely using the conventional policy instrument \(i_t\); there are no unconventional instruments such as balance-sheet policies. Lump-sum taxation is infeasible, and the central bank does not control any other fiscal policy instruments (from which the model abstracts).

In equilibrium, the goods, housing, and bond markets must clear:

\[
C_t = Y_t, \quad \text{where} \quad C_t = N_{y,t}C_{y,t} + N_{m,t}C_{m,t} + N_{o,t}C_{o,t}; \quad [2.11]
\]

\[
N_{y,t}H_{m,t+1} = H_t; \quad \text{and} \quad [2.12]
\]

\[
N_{y,t}B_{y,t} + N_{m,t}B_{m,t} = 0. \quad [2.13]
\]

\(^4\)A collateral constraint where borrowing is restricted to a fraction of housing values is not imposed. However, lending will be limited by the net worth of lenders, which is positively related to house prices. It also turns out that borrowing will be proportional to housing values in equilibrium even without a binding collateral constraint.

\(^5\)This could be done by paying interest on reserves that are perfect substitutes for nominal bonds.
Total consumption $C_t$ must equal the economy’s endowment of goods. House purchases by the young generation must equal the total housing stock, which is sold by the previous generation. The pension savings accumulated by the middle-aged must be equal to the loans taken out by the young.

The following definitions are made. The consumption shares of GDP of each generation $\tau \in \{y, m, o\}$ are denoted by $c_{\tau,t}$, and the value of all houses relative to GDP by $h_t$. Let $l_t$ denote the value of new lending relative to GDP, and $d_t$ the amount of outstanding debt relative to the value of all houses at time $t$:

$$c_{\tau,t} \equiv \frac{N_{\tau,t}C_{\tau,t}}{Y_t}, \quad h_t \equiv \frac{V_tH_t}{P_tY_t}, \quad l_t \equiv -\frac{Q_tN_{y,t}B_{y,t}}{P_tY_t}, \quad \text{and} \quad d_t \equiv -\frac{N_{y,t-1}B_{y,t-1}}{V_tH_t}. \quad [2.14]$$

The variables $c_{\tau,t}$ and $h_t$ must be non-negative, as must $l_t$ and $d_t$ because of $B_{y,t} \leq 0$.

Goods-price inflation is denoted by $\varpi_t$ and house-price inflation by $\pi_t$:

$$\varpi_t \equiv \frac{P_t - P_{t-1}}{P_{t-1}}, \quad \text{and} \quad \pi_t \equiv \frac{V_t - V_{t-1}}{V_{t-1}} + (\gamma - 1) \frac{V_t}{V_{t-1}}, \quad [2.15]$$

where the definition of the percentage change in the value of a house $\pi_t$ adds a term $(\gamma - 1)V_t/V_{t-1}$ because increments to the stock of houses accrue to existing owners. This means that $\pi_t$ is the correct measure of nominal capital gains or losses on housing. In any case, $1 + \pi_t = \gamma V_t/V_{t-1}$ is proportional to a simple measure of the change in house prices $V_t/V_{t-1}$.

The imputed nominal rent $Z_t$ received by homeowners at date $t$, the total nominal ex-post return on housing $\hat{R}_t$ between $t-1$ and $t$ inclusive of imputed rent, and the ex-post and ex-ante real returns on housing $\hat{r}_t$ and $\hat{\varrho}_t$ (in terms of consumption goods) are defined by:

$$Z_t \equiv \Theta'(H_{m,t})P_tC_{m,t}, \quad \hat{R}_t \equiv \pi_t + \frac{Z_t}{V_{t-1}}, \quad \hat{r}_t \equiv \frac{1 + \hat{R}_t}{1 + \varpi_t} - 1, \quad \text{and} \quad \hat{\varrho}_t \equiv \mathbb{E}_t\hat{r}_{t+1}. \quad [2.16]$$

That the imputed rent and thus the return on housing are defined appropriately in (2.16) can be seen by noting that the following Euler equation satisfied by the housing return is equivalent to the housing demand equation in (2.8):

$$\frac{1}{C_{y,t}} = \delta \mathbb{E}_t \left[ \frac{1 + \hat{R}_{t+1}}{(1 + \varpi_{t+1})C_{m,t+1}} \right].$$

For nominal bonds, the definitions of the nominal return $R_t$, the ex-post and ex-ante real returns $r_t$ and $\varrho_t$, and the excess expected return $\xi_t$ of housing over nominal bonds are:

$$R_t \equiv \frac{1}{Q_{t-1}} - 1, \quad r_t \equiv \frac{1 + R_t}{1 + \varpi_t} - 1, \quad \varrho_t \equiv \mathbb{E}_t r_{t+1}, \quad \text{and} \quad \xi_t \equiv \frac{1 + \mathbb{E}_t \hat{R}_{t+1}}{1 + \mathbb{E}_t R_{t+1}} - 1. \quad [2.17]$$

The Euler equations (2.8) and (2.9) for housing and bonds imply that the excess expected return $\xi_t$
satisfies the usual conditional covariance formula for a risk premium:

$$\xi_t = -\text{Cov}_t \left[ \hat{R}_{t+1}, \frac{\delta C_{y,t}}{1 + \omega_{t+1} C_{m,t+1}} \right].$$

The formula is in terms of the covariance between the nominal return on housing and the nominal stochastic discount factor of buyers of housing because bonds are risk free in nominal terms, but not necessarily in real terms.

### 2.1 Equilibrium conditions

The definition of the consumption shares $c_{t,t}$ in (2.14) implies the goods-market clearing condition (2.11) can be written as:

$$c_y,t + c_m,t + c_o,t = 1. \quad \text{[2.18]}$$

Using the population growth rate from (2.1), the stock of houses from (2.4) and real GDP growth (2.5), the definition of the value of houses to GDP and the new and outstanding debt ratios from (2.14), the inflation rates from (2.15), and the real return from (2.17), the following two accounting identities are obtained:

$$d_t = \frac{(1 + r_t)h_{t-1}}{(1 + g_t)h_t}, \quad \text{and} \quad \frac{1 + \pi_t}{1 + \omega_t} = \frac{(1 + g_t)h_t}{h_{t-1}}. \quad \text{[2.19a]}$$

The housing stock in (2.4) and the housing-market clearing condition (2.12) require $H_{m,t} = 1$. Imposing bond-market clearing (2.13) and using the age distribution in (2.1), the consumption and housing ratios and the loans and debt ratios from (2.14), the three budget identities in (2.6) can be written as:

$$c_y,t + h_t = l_t, \quad c_{m,t} + l_t = 1 + (1 - d_t)h_t, \quad \text{and} \quad c_{o,t} = d_t h_t. \quad \text{[2.19b]}$$

The housing Euler equation (2.8) and the housing-market clearing condition (2.12) together with the age distribution from (2.1), the housing stock (2.4), and the definitions (2.14), imply the Euler equation for housing is:

$$\frac{h_t}{c_{y,t}} = \delta \theta + \delta E_t \left[ \frac{h_{t+1}}{c_{m,t+1}} \right], \quad \text{where} \quad \theta \equiv \Theta'(1), \quad \text{[2.19c]}$$

with the restrictions on the function $\Theta(H)$ implying that the coefficient $\theta$ (the marginal utility from housing services at $H = 1$) satisfies $0 < \theta < \infty$. The bond Euler equations (2.9) together with the age distribution from (2.1), real GDP growth from (2.5), the consumption shares in (2.14), and the real return in (2.17) lead to:

$$\frac{1}{c_{y,t}} = \delta E_t \left[ \frac{1 + r_{t+1}}{(1 + g_{t+1})c_{m,t+1}} \right], \quad \text{and} \quad \frac{1}{c_{m,t}} = \delta E_t \left[ \frac{1 + r_{t+1}}{(1 + g_{t+1})c_{o,t+1}} \right]. \quad \text{[2.19d]}$$
The bond price equation (2.10), the definition of inflation (2.15), and the real return on bonds in (2.17) imply the ex-post Fisher equation:

\[ 1 + r_t = \frac{1 + i_{t-1}}{1 + \varpi_t}. \]  

[2.19e]

The no-default constraint (2.7) is equivalent to the following holding in all states of the world:

\[ 1 + (1 - d_t)h_t \geq 0. \]  

[2.20]

In equilibrium, \( \pi_t \geq -1 \), \( \varpi_t \geq -1 \), \( r_t \geq -1 \), and \( i_t \geq -1 \), and \( d_t \), \( h_t \), \( l_t \), and \( \pi_t \) must all be non-negative. The system of equations above contains only two relevant parameters: time preference \( \delta \) and the preference for housing relative to goods \( \theta \). Population growth \( \gamma \) does not appear, and the parameter \( \theta \) (from equation 2.19c) summarizes all that needs to be known about the housing utility function \( \Theta(H) \).

6 If utility from consumption of goods is additively separable from housing but were to have the general isoelastic form \( C^{1-\iota}/(1-\iota) \) then (2.19c) and the first equation in (2.19d) would be replaced by:

\[ \frac{h_t}{c_{y,t}} = \delta \gamma^{1-\iota} \left( \frac{1}{y_{m,t}} \right)^{1-\iota} \frac{h_{t+1}}{c_{m,t+1}} + \delta E_t \left[ \frac{1 + r_{t+1}}{(1 + g_{t+1})c_{m,t+1}} \right], \]

and

\[ \frac{1}{c_{y,t}} = \delta E_t \left[ \frac{1 + r_{t+1}}{(1 + g_{t+1})c_{m,t+1}} \right], \]

where \( \iota \) is the inverse of the elasticity of intertemporal substitution. With positive trend growth in income per worker \( y_{m,t} \), if \( \iota \neq 1 \) then there is generally no non-stochastic steady state where the ratios \( h_t \), \( c_{y,t} \), and \( c_{m,t} \) have finite positive values. Conditional on additively separable utility, this argument provides a reason for the restriction to logarithmic utility in consumption (\( \iota = 1 \)).

2.2 Analysis of the equilibrium

The equilibrium of the economy is found in a series of steps that characterize the optimal behaviour of savers and borrowers, and the interaction between the two in determining interest rates and house prices. The first step is the behaviour of the middle-aged, who are the lenders in the economy.

Step 1 (Lenders’ supply of loans) The debt accounting identity in (2.19a), the budget identity of the old in (2.19b), and the bond Euler equation of the middle-aged in (2.19d) imply:

\[ l_t = \delta c_{m,t}. \]  

[2.21]
Lending $l_t$ is therefore proportional to the consumption $c_{m,t}$ of the middle aged.

**Proof** See appendix A.1. ■

The lending of the middle aged being proportional to their consumption means that they save a constant fraction of their net worth. As the utility function is logarithmic in consumption, income and substitution effects exactly cancel out for those who will be retirees in the future (that is, have no non-financial income), so lending does not depend on either the expected real return on bonds $\varrho_t = E_t r_{t+1}$ or uncertainty about the real return $r_{t+1}$.

The next step is to find how the young, who are the borrowers in the economy, will allocate borrowed funds between financing house purchases and paying for consumption goods. The question of how much is borrowed is deferred until later.

**Step 2 (Borrowers’ use of loans)** The budget identity of the young in (2.19b) and the housing Euler equation (2.19c), together with lenders’ optimal behaviour in Step 1, imply that house prices $h_t$ and the consumption of the young $c_{y,t}$ are fixed multiples of total lending $l_t$:

$$h_t = (1 - \kappa)l_t, \quad \text{and} \quad c_{y,t} = \kappa l_t,$$

where the coefficient $\kappa$ is:

$$\kappa = \frac{2}{1 + \delta^2 + \delta \theta + \sqrt{((1 - \delta)^2 + \delta \theta)((1 + \delta)^2 + \delta \theta)}}, \quad \text{with} \quad 0 < \kappa < 1. \tag{2.23}$$

The coefficient $\kappa$ is decreasing in both patience $\delta$ and housing preference $\theta$.

**Proof** See appendix A.2. ■

Young borrowers use a constant fraction $\kappa$ of the loans $l_t$ they receive to pay for consumption $c_{y,t}$, where $\kappa$ depends only on the parameters $\delta$ and $\theta$. Since the total supply of houses is inelastic and all houses are sold to the next generation, the value of houses $h_t$ is equal to the amount of funds loaned to borrowers for house purchases ($h_t = l_t - c_{y,t}$ according to 2.19b), which is a constant fraction $1 - \kappa$ of total credit. The share of credit allocated to housing increases if the preference for housing increases (higher $\theta$) or individuals become more patient (higher $\delta$).

For given parameters, the ratio of lending to house prices $l_t/h_t$ is equal to a constant $\lambda$:

$$l_t = \lambda h_t, \quad \text{where} \quad \lambda = \frac{1}{1 - \kappa}. \tag{2.24}$$

An increase in the real interest rate $\varrho_t$ increases the incentive to defer consumption, but also makes savers better off, leading them to demand higher consumption. An increase in uncertainty about $r_{t+1}$ makes saving less attractive because of aversion to risk, but also makes risk-averse savers worse off, leading to them to demand lower consumption now. This interpretation of the results can be confirmed by solving the expenditure minimization problem for the middle aged subject to achieving a given level of expected continuation utility. The compensated demand for consumption by the middle aged is proportional to $\exp\left(-\delta(1 + \delta)^{-1}E_t \log(1 + r_{t+1})\right)$. This decreases following an increase in $\varrho_t = E_t r_{t+1}$ (holding constant the dispersion of $r_{t+1}$ around $\varrho_t$), and decreases following a mean-preserving spread of $r_{t+1}$ around $\varrho_t$ owing to the strict concavity of the logarithm function.
Lending and house prices move together, with causation going from lending to house prices. This reflects a key feature of the model: changes in credit growth explain changes in house prices.

The constant shares of borrowed funds used to purchase houses and consumption goods are due to logarithmic utility and the dependence of the young on credit to make purchases. Since the supply of houses is inelastic, the marginal utility of housing will be constant in equilibrium (and equal to $\theta$). If there were a rental market, logarithmic utility would ensure that expenditure on rents equals to a fixed multiple of consumption expenditure ($Z_t = \theta P_t C_{m,t}$ according to 2.16).

However, housing services are acquired by buying houses, so there is also the return on housing as an asset to consider. With logarithmic utility, the marginal utility of consumption is inversely proportional to consumption, so the utility cost of buying a unit of housing is $h_t/c_{y,t}$ and the future value of the housing asset is $h_{t+1}/c_{m,t+1}$ (see equation 2.19c). As shown in Step 1, future lending $l_{t+1}$ is proportional to $c_{m,t+1}$ because income and substitution effects cancel out for savers, hence the future value of housing $h_{t+1}/c_{m,t+1}$ in utility units depends only on the future share $h_{t+1}/l_{t+1}$ of credit allocated to house purchases. Since the young are dependent on credit to make purchases ($c_{y,t} + h_t = l_t$), the utility cost of buying a unit of housing $h_t/c_{y,t}$ is determined only by the current share $h_t/l_t$ of credit allocated to house purchases. The Euler equation for housing (2.19c) then implies it is optimal for individuals to allocate a fixed share of borrowing to house purchases irrespective of house prices and interest rates.

This result is consistent with housing values that perfectly co-move with consumption at the times when houses are bought and sold because it means the motive of hedging consumption risk does not affect the demand for housing relative to the demand for consumption goods. However, it is important to remember the analysis so far is conditional on the total amount of borrowing $l_t$ for both house purchases and consumption. Concerns about risk might still affect the total demand for borrowing and thus the demands for housing and consumption. Taking this into account, the next step is to determine the equilibrium interest rate and the rate of return on houses relative to bonds.

**Step 3 (Equilibrium interest rate and risk premium)** Conditional on the probability distribution of future nominal house-price inflation $\pi_{t+1}$, the bond Euler equation of borrowers in (2.19d) and the optimal allocation of loans characterized in Step 2 imply that the equilibrium nominal interest rate is:

$$i_t = \frac{1}{\beta E_t [(1 + \pi_{t+1})^{-1}] - 1}, \quad \text{where } \beta = \delta^2 \kappa \text{ with } 0 < \beta < 1. \quad [2.25]$$

The coefficient $\beta$ is increasing in patience $\delta$ and decreasing in housing preference $\theta$. There is no equilibrium where $\pi_{t+1} = -1$ has positive probability.

The equilibrium ex-post nominal returns on bonds and houses, and the excess expected return of houses over bonds, are:

$$R_t = i_{t-1}, \quad \hat{R}_t = \frac{1 + \pi_t}{\beta} - 1, \quad \text{and } \xi_t = E_t[1 + \pi_{t+1}]E_t[(1 + \pi_{t+1})^{-1}] - 1, \quad [2.26]$$

where these variables are defined in (2.16) and (2.17). The ex-post nominal return on houses is
proportional to house-price inflation \( \pi_t \). The housing risk premium \( \xi_t \) is strictly positive for any non-degenerate probability distribution of house-price inflation \( \pi_{t+1} \) and increases with a mean-preserving spread of \( \pi_{t+1} \); it would be zero if \( \pi_{t+1} \) were perfectly predictable.

**Proof** See appendix A.3.

The results above take as given the path of nominal house-price inflation \( \pi_t \) because the equilibrium inflation rate cannot be determined until monetary policy is specified. Nonetheless, conditional on a particular state-contingent path of nominal house-price inflation, the optimal behaviour of lenders and borrowers in Step 1 and Step 2 and market clearing for bonds and houses determine the equilibrium nominal interest rate and nominal return on housing relative to nominal bonds.\(^8\)

Since nominal bonds are risk free, the nominal return \( R_{t+1} \) between \( t \) and \( t+1 \) is exactly equal to the nominal interest rate \( i_t \). The nominal return on housing \( \hat{R}_{t+1} \) comprises nominal capital gains or losses \( \pi_{t+1} \) and imputed rents (see 2.16). According to (2.26), the return on housing always moves in proportion to house-price inflation, which reflects the tight link between house prices and consumption, and from (2.16), imputed rents. The co-movement of house prices and the consumption of homeowners that comes from Step 1 and Step 2 shows that housing as an asset provides no insurance against consumption risk. On the other hand, the fixed nominal payments on bonds are uncorrelated with the unpredictable changes in nominal house prices and nominal consumption expenditure. This means that taking on nominal mortgage debt to buy houses is doing the opposite of hedging consumption risk.

It follows that the expected return on housing must be greater than the return on bonds for individuals to be willing to borrow money to buy houses. This excess expected return of houses over bonds is the housing risk premium, which can be interpreted as the compensation to borrowers for absorbing house-price risk while promising to make fixed nominal debt repayments. According to (2.26), the risk premium \( \xi_t \) depends only on the probability distribution of nominal house-price inflation \( \pi_{t+1} \), a sufficient statistic for the risk borne by borrowers due to the co-movement of nominal house-prices and the nominal consumption expenditures of homeowners. The risk premium is positive for any non-degenerate distribution of \( \pi_{t+1} \), while in the special case where nominal house-price inflation is perfectly predictable, houses and bonds are equivalent as assets and the risk premium \( \xi_t \) must be zero. As the monetary policy regime affects the probability distribution of nominal house prices, this logic points to a novel portfolio balance channel of monetary policy transmission through risk premiums even in the absence of any asset purchases by the central bank.

Conditional on the probability distribution of \( \pi_{t+1} \), the equilibrium nominal interest rate \( i_t \) is given in (2.25). This can be written in terms of the risk premium \( \xi_t \) from (2.26):

\[
\frac{1 + i_t}{1 + E_t \pi_{t+1}} = \frac{1}{\beta(1 + \xi_t)}.
\]

\(^8\)Although equivalent, it is analytically convenient to think of house-price inflation rather than goods-price inflation as the nominal variable that will ultimately be determined by monetary policy. This is because it is more difficult to characterize the set of state-contingent paths of goods-price inflation such that there is no involuntary default on nominal debt, which can affect the existence of an equilibrium when markets are incomplete. In terms of house-price inflation, ruling out non-existence of an equilibrium requires only that \( \pi_t > -1 \) with probability one.
Given expected nominal house-price inflation $E_t \pi_{t+1}$, the equilibrium nominal interest rate is lower when the housing risk premium $\xi_t$ is higher. The equilibrium interest rate also depends on the parameters for patience $\delta$ and housing preference $\theta$ through the term $\beta$ defined in (2.25). This term plays a role analogous to the subjective discount factor in a representative-agent model of interest-rate determination. All else equal, the equilibrium interest rate is increasing in housing preference $\theta$ (greater demand for mortgages) and decreasing in patience $\delta$ (less demand for borrowing).

The next step is to determine the total volume of lending $l_t$ in equilibrium and the relative price of housing in terms of consumption $V_t/P_t$, which is equivalent to determining the ratio $h_t = V_t H_t / P_t Y_t$ given that $H_t$ and $Y_t$ are exogenous. With an inelastic supply of houses, Step 2 has already shown that the amount of lending affects the relative price of housing. There is also an effect from house prices to the volume of credit that works through the net worth of lenders. The feedback loop arising from the interaction of the two effects is the ‘financial accelerator’ (Bernanke and Gertler, 1989).

The lenders in the economy are the middle aged, and their behaviour was analysed in Step 1. Substituting (2.21) into the so-far-unused budget identity of the middle aged from (2.19b):

$$ l_t = \frac{\delta}{1 + \delta} n_t, \text{ where } n_t = 1 + (1 - d_t) h_t, \quad [2.28] $$

which shows that lending $l_t$ is a fixed multiple of lenders’ net worth $n_t$ (all relative to GDP). Net worth is given by the right-hand side of the budget identity of the middle aged, and this depends on the value of houses $h_t$ and the burden of debt $d_t$ relative to house values. Using the equations in (2.28) together with $h_t = (1 - \kappa) l_t$ from (2.22), the equilibrium level of net worth $n_t$ resulting from the financial accelerator feedback loop is the solution of the equation:

$$ n_t = 1 + \frac{\delta(1 - \kappa)}{1 + \delta} (1 - d_t) n_t, \quad \text{because } h_t = \frac{\delta(1 - \kappa)}{1 + \delta} n_t. \quad [2.29] $$

The solution for $n_t$ depends on $d_t$, which is the ratio of debt to assets on lenders’ balance sheets.

Since loans are proportional to housing values when they are taken out (see 2.24), the ex-post burden of debt repayment (principal and interest) relative to the value of houses varies only with the interest rate $i_{t-1}$ and the realized rate of house-price inflation $\pi_t$. Using the definitions in (2.19a), the ex-post real return on bonds from (2.19e), and the constant ratio $\lambda$ from (2.24):

$$ d_t = \lambda \frac{1 + i_{t-1}}{1 + \pi_t}. \quad [2.30] $$

---

9The negative equilibrium relationship between interest rates and house-price risk can also be understood in terms of income and substitution effects. Unlike savers, income and substitution effects are reinforcing for borrowers in respect of both risk and interest rates. An increase in house-price risk makes buying housing with a mortgage less attractive and reduces the demand for housing. It also makes risk-averse borrowers worse off, which further reduces the demand for housing. With less demand for housing and mortgage finance, equilibrium needs to be restored by an adjustment of interest rates. Lower interest rates encourage borrowing and make borrowers better off, with both effects increasing the demand for loans and restoring equilibrium.

10The overlapping generations structure and the incompleteness of markets naturally imply that lending can only be done by the middle aged, that their net worth is a constraint on lending, and that their balance sheets are exposed to house prices. It is not necessary to introduce a separate exogenous group of ‘bankers’ into the model to capture a financial accelerator effect working through lenders’ balance sheets.
The ratio \( d_t \) of debt to assets is linearly related to the following measure of financial conditions \( \nu_t \):

\[
\nu_t = 1 - \beta \frac{1 + i_{t-1}}{1 + \pi_t} \text{ satisfying } -\infty \leq \nu_t \leq 1, \quad \text{with } d_t = \frac{\lambda}{\beta} (1 - \nu_t). \tag{2.31}
\]

The ratio \( d_t \) is decreasing in \( \nu_t \), which depends positively on house-price inflation \( \pi_t \) and negatively on the nominal interest rate \( i_{t-1} \). The measure of financial conditions \( \nu_t \) provides a sufficient statistic for the state of balance sheets. It is appropriate to measure financial conditions in terms of nominal variables because borrowing takes the form of nominal debt.

Using (2.29) and (2.31), the functional relationship \( n_t = A(\nu_t) \) between net worth \( n_t \) and financial conditions \( \nu_t \) is implicitly defined by:

\[
A(\nu) = 1 + \frac{\delta (1 - \kappa)}{1 + \delta} \left( 1 - \frac{\lambda}{\beta} + \frac{\lambda}{\beta} \nu \right) A(\nu), \quad \text{and thus } A'(\nu) > 0 \tag{2.32}
\]

Net worth \( A(\nu) \) appears on both sides of the equation because it affects lending, which affects the relative price of housing, which in turn affects net worth. The coefficient \( \delta (1 - \kappa)/(1 + \delta) \) is the product of \( \delta/(1 + \delta) \), the sensitivity of total lending to net worth, and \( 1 - \kappa \), the response of mortgage lending to a change in total lending. This coefficient lies between 0 and 1 because \( 0 < \delta < \infty \) and \( 0 < \kappa < 1 \). As the measure of financial conditions satisfies \( \nu \leq 1 \), the equilibrium net worth that solves (2.32) must always be positive. Given the presence of debt, this finding seems surprising. But without positive net worth, there would be no mortgage lending, which would lead to a collapse in the relative price of housing and thus raise goods-price inflation for a given level of house-price inflation, reducing the real value of the debt that appears in the calculation of net worth.

Financial conditions \( \nu \) measure the value of assets relative to liabilities on lenders’ balance sheets, so intuitively, net worth should be increasing in \( \nu \). This variable appears positively on the right-hand side of (2.32), and the claim can be confirmed by differentiating both sides of the equation:

\[
A'(\nu) = \frac{\delta (1 - \kappa)}{1 + \delta} \left( 1 - \frac{\lambda}{\beta} + \frac{\lambda}{\beta} \nu \right) A'(\nu) + \frac{\delta (1 - \kappa) \lambda}{1 + \delta} \beta A(\nu), \quad \text{and thus } A'(\nu) > 0 \tag{2.33}
\]

given that \( A(\nu) > 0 \). The strength of the financial accelerator is measured by the semi-elasticity of equilibrium net worth to financial conditions \( \nu \), which evaluated at \( \nu = 0 \) is denoted by \( \alpha \):

\[
\alpha = \frac{A'(0)}{A(0)}, \quad \text{and given by } \alpha = \frac{\frac{\delta (1 - \kappa) \lambda}{1 + \delta} \beta}{1 - \frac{\delta (1 - \kappa)}{1 + \delta} + \frac{\delta (1 - \kappa) \lambda}{1 + \delta}}, \tag{2.34}
\]

the expression for \( \alpha \) being obtained from (2.33). It is intuitive that the semi-elasticity \( \alpha \) is increasing in \( \delta (1 - \kappa)/(1 + \delta) \), the sensitivity of mortgage lending to net worth (see 2.29), and in \( \lambda/\beta \), the debt-to-assets ratio at financial conditions \( \nu = 0 \) (see 2.31).\(^\text{11}\) The effects of the financial accelerator in equilibrium are stated below.

\(^{11}\)Using Step 2, the first of these terms is increasing in patience \( \delta \) and housing preference \( \theta \), but those parameters have an ambiguous effect on the second term (see equations 2.24 and 2.25). Hence, the overall effects of the parameters \( \delta \) and \( \theta \) on the semi-elasticity \( \alpha \) are not immediately clear. See the equivalent expression for \( \alpha \) in (2.35) below.
Step 4 (Financial accelerator) The budget identity of the middle aged in (2.19b) (or equivalently, the goods-market clearing condition 2.18) and the behaviour of savers and borrowers in Step 1 and Step 2 imply that total lending $l_t$ and the value of houses $h_t$ relative to GDP are the following increasing functions of the measure of financial conditions $\nu_t$ defined in (2.31):

$$l_t = \frac{\alpha \beta}{1 - \alpha \nu_t}, \quad \text{and} \quad h_t = \frac{\alpha \beta}{\lambda (1 - \alpha \nu_t)},$$

where $\alpha = \frac{1}{1 + \delta \kappa + \delta^2 \kappa^2}$ with $\frac{1}{3} < \alpha < 1$. [2.35]

The resulting level of net worth $n_t$ (from 2.28) is non-negative for all $\nu_t$, confirming the no-default condition (2.20) holds. The results imply goods-price inflation $\varpi_t$ is related in equilibrium to house-price inflation $\pi_t$, the nominal interest rate $i_t$, and the real GDP growth rate $g_t$ as follows:

$$\varpi_t = \left( (1 - \alpha)(1 + \pi_t) + \alpha \beta (1 + i_t) \right) (1 + \pi_{t-1}) - 1.$$  \[2.36\]

The coefficient $\alpha$ is as defined by (2.34), with an equivalent expression given above in (2.35). It is increasing in housing preference $\theta$ and the distance of the discount factor $\delta$ from 1.

**Proof** See appendix A.4.

With net worth determining the relative price of housing and the amount of lending relative to the size of the economy, the fact that net worth is increasing in financial conditions $\nu_t$ implies that $h_t$ and $l_t$ are also increasing in $\nu_t$, as seen in (2.35). The semi-elasticity $\alpha$ is also the semi-elasticity (at $\nu_t = 0$) of both lending and the relative price of housing with respect to financial conditions, and this semi-elasticity lies somewhere between $1/3$ and 1 depending on the parameters $\delta$ and $\theta$.

The link between financial conditions and the relative price of housing in (2.35) can also be stated as the relationship between goods-price inflation $\varpi_t$, house-price inflation $\pi_t$, and the nominal interest rate $i_{t-1}$ in equation (2.36). Given the past values of these variables and real GDP growth $g_t$, a 1% increase in house-price inflation $\pi_t$ with no change in the nominal interest rate $i_{t-1}$ raises goods-price inflation $\varpi_t$ by a smaller amount $(1 - \alpha)\%$, reflecting the rise in the relative price of housing when asset-price inflation improves lenders’ net worth. A 1% fall in the nominal interest rate $i_{t-1}$ similarly leads to a rise in the relative price of housing, so for a given realization of nominal house-price inflation, goods-price inflation $\varpi_t$ will fall by only $\alpha \beta\%$.

Finding the equilibrium real returns on the assets in the economy and the resulting consumption allocation requires putting together all the findings from Step 1 through to Step 4. Using the equilibrium nominal interest rate conditional on the path of nominal house-price inflation from (2.25), the measure of financial conditions $\nu_t$ defined in (2.31) must satisfy:

$$\nu_t = \frac{E_{t-1}[(1 + \pi_t)^{-1}] - (1 + \pi_t)^{-1}}{E_{t-1}[(1 + \pi_t)^{-1}]} \text{, and therefore } E_{t-1}\nu_t = 0.$$  \[2.37\]

With rational expectations and competitive equilibrium, financial conditions $\nu_t$ must be a martingale difference sequence ($E_{t-1}\nu_t = 0$), which implies $\nu_t$ has mean zero and is serially uncorrelated. Equation (2.37) shows that financial conditions are entirely determined by the stochastic process for
nominal house-price inflation, and as will be seen, the real equilibrium of the economy is affected by inflation only through the variable $\nu_t$. Financial conditions cannot be determined until monetary policy is specified, but conditional on $\nu_t$, the real equilibrium of the economy is given below.

**Result 1 (Equilibrium conditional on financial conditions)** Conditional on the path of nominal house-price inflation $\pi_t$, Step 3 implies financial conditions $\nu_t$ are determined by (2.37). Given the implied probability distribution of $\nu_t$ with $E_{t-1}\nu_t = 0$, there exists an equilibrium (which is unique) for the real variables of the economy if and only if $-\infty < \nu_t \leq 1$ with probability one.

Given the definition of $\nu_t$ in (2.31) and the financial accelerator effects of Step 4, the realized real returns on bonds $r_{t+1}$ and housing $\hat{r}_{t+1}$, and the housing risk premium $\xi_t$ are:

$$ \frac{1 + r_{t+1}}{1 + g_{t+1}} = \frac{(1 - \alpha \nu_t)(1 - \nu_{t+1})}{\beta (1 - \alpha \nu_{t+1})}, \quad \frac{1 + \hat{r}_{t+1}}{1 + g_{t+1}} = \frac{(1 - \alpha \nu_t)}{\beta (1 - \alpha \nu_{t+1})}, \quad \text{and} \quad \xi_t = E_t \left[ \frac{\nu_{t+1}}{1 - \nu_{t+1}} \right]. \quad [2.38] $$

The risk premium $\xi_t$ is positive whenever financial conditions $\nu_{t+1}$ are not perfectly predictable, and increases with any spread of $\nu_{t+1}$ around its mean $E_t\nu_{t+1} = 0$. A spread of $\nu_{t+1}$ decreases the expected real return on bonds $E_t[(1 + r_{t+1})/(1 + g_{t+1})]$ and increases the expected real return on housing $E_t[(1 + \hat{r}_{t+1})/(1 + g_{t+1})]$. Whenever financial conditions are not perfectly predictable:

$$ E \left[ \frac{1 + r_{t+1}}{1 + g_{t+1}} \right] < \frac{1}{\beta}, \quad \text{and} \quad E \left[ \frac{1 + \hat{r}_{t+1}}{1 + g_{t+1}} \right] > \frac{1}{\beta}, \quad [2.39] $$

which would both hold with equality were there no uncertainty about financial conditions. The expected real returns on bonds $\varrho_t$ and housing $\hat{\varrho}_t$ are always related as follows:

$$ \alpha E \left[ \frac{1 + \varrho_t}{1 + E_t g_{t+1}} \right] + (1 - \alpha) E \left[ \frac{1 + \hat{\varrho}_t}{1 + E_t g_{t+1}} \right] = \frac{1}{\beta}, \quad [2.40] $$

and similarly, the two expectations in (2.39) weighted by $\alpha$ and $1 - \alpha$ always sum to $1/\beta$.

These results together with the behaviour of savers and borrowers in Step 1 and Step 2 imply that the equilibrium consumption allocation is:

$$ c_{y,t} = \frac{\alpha \delta^2 \kappa^2}{1 - \alpha \nu_t}, \quad c_{m,t} = \frac{\alpha \delta \kappa}{1 - \alpha \nu_t}, \quad \text{and} \quad c_{o,t} = \frac{\alpha (1 - \nu_t)}{1 - \alpha \nu_t}. \quad [2.41] $$

If financial conditions $\nu_t$ are not perfectly predictable then:

$$ E c_{y,t} > c^*_y, \quad E c_{m,t} > c^*_m, \quad E c_{o,t} < c^*_o, \quad \text{where} \quad c^*_y = \alpha \delta^2 \kappa^2, \quad c^*_m = \alpha \delta \kappa, \quad c^*_o = \alpha, \quad [2.42] $$

with $c^*_y$, $c^*_m$, and $c^*_o$ denoting the consumption allocation when there is no uncertainty about financial conditions. A spread of $\nu_t$ increases $E c_{y,t}$ and $E c_{m,t}$ and decreases $E c_{o,t}$.

**Proof** See appendix A.5.

Before discussing the findings in Result 1, the question of the determination of financial conditions $\nu_t$ must be addressed. Financial conditions are related to the state-contingent path of nominal house-
price inflation according to (2.37), or more precisely, the unpredictable component of inflation:

\[
\pi_t = \mathbb{E}_{t-1} \pi_t + (1 + \mathbb{E}_{t-1} \pi_t) \frac{\nu_t}{1 - \nu_t} - \mathbb{E}_{t-1} \left[ \frac{\nu_t}{1 - \nu_t} \right].
\]  \hspace{1cm} [2.43]

This equation is derived from (2.37) and shows that specifying the full state-contingent path of nominal house-price inflation \( \pi_t \) is equivalent to specifying the predictable component of inflation \( \mathbb{E}_{t-1} \pi_t \) and financial conditions \( \nu_t \) satisfying \( \mathbb{E}_{t-1} \nu_t = 0 \). The equilibrium conditions (2.19a)–(2.19e) therefore cannot determine financial conditions without adding a description of monetary policy.

Monetary policy is implemented using any interest-rate feedback rule. One example is:

\[
i_t = \frac{\frac{1 + \pi_t^*}{1 + \pi_t^* + \mu_t}}{\beta E_t [(1 + \mu + 1)^{-1}] - 1},
\]  \hspace{1cm} [2.44]

where \( \mu_t \) is an exogenous martingale difference sequence \( (E_{t-1} \mu_t = 0) \) that depends only on the exogenous variables \( g_t \) and \( \psi_t \), and \( \pi_t^* \) is an exogenous predictable sequence \( (\pi_t^* = E_{t-1} \pi_t^* \text{ with probability one}) \). This interest-rate rule has \( i_t \) react positively to deviations of house-price inflation \( \pi_t \) from \( \pi_t^* \), with the coefficient \( \zeta \) measuring the size of the response. The interest-rate rule allows for the stance of monetary policy to respond to changes in the exogenous variables \( g_t \) and \( \psi_t \) through the term \( \mu_t \), with a rise in \( \mu_t \) increasing the nominal interest rate all else equal.

The equilibrium conditions stated earlier include an as-yet-unused equilibrium selection criterion: paths of inflation must satisfy \( \pi_t < \infty \) with probability one, and there must be no positive-probability realizations of \( \pi_t \) converging to \(-1 \) or \( \infty \) as \( t \to \infty \). Subject to this, inflation can be determined.

**Step 5 (Implementation of monetary policy using a feedback rule)** If monetary policy is conducted according to interest-rate rule (2.44) with \( \zeta > 1 \), \( \mu_t \) satisfying \(-1 < \mu_t < \infty \) with probability one and \( E_{t-1} \mu_t = 0 \), and \(-1 < \pi_t^* = E_{t-1} \pi_t^* < \infty \) with probability one that does not converge to \(-1 \) or \( \infty \), then there exists a unique equilibrium with financial conditions \( \nu_t = -\mu_t \), expected inflation \( E_{t-1} \pi_t = \pi_t^* \), and the full state-contingent path of inflation \( \pi_t \) given by (2.43).

Any equilibrium that could be implemented using some other interest-rate feedback rule can also be implemented as the unique equilibrium using the feedback rule (2.44) with any \( \zeta > 1 \) for some exogenous \( \pi_t^* \) and \( \mu_t \) satisfying \(-1 < \mu_t < \infty \).

**Proof** See appendix A.6.

At first glance, the interest-rate feedback rule (2.44) seems restrictive in that the interest rate reacts to only one endogenous variable \( \pi_t \). However, Step 5 demonstrates that (2.44) is sufficiently flexible in the sense that any equilibrium that could be implemented with an alternative interest-rate rule (for example, a conventional Taylor rule) can also be implemented using (2.44) for some choice of the exogenous \( \mu_t \) and \( \pi_t^* \). This means that monetary policies such as a strict goods-price inflation target could be implemented using a \( \mu_t \) and \( \pi_t^* \) that depend on the economy’s exogenous fundamentals (real
GDP growth $g_t$). The particular form of the interest-rate feedback rule only matters for questions of the determinacy of the equilibrium, not what can actually be achieved using monetary policy. For (2.44), the equilibrium is determinate if the coefficient $\zeta$ satisfies $\zeta > 1$, analogous to the usual ‘Taylor principle’. In what follows, none of the results makes use of the functional form (2.44).\footnote{Although (2.44) has an unusual form and is specified in terms of house-price inflation $\pi_t$, it is actually the simplest feedback rule when deriving an explicit solution of the non-linear equations describing the equilibrium of the economy.}

Different monetary policy strategies have different implications for the stochastic process of financial conditions $\nu_t$. Using the findings in Step 5, monetary policy can effectively choose from among all mappings from the exogenous variables to financial conditions $\nu_t$ where the probability distribution of $\nu_t$ satisfies:

$$E_{t-1} \nu_t = 0, \quad \text{and} \quad -\infty < \nu_t < 1 \text{ with probability 1.} \tag{2.45}$$

The actual realizations of $\nu_t$ will be linked to realizations of the exogenous fundamentals of the economy (real GDP growth $g_t$) or even exogenous ‘sunspots’ (the stochastic process $\psi_t$), but the way in which $\nu_t$ responds to the exogenous variables can be controlled by monetary policy subject to (2.45). As shown in Result 1, the real equilibrium of the economy depends on financial conditions $\nu_t$, and it is through financial conditions that monetary policy has its real effects. Monetary policy can also choose the predictable component of inflation $\pi_t^* = E_{t-1} \pi_t$, but since that choice has no real implications, the focus is on $\nu_t$ in what follows.

Consider the effects of a particular realization of financial conditions $\nu_t$ from a given probability distribution. A positive value of $\nu_t$ means better financial conditions than expected, which from (2.31) corresponds to higher nominal house-price inflation $\pi_t$ relative to borrowing costs $i_{t-1}$. Nominal goods-price inflation $\varpi_t$ will also increase according to (2.36), but by less than house-price inflation, reflecting the rise in the relative price of housing. This explains why in equation (2.38), the real return on housing $\hat{r}_t$ will increase with $\nu_t$, but the real return on bonds $r_t$ will decline. These changes in asset returns affect the consumption levels of each generation relative to GDP, the equilibrium values of which are given in (2.41). The consumption of the middle aged is positively related to financial conditions $\nu_t$, which works directly through an improvement of their balance sheets. The positive effect on the consumption of the young is indirect and works through the expansion of lending. The consumption of the old is negatively related to $\nu_t$ and works through the erosion of nominal pension wealth by goods-price inflation.

More interestingly, Result 1 shows that monetary policy actually has permanent real effects on the economy in the sense that the monetary policy regime affects the average real returns on bonds and houses, and consequently the average levels of consumption of different age groups. Formally, different monetary policy regimes imply different probability distributions of financial conditions $\nu_t$, and this has implications for the average real values of asset returns and consumption. The explanation for these permanent real effects is found in the impact of the monetary policy regime on the expected return on nominal assets relative to real assets, and in the use of nominal debt by borrowers owing to the incompleteness of markets.
First consider the mechanism through which monetary policy affects risk premiums. The ex-post nominal return on a risk-free one-period nominal bond is by definition perfectly predictable at the time the bond is issued. On the other hand, the ex-post nominal return on a real asset depends on how the nominal price of the asset changes in the future. Since monetary policy can in principle always set one nominal price of something real, different monetary policy regimes will lead to different probability distributions of the future nominal value of the real asset. Standard asset-pricing theory then implies that this will generally change the equilibrium relative returns on the two assets because monetary policy affects their relative riskiness.\footnote{An example of this in a different context is inflation targeting reducing the risk premium on nominal bonds relative to inflation-indexed bonds (a real asset).} The logic is related to the portfolio balance channel because the relative asset return must adjust to ensure that financial markets are in equilibrium after a change in the monetary policy regime, but the central bank does not need to make asset purchases for this channel to be operative. Note that the mechanism only allows monetary policy to affect the return on nominal assets compared to real assets, not the relative returns on two real assets.

In the model here, the real asset is housing and the nominal asset is mortgage debt and other nominal debt contracts. The expected return on housing relative to nominal bonds is the risk premium $\xi_t$ from (2.26). This depends on the probability distribution of nominal house-price inflation $\pi_{t+1}$, which is different under different monetary policy regimes. The general argument that monetary policy can affect the relative returns on nominal and real assets does not depend on market incompleteness. In a complete-markets version of the model presented in section 5.1, it turns out that the formula for the risk premium $\xi_t$ in terms of the probability distribution of nominal house-price inflation is exactly the same as (2.26). But if markets were complete, this risk premium would have no real consequences because borrowers do not need to issue nominal debt. The risk premium matters for real variables when market incompleteness restricts borrowers to nominal debt contracts because it affects the real borrowing cost they face.

Result 1 shows that a high expected excess return $\xi_t$ on housing over nominal bonds implies a lower expected real return on nominal bonds and a higher expected real return on housing (see 2.39), where real return refers to the returns on these assets measured in terms of consumption goods. To understand the intuition, observe from equation (2.27) that conditional on a given average rate of house-price inflation, an increase in the risk premium lowers the nominal interest rate. With incomplete markets and nominal debt contracts, there is a financial accelerator effect analysed in Step 4. A lower nominal interest rate raises the relative price of housing, so goods-price inflation falls by less than the nominal interest rate (see 2.36). Consequently, the real return on nominal bonds is lower.\footnote{Recall that the model has fully flexible prices, so there is no role for the usual New Keynesian transmission mechanism of monetary policy to real interest rates.}

The higher expected real return on housing is due to the convexity of the real housing return in financial conditions $\nu_t$. With credit determining real house prices, and the supply of credit determined by lenders’ net worth, the convexity of the housing return comes from the convexity of net worth in financial conditions. The source of this convexity can be seen mathematically by
differentiating both sides of (2.33):

$$A''(\nu) = \frac{\delta(1 - \kappa)}{1 + \delta} \left(1 - \frac{\lambda}{\bar{\beta}} + \frac{\lambda}{\bar{\beta}} \nu\right) A''(\nu) + 2 \frac{\delta(1 - \kappa)}{1 + \delta} \frac{\lambda}{\bar{\beta}} A'(\nu), \quad \text{and thus } A''(\nu) > 0$$

given that $A'(\nu) > 0$. The convexity therefore comes from the relative price of housing being an increasing function of financial conditions $\nu$ (compare equations 2.29 and 2.32). Intuitively, when house-price inflation is unexpectedly high, nominal net worth increases because nominal debt is predetermined. This improvement in net worth pushes up the relative price of housing, so in real terms, net worth rises by even more. On the other hand, an unexpected fall in nominal house-price inflation lowers nominal net worth, but also pushes down the relative price of housing. Net worth thus declines by less in real terms than in nominal terms.

It follows that a monetary policy with a high housing risk premium will imply cheap real mortgage rates and high expected real capital gains on housing. On average, this is good for those who are borrowing to buy houses, so the average consumption of the young and middle aged will be higher. On the other hand, low real interest rates are bad for pensions, so the average consumption of the old will be lower. This is what lies behind the results in (2.42).

A consequence of these findings is that there is no ‘natural real rate of interest’, in the sense of an average real return on nominal bonds that is independent of monetary policy.\textsuperscript{15} Such a natural rate would exist if all the bonds issued by borrowers were indexed to inflation. However, when debt contracts are written in nominal terms, monetary policy can have a permanent effect on risk premiums, and therefore on the average real return on bonds.

While monetary policy is able to have permanent real effects on interest rates and housing returns, this does not mean that monetary policy can raise or lower the real returns on all assets. Equation (2.40) shows that a weighted average of the real returns on nominal bonds and housing is independent of monetary policy, being determined only by the economy’s expected real growth rate $E_t g_{t+1}$ and parameters through the term $\bar{\beta}$. This ‘natural real return’ across all assets in the economy is still determined purely by real factors. Therefore, greater real gains received by homeowners as a result of monetary policy must come at the expense of lower real returns for pensioners.

\section{Monetary policy and financial crises}

Having set up the model of the economy, this section develops the key claim that the interaction between incomplete markets and the political economy of monetary policy creates the conditions for financial crises to occur in equilibrium.

\textsuperscript{15}The model here has fully flexible prices and wages, so the actual real interest rate corresponds to what is defined as the natural rate of interest in models with sticky prices or wages.
3.1 The political economy of monetary policy

It is important to consider how monetary policy through its impact on interest rates and the prices of assets and goods affects the welfare of different individuals in the economy. Continuation expected utilities $U_y,t$, $U_m,t$, and $U_o,t$ of the date-$t$ cohorts of young, middle-aged, and old individuals are given below with reference to the lifetime utility function in (2.2):

$$U_y,t = U_t, \quad U_m,t = \log C_{m,t} + \Theta(H_m,t) + \delta E_t \log C_{o,t+1}, \quad \text{and} \quad U_o,t = \log C_{o,t},$$

where the continuation utility of the young is simply equal to the lifetime expected utility function $U_t$ from (2.2). These continuation utilities give current and expected future payoffs conditional on having reached a particular stage of life.

Result 2 (Political economy of monetary policy) The expected continuation utilities in (3.1) can be written in terms of two expected payoffs $U_{b,t}$ and $U_{s,t}$:

$$E_{t-1}U_y,t = U_{b,t} + \delta E_t U_{b,t+1} + \delta^2 E_t U_{s,t+2} + \text{t.i.p.}, \quad E_{t-1}U_m,t = U_{b,t} + \delta E_t U_{s,t+1} + \text{t.i.p.}, \quad \text{and} \quad E_{t-1}U_o,t = U_{s,t} + \text{t.i.p.},$$

where t.i.p. denotes terms independent of monetary policy. The expected payoffs $U_{b,t}$ and $U_{s,t}$ are:

$$U_{b,t} = -E_{t-1} \log(1 - \alpha \nu_t), \quad \text{and} \quad U_{s,t} = E_{t-1} [\log(1 - \nu_t) - \log(1 - \alpha \nu_t)],$$

so the payoffs $U_{b,t}$ and $U_{s,t}$ depend only on the probability distribution of the financial conditions variable $\nu_t$ defined in (2.31) with conditional mean $E_{t-1} \nu_t = 0$. If financial conditions are predictable ($\nu_t = 0$ with probability one) then $U_{b,t} = U_{s,t} = 0$. For any non-degenerate probability distribution of financial conditions, $U_{b,t} > 0 > U_{s,t}$ because $U_{b,t}$ is the expectation of a convex function of $\nu_t$, while $U_{s,t}$ is the expectation of a concave function of $\nu_t$. A spread of $\nu_t$ around its mean $E_{t-1} \nu_t = 0$ increases $U_{b,t}$ and decreases $U_{s,t}$.

Proof See appendix A.7.

It was shown in Result 1 that the ratios of consumption to GDP for each age group are functions of the path of financial conditions $\nu_t$ and parameters (see 2.41). A consequence of this together with (2.3), (2.4), (2.11), and (2.12) is that the continuation utilities (3.1) depend on monetary policy only through the probability distribution of financial conditions $\nu_t$. Recall from (2.31) that financial conditions $\nu_t$ are positively related to nominal house-price inflation $\pi_t$ and negatively related to the nominal cost of borrowing $i_{t-1}$. In equilibrium, $\nu_t$ is determined entirely by the probability distribution of inflation (see 2.37).

First consider the old, for whom the welfare effects of monetary policy are straightforward. The consumption of the old is paid for from their pension savings, which are accumulated during middle age. Unpredictability in financial conditions $\nu_t$ is caused by unpredictability in house-price inflation $\pi_t$, and with the link between house-price inflation $\pi_t$ and goods-price inflation $\omega_t$ in (2.36),
unpredictability in $\pi_t$ means uncertainty about the real return (measured in units of goods) from investing in nominal bonds. Given that the pensions of the old are held as nominal bonds, risk aversion means that the old dislike unpredictability in financial conditions.

Furthermore, as established in Result 1, the housing risk premium that arises owing to uncertainty about financial conditions has the general-equilibrium effect of lowering the expected real return on nominal bonds. With both lower expected real returns and greater risk, old savers are unambiguously worse off in terms of expected utility. Note that there is no inconsistency between lower expected real returns for savers and greater risk given the analysis in Step 1: income and substitution effects of both interest rates and risk cancel out for savers.

Thus, as shown in (3.2) and (3.3), the part of the expected utility of the old that is influenced by monetary policy is the expectation of a strictly concave function of financial conditions $\nu_t$. This is the variable $U_{s,t}$, representing the interests of savers. Savers are made strictly worse off by an increase in uncertainty about financial conditions as measured by a mean-preserving spread of $\nu_t$.

Now consider the effects of monetary policy on the welfare of borrowers. As discussed earlier, when houses are purchased using nominal mortgage debt, borrowers take on risk by committing to fixed repayments while acquiring an asset whose future value is uncertain. For this reason, borrowers dislike uncertainty about financial conditions, all else equal. However, this logic also justifies the existence of a housing risk premium when financial conditions are unpredictable, as seen earlier in Step 3. This risk premium is sufficient to compensate borrowers at the margin for taking on house-price risk. Differently from savers, income and substitution effects are reinforcing for borrowers and hence mortgage interest rates must fall to reflect the risk of a leveraged investment in housing.

Crucially, changes in nominal house prices and borrowing costs also have general-equilibrium effects that arise owing to the interaction between nominal debt and the financial accelerator feedback loop from balance sheets to lending to house prices and back to balance sheets. As discussed earlier, these general-equilibrium effects of changes in financial conditions have an impact on the relative price of housing and goods, and Result 1 shows that increased uncertainty about financial conditions lowers the expected real return on bonds and raises the expected real return on housing. Mathematically, this is reflected in net worth being a convex function of financial conditions.

While the expected real return from a leveraged investment in housing is higher when there is greater uncertainty about financial conditions, borrowers still directly dislike the additional risk. However, the financial accelerator feedback loop implies net worth is actually a superconvex function of financial conditions (the logarithm of net worth, a concave transformation, remains a convex function of financial conditions). By dividing both sides of (2.33) by $A(\nu)$ and differentiating:

$$\frac{dA'(\nu)}{d\nu} = \frac{\delta(1 - \kappa)}{1 + \delta} \left( 1 - \frac{\lambda}{\beta} + \frac{\lambda}{\beta} \nu \right) \frac{dA'(\nu)}{d\nu} + \frac{\delta(1 - \kappa)}{1 + \delta} \frac{\lambda A'(\nu)}{\beta A(\nu)},$$

and hence $\frac{d^2 \log A(\nu)}{d\nu^2} > 0$, given that $A'(\nu) > 0$ and $A(\nu) > 0$. The superconvexity stems from the presence of the term $\nu A'(\nu)$ in (2.33), the interaction between the ratio of nominal debt to asset values as measured by financial conditions $\nu$ and the effect of financial conditions on the relative price of housing $A'(\nu)$.  

23
The superconvexity of net worth means that even though borrowers are risk averse, when taking into account the general-equilibrium effects, their expected utility is higher for monetary policies with greater financial risk but lower real interest rates. Formally, (3.2) and (3.3) show that for the middle aged, who are at the point where their mortgages must be repaid, the part of expected utility that depends on monetary policy is the expectation of a strictly convex function of $\nu_t$. This is the variable $U_{b,t}$, representing the interests of borrowers.

It might be wondered why the equilibrium risk premium is effectively overcompensating borrowers for bearing house-price risk. The key point is that no individual internalizes the financial accelerator effects on the real returns on housing and bonds that result from nominal house-price risk. These pecuniary externalities work in the favour of borrowers and imply that borrowers actually lose in equilibrium from monetary policies that would make house prices more stable.

To confirm this interpretation of the results, observe that the equilibrium housing risk premium is determined in Step 3 with reference only to the probability distribution of nominal house prices. The relative price of housing plays no role and is not even determined itself until the full set of market-clearing conditions is imposed later in Step 4. Hypothetically, suppose these equilibrium conditions were altered to make the supply of housing or goods perfectly elastic so that the relative price of houses in terms of goods was always equal to an exogenous constant ($h_t = h$). Using equations (2.28) and (2.31), net worth would be $1 + (1 - (\lambda/\beta) + (\lambda/\beta)\nu)h$, a linear function of financial conditions $\nu$. The consumption of the middle aged is proportional to net worth, hence the utility of borrowers would be the expectation of a strictly concave function of financial conditions. This means that uncertainty about financial conditions would be bad for borrowers in the absence of the financial accelerator effect.

The results in (3.2) show that the variable $U_{b,t}$, which is the policy-sensitive component of the expected utility of the middle aged, is also the component of the ex-ante expected utility of the young that depends on monetary policy. This means that in equilibrium, the interests of the young are exactly aligned with the interests of the current generation of middle-aged individuals. The explanation for this lies in the link between balance sheets and lending that was key in the analysis of the financial accelerator. Combined with the dependence of the young on credit to buy houses and pay for consumption goods, the young therefore have a stake in the financial health of the previous cohort of borrowers, the current middle aged.

To summarize the findings of Result 2, there are two distinct interest groups in respect of monetary policy at any point in time. A group of ‘borrowers’ who gain from low interest rates and high house-price inflation, and a group of ‘savers’ who lose from low interest rates and high house-price inflation. The group of ‘borrowers’ comprises the current young and middle aged, and the group of ‘savers’ comprises the current old generation.

In what follows, rather than start by specifying some arbitrary rule for monetary policy, it is natural to think of monetary policy as being the outcome of some collective choice process reflecting the interests of different individuals in the economy. In other words, monetary policy will maximize a social welfare function, that is, some function of all individuals’ utilities. Considering the economy
at a given date $t_0$, the following additive social welfare function is used:

$$
\mathcal{W}_{t_0} = E_{t_0-1} \left[ N_{a,t_0} \Omega_{t_0-2} \mathcal{U}_{a,t_0} + N_{m,t_0} \Omega_{t_0-1} \mathcal{U}_{m,t_0} + \sum_{t=t_0}^{\infty} N_{y,t} \Omega_{t} \mathcal{U}_{y,t} \right], \text{ with } \Omega_{t} = E_{t_{0}-1}\Omega_{t}, \ [3.4]
$$

where $N_{t,t}$ is the population of age-$\tau$ individuals at date $t$ (see 2.1), $\Omega_{t}$ denotes the positive weight assigned to each individual born at date $t \geq t_0 - 2$, and $\mathcal{U}_{t,t}$ is the continuation utility of age-$\tau$ individuals (see 3.1). Social welfare is calculated using ex-ante expectations of utility, hence the presence of the conditional expectation dated $t_0 - 1$. The weights $\Omega_{t}$ must be non-stochastic ($\Omega_{t} = E_{t_{0}-1}\Omega_{t}$ for all $t \geq t_0 - 2$) to ensure that people are treated as autonomous individuals for the period of time covered by the social welfare function (3.4).

Monetary policy has only a single instrument, the nominal interest rate $i_{t}$. Since the central bank does not have access to a complete set of policy instruments including lump-sum taxes and transfers, monetary policy is the solution of a Ramsey problem. That is, the central bank maximizes social welfare function $\mathcal{W}_{t_0}$ from (3.4) subject to the utility-maximizing behaviour of individuals and market-clearing conditions, not only resource constraints. The Ramsey problem is:

$$
\sup \mathcal{W}_{t_0} \text{ subject to (2.19a)--(2.19e),} \ [3.5]
$$

for a given sequence of weights $\Omega_{t} > 0$, and where sup refers to the supremum of $\mathcal{W}_{t_0}$ from (3.4).

**Step 6 (Ramsey problem)** The Ramsey problem stated in (3.5) has the following features:

(i) An allocation can be implemented by monetary policy subject to the constraints (2.19a)--(2.19e) if and only if it is given by (2.41) for some sequence $\nu_{t}$ satisfying (2.45).

(ii) A sufficient condition for existence of the constrained supremum of $\mathcal{W}_{t_0}$ is $\sum_{t=t_0}^{\infty} N_{y,t} \Omega_{t} < \infty$ and $\sum_{t=t_0}^{\infty} tN_{y,t} \Omega_{t} < \infty$, with convergence of the first series being a necessary condition.

(iii) The social welfare function (3.4) can be written as:

$$
\mathcal{W}_{t_0} = \sum_{t=t_0}^{\infty} \Delta_{t} E_{t_{0}-1} W_{t} + \text{t.i.p.}, \text{ where } W_{t} = (1 - \omega_{t}) U_{b,t} + \omega_{t} U_{s,t}. \ [3.6a]
$$

The term $W_{t}$ is a weighted average of the date-$t$ borrower and saver expected utilities $U_{b,t}$ and $U_{s,t}$ from (3.3), where the weight $\omega_{t}$ on savers ($0 < \omega_{t} < 1$) is given by:

$$
\omega_{t} = \frac{\delta_{\min(2,t-t_{0})} N_{a,t} \Omega_{t-2}}{N_{y,t} \Omega_{t} + \delta_{\min(1,t-t_{0})} N_{m,t} \Omega_{t-1} + \delta_{\min(2,t-t_{0})} N_{a,t}}. \ [3.6b]
$$

Social welfare $\mathcal{W}_{t_0}$ is an expected discounted sum of date-$t$ utilities $W_{t}$, with the discount factor applied to $W_{t}$ being $\Delta_{t}/\Delta_{t_{0}}$, where $\Delta_{t} = N_{y,t} \Omega_{t} + \delta_{\min(1,t-t_{0})} N_{y,t-1} \Omega_{t-1} + \delta_{\min(2,t-t_{0})} N_{y,t-2} \Omega_{t-2}$.

(iv) The date-$t$ weighted expected utility $W_{t}$ from (3.6a) depends only on the probability distribu-
tion of financial conditions \( \nu_t \) (with \( 2.45 \) as the only implementability constraint):

\[
W_t = E_{t-1} \left[ \omega_t \log(1 - \nu_t) - \log(1 - \alpha \nu_t) \right]. \tag{3.7}
\]

(v) In finding \( \sup \nu_t \), irrespective of whether \( \{ \nu_t \}_t^{\infty} \) is chosen (full commitment), or only \( \{ \nu_t \}_{t=0}^{t_0+\ell} \) for some \( 0 \leq \ell < \infty \) with \( \{ \nu_t \}_{t=t_0+\ell+1}^{\infty} \) taken as given and assumed to depend on exogenous variables (\( \ell \)-period ahead commitment), a solution for any probability distribution \( \nu_t \) chosen must also be a solution of the simpler problem \( \sup \nu_t W_t \) subject to \( 2.45 \), with \( W_t \) from \( 3.7 \).

**Proof** See appendix A.8.

Result 1 has shown that any competitive equilibrium has a consumption allocation given by \( 2.41 \) for some sequence of probability distributions of financial conditions \( \nu_t \) satisfying \( 2.45 \), and these findings hold under any monetary policy regime. Step 5 further showed that monetary policy can be set to achieve as the unique equilibrium of the economy any specific probability distribution of financial conditions \( \nu_t \) satisfying \( 2.45 \). Consequently, the only ‘implementability constraints’ are \( 2.41 \) and \( 2.45 \) when solving the Ramsey problem.

16 The analysis of the political economy of monetary policy in Result 2 shows that up to terms independent of monetary policy, continuation utilities of all generations are sums of current and future expected utilities for a group of ‘borrowers’ and a group of ‘savers’. Substituting the continuation utilities \( 3.1 \) into the social welfare function \( 3.4 \) and changing the order of summation leads to social welfare in \( 3.6a \) expressed as the sum of a series of terms \( W_t \) (multiplied by \( \Delta_t > 0 \)), where \( W_t \) denotes a weighted average of borrower and saver expected utilities from consumption at date \( t \). The weight \( \omega_t \) attached to the interests of savers is given in \( 3.6b \) and depends on the number of old individuals relative to the number of young and middle aged and the welfare weights of the individuals in these age groups. It is also known that the expected payoffs of borrowers and savers \( U_{b,t} \) and \( U_{s,t} \) from \( 3.3 \) depend only on the probability distribution of financial conditions \( \nu_t \) at date \( t \), so this property is inherited by \( W_t \). An explicit expression for \( W_t \) is given in \( 3.7 \).

The final point in Step 6 analyses the role of commitment in the Ramsey problem, where it is standard to assume full commitment to a state-contingent path of variables from date \( t_0 \) onwards. However, since \( W_t \) depends only on \( \nu_t \) and parameters, and as the only implementability constraint on \( \nu_t \) is \( E_{t-1} \nu_t = 0 \), there is no difference between maximizing \( W_t \) period-by-period and a state-contingent commitment to maximize \( \nu_t \) over a number of future dates or for all future dates, as long as the date-\( t \) relative saver-borrower weight \( \omega_t \) is the same in each case.

17 The second point in Step 6 concerns the conditions on the social welfare weights \( \Omega_t \) such that a solution to the Ramsey problem actually exists. The social welfare function \( 3.4 \) is a sum of utilities for infinitely many generations of individuals all with positive weights. In order for social welfare to be a well-defined criterion for choosing a monetary policy, it is necessary to place restrictions on the behaviour of the weights \( \Omega_t \) for large \( t \) so that the social welfare function remains finite. For the purposes of this paper, these restrictions are not relevant for any of the results.

18 Strictly speaking, in the absence of a perpetual commitment, this argument assumes a ‘Markovian’ property for future monetary policy whereby the probability distribution of future financial conditions depends only on exogenous variables (no endogenous variables are fundamental state variables). In all cases, a weak degree of commitment is still implicitly assumed because the central bank cannot choose ex-post outcomes for \( \nu_t \) that violate \( 2.45 \).
The baseline case in what follows is ‘democratic’ equal weights assigned to all individuals currently alive. Formally, this means the central bank maximizes \( W_t \) period-by-period with no commitment to future policies and with the weights \( \Omega_t = \Omega_{t-1} = \Omega_{t-2} \) used to determine policy at date \( t \) (the weights attached to future generations do not matter for \( \omega_t \); and any geometric series for subsequent \( \Omega_t \) with ratio less than \( \gamma \) satisfies the conditions in Step 6 for social welfare to be bounded). Using (3.6b), the relative weight on savers is given by the following for all \( t \):

\[
\omega_t = \frac{1}{1 + \gamma + \gamma^2}.
\]

[3.8]

With \( \omega_t \) equal to the constant above, monetary policy maximizes the mean welfare of all individuals currently alive, which is a natural starting point for the analysis. It is also possible to consider the case where monetary policy maximizes the welfare of the median individual in the economy, an application of the median voter theorem which has stronger political-economy foundations. This extension is taken up in section 5.

### 3.2 Financial crises

This section shows how the two key ingredients of the model, incomplete markets and political economy, interact to give rise to financial crises. The outcome for the incomplete-markets economy where the central bank has the social welfare function (3.4) as its goal is found by solving the Ramsey problem sup

\[
\nu_t W_t \quad \text{subject to} \quad E_{t-1} \nu_t = 0.
\]

**Result 3 (Financial crises)** If \( \omega_t < \alpha \) then for any \( \bar{\epsilon} > 0 \), there exists an \( 0 < \epsilon < \bar{\epsilon} \) such that the following probability distribution of \( \nu_t \) implies that \( W_t \) is arbitrarily close to sup\( \nu_t W_t \):

\[
\nu_t = \begin{cases} 
\frac{x}{1 + x} & \text{with probability } 1 - \epsilon, \\
\frac{(1 - \epsilon)x}{\epsilon(1 + x)} & \text{with probability } \epsilon,
\end{cases}
\]

[3.9]

and where \( x \) is equal to \( x_t = (\alpha - \omega_t)/(1 - \alpha) \omega_t \) at date \( t \). There are no restrictions on the correspondence between the realizations of the exogenous variables \( g_t \) and \( \psi_t \) and the two states of the distribution of \( \nu_t \) except consistency with the probabilities of the states in (3.9). With democratic weights (3.8), the condition \( \omega_t < \alpha \) is equivalent to \( \gamma > \delta \kappa \) where \( \delta \kappa < 1 \).

The distribution (3.9) corresponds to two realizations of nominal house-price inflation \( \pi_t \) above and below its expected value \( E_{t-1} \pi_t \). The high realization of \( \pi_t \) implies that \( h_t = \bar{h} \) and \( l_t = \bar{l} \) where for small \( \epsilon \):

\[
\bar{h} = \frac{\alpha \beta (1 + x)}{\lambda (1 + (1 - \alpha) x)}, \quad \text{and} \quad \bar{l} = \frac{\alpha \beta (1 + x)}{1 + (1 - \alpha) x},
\]

[3.10]

and the low realization of \( \pi_t \) implies \( h_t = \underline{h} \) and \( l_t = \underline{l} \), where \( \underline{h} \) and \( \underline{l} \) become small as \( \epsilon \) falls to zero.
The equilibrium nominal interest rate is:

\[ i_{t-1} = \frac{1 + E_{t-1}\pi_t}{\beta(1+x_t)} - 1. \]  

[3.11]

Given an arbitrary initial monetary policy where financial conditions \( \nu_t \) has a non-degenerate probability distribution at some date \( t \), there exists an alternative monetary policy at date \( t \) that makes all individuals weakly better off (and some strictly better off) and implies a probability distribution of \( \nu_t \) of the form (3.9) with a lower housing risk premium \( \xi_{t-1} \).

**Proof** See appendix A.9.

The findings above characterize the probability distribution of financial conditions \( \nu_t \) and hence the behaviour of the unpredictable component of nominal house-price inflation \( \pi_t \) that results when monetary policy aims to maximize the social welfare function. The analysis applies to the case where the relative weight on savers \( \omega_t \) in the social welfare function is less than \( \alpha \), the constant given in (2.35) that measures the strength of the financial accelerator effect and hence the real effects of monetary policy. With equal democratic weights on all individuals alive (3.8), the condition \( \omega_t < \alpha \) is satisfied whenever the population is stable (\( \gamma = 1 \)), rising (\( \gamma > 1 \)), or not declining too fast (\( \gamma > \delta \kappa < 1 \)).

In this case, occasional financial crises are the consequence of the monetary policy that is chosen. Formally, the zero-mean random variable \( \nu_t \) has the two-point probability distribution in (3.9). One realization of \( \nu_t \) is positive, corresponding to nominal house-price inflation \( \pi_t \) above its mean value \( E_{t-1}\pi_t \). The other realization is negative, meaning that \( \pi_t \) is below its mean value. In line with Step 4, the two realizations of \( \nu_t \) lead to different real outcomes in the economy. The high realization of \( \nu_t \) features a high value \( h \) of the ratio of house prices to income, and a high value \( l \) of new lending relative to GDP, while the low realization of \( \nu_t \) leads to low values \( h \) and \( l \) of \( h_t \) and \( l_t \) respectively.

The two realizations of \( \nu_t \) in (3.9) can be thought of respectively as a long credit boom and a financial crisis. This is because the probability of the credit boom state will be high and the probability of the financial crisis state, denoted by \( \epsilon \), will be low. As \( \epsilon \) becomes small, and negative realizations of house-price inflation become accordingly rare, there is a larger drop in house prices when prices do eventually fall. The larger the fall in nominal house prices associated with the negative realization of \( \nu_t \), the smaller the ratio of lending to GDP \( l \) and the house price-income ratio \( h \) become in that state of the world. For arbitrarily small \( \epsilon \), the fall in nominal house prices and the declines in the house price-income ratio and the ratio of lending to GDP can be arbitrarily large. Formally, the supremum of the social welfare function is approached as the positive probability \( \epsilon \) of a financial crisis becomes small, but is never equal to zero.\(^\text{18}\)

Thus, with small but positive \( \epsilon \), most of the time the economy experiences a long credit boom with a high ratio of house prices to income and a high ratio of credit to GDP. But on rare occasions with probability \( \epsilon \), the economy undergoes a financial crisis triggered by a collapse of house prices.

\(^{18}\)Technically, the social welfare function has a finite supremum, but not a maximum.
in both nominal and real terms, and also relative to income. The crisis is marked by an abrupt deleveraging as the ratio of lending to GDP falls sharply.

The probability distribution (3.9) is indexed by a number $x$ that gives the size of the positive realization of $\nu_t$. Conditional on the expected rate of nominal house-price inflation, $x$ is higher when the equilibrium nominal interest rate set by the central bank is lower. A larger value of $x$ means that the credit to GDP and house price to income ratios during the long credit boom ($\bar{I}$ and $\bar{h}$) are both higher, so $x$ can be interpreted as the size of the expansion in credit and the growth of asset prices during the boom. Following a credit boom with a higher $x$, the subsequent financial crisis is worse for a given probability $\epsilon$ in that the drops in house prices and credit are more severe.

A typical realization of the path of nominal house-price inflation for small but positive $\epsilon$ is shown in Figure 1. Interestingly, the equilibrium of the economy has a probability distribution with the type of ‘rare events’ that have been emphasized in the literature on asset pricing (see, for example, Barro, 2006). Unlike many models studying imperfections in financial markets (Bernanke, Gertler and Gilchrist, 1999, Kiyotaki and Moore, 1997), the economy undergoes occasional large discrete jumps rather than continuous small fluctuations in the neighbourhood of a steady state. It is important to note that the probability distribution (3.9) is endogenously generated by the interaction of incomplete markets and monetary policy. Its features do not derive from the exogenous shocks $g_t$ and $\psi_t$, which can have essentially any non-degenerate probability distributions (see 2.5).

**Figure 1: Equilibrium with financial crises**

![Equilibrium with financial crises](image)

**Notes:** The graph depicts a typical realization of the path of (log) nominal house prices when $\epsilon$ is small but positive, and the average rate of nominal house-price inflation is constant.

The probability distribution of financial conditions $\nu_t$ in (3.9) features negative skewness and excess kurtosis for small $\epsilon$. These properties are also shared by the implied probability distribution of $\pi_t - E_{t-1}\pi_t$. Excessively loose monetary policy owing to political pressure thus leads to disproportionately large fluctuations in asset prices relative to a Normal distribution, and asymmetry in that these large shocks are found in the left tail of the distribution. The probability distribution of $\nu_t$ is also ‘heavy tailed’ in the sense that the variance and all higher moments become arbitrarily large as $\epsilon$ becomes small.

It should also be observed that while the form of the probability distribution (3.9) is determined
(the specific value of $x$ and the requirement that $\epsilon$ should be small), the exogenous trigger for switching from the credit boom to the financial crisis state is not pinned down. The state of the economy can be any function of the exogenous fundamentals $g_t$ (real GDP growth) or ‘sunspots’ $\psi_t$, subject to the probabilities of the two states being consistent with (3.9). Thus, the interaction of incomplete markets and the political economy of monetary policy creates the conditions for financial crises to occur with some positive probability, but does not give a definite answer for the precise circumstances that will lead to the onset of a crisis.

The nature of the equilibrium probability distribution of $\nu_t$ highlights the advantage of being able to give a full non-linear solution of the equilibrium analytically. This equilibrium could not be found through the use of perturbation methods, no matter what order of Taylor polynomial around the model’s non-stochastic steady state is used. While other numerical methods could in principle work, the computational challenges would be non-trivial.

The intuition for why the central bank pursues a monetary policy that causes occasional financial crises lies in the analysis of the political economy of monetary policy conducted earlier. Owing to the financial accelerator feedback loop from house prices and interest rates to balance sheets and then back into lending and house prices, borrowers gain overall from low interest rates even though low interest rates are associated with greater uncertainty about future house prices. This means that borrowers would not want the central bank to strive for financial stability by setting high interest rates. In contrast, savers dislike both low interest rates and financial instability. If the interests of borrowers have sufficient sway over the central bank relative to savers, it follows that monetary policy will not be tight enough to keep house prices under control. The lower the relative weight attached to savers ($\omega_t$) or the stronger is the financial accelerator effect ($\alpha$), the larger the size $x$ of the credit boom that precedes a financial crisis.

Given that the actual savers and borrowers in the economy are just individuals at different stages of their symmetric life-cycles, it might be expected that with zero population growth and equal democratic weighting of all individuals, borrower and saver interests are exactly in balance. This is not the case because the amount of loans that will be received by the next generation of borrowers depends on the financial health of the existing generation of borrowers. The interests of the young are thus aligned with existing middle-aged borrowers before the young have even taken out any loans. This means that overall political power rests with borrowers unless demographics are such that the age distribution of the population is heavily skewed towards old savers.

The political economy argument above explains why there will be house-price risk in equilibrium, but it does not account for the extreme concentration of risk in the left tail of the house-price

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19The reason is that equilibrium consumption $c_{\tau,t}$ from (2.41) and the implied levels of utility $\log c_{\tau,t}$ do not converge to their Taylor series expansions in financial conditions $\nu_t$ over the support of the equilibrium probability distribution of $\nu_t$. Since $E_{\nu_{t-1}}\nu_t = 0$, the non-stochastic steady state of the model must be $\nu_t = 0$. Although $c_{\tau,t}$ and $\log c_{\tau,t}$ are infinitely differentiable functions of financial conditions for all $-\infty < \nu_t < 1$, their Taylor series at 0 do not converge for $\nu_t < -\alpha^{-1}$.

20The utility-maximization problems of individuals require solving a portfolio choice problem because of the presence of housing as an asset. On top of this, finding the equilibrium requires solving a Ramsey problem where the social welfare function is not globally concave. Finally, the extent of aggregate risk is non-negligible because the equilibrium features rare events with large aggregate fluctuations.
distribution. To understand this, note that the literature on asset pricing (starting from Rietz, 1988) has suggested that ‘rare events’ (huge negative shocks with small probabilities) can be successful in explaining large risk premiums. Since a financial crisis has the properties of a ‘rare event’, and as the transmission mechanism of monetary policy here works through risk premiums, the same logic suggests that eliminating extreme left-tail events from the distribution of house prices requires very significantly higher interest rates. That would be unpalatable to borrowers. On the other hand, reducing house price risk away from the left tail does not make as much difference to risk premiums and hence does not require as large an increase in interest rates. Therefore, when the central bank puts some weight on savers, but sufficiently high weight on borrowers, interest rates are set too low to eliminate the risk of financial crises, even though the economy enjoys long periods of apparent stability with steadily rising house prices during the credit boom that precedes a crisis.

Another way to understand the result about the concentration of risk in the left tail of the distribution is the following. Consider a central bank starting from some arbitrary initial monetary policy for which the resulting probability distribution of financial conditions is a non-degenerate random variable. Rather than maximize the social welfare function, suppose the central bank seeks changes to its monetary policy (at just one date for simplicity) that would be agreeable to both borrowers and savers. According to Result 3, there always exists an alternative policy that both borrowers and savers would prefer which results in an equilibrium with financial crises of the form (3.9) for some positive $x$. This alternative policy actually reduces the housing risk premium. The policy change has the effect of smoothing out house-price growth during ‘normal times’ at the expense of concentrating risk in low-probability states where house prices drop sharply. There are mutual gains from somewhat higher interest rates to reduce house-price risk in ‘normal times’, but not from interest rates high enough to reduce house-price risk in the left tail of the distribution.

4 Policy implications

The previous section argued that financial crises will occur in equilibrium in an incomplete-markets economy where monetary policy is set to maximize a democratic unweighted social welfare function. For this to be a theory of financial crises caused by loose monetary policy, this section will show that such crises would not occur if monetary policy were conducted differently, in particular, if monetary policy were systematically tighter. Furthermore, since financial crises occur even though monetary policy is set to maximize a social welfare function, it might be wondered in what sense such crises are actually a bad thing. To address that point, this section will also shows that financial crises never result in a first-best (Pareto efficient) allocation of resources, while having sufficiently tight monetary policy to rule out crises does achieve first best. In this sense, there is a political-economy problem because a central bank subject to democratic pressures chooses a sub-optimal monetary policy.
4.1 The social planner benchmark

Before analysing alternative monetary policies, it is helpful to consider a hypothetical social planner. The social planner can directly specify a state-contingent allocation of consumption and housing across all individuals subject only to the economy’s resource constraints. The social planner effectively has a complete set of policy instruments including direct lump-sum taxes and transfers, unlike a central banker who operates by setting the nominal interest rate in a market economy.

The resource constraints are that total consumption from (2.11) must add up to the economy’s endowment of goods $Y_t$ from (2.3), and total consumption of housing services (2.12) must equal the supply of houses $H_t$ from (2.4). Mathematically, these are the same as the market-clearing conditions for goods and housing from (2.11) and (2.12):

$$N_{y,t}C_{y,t} + N_{m,t}C_{m,t} + N_{o,t}C_{o,t} = Y_t, \quad \text{and} \quad N_{y,t-1}H_{m,t} = H_{t-1}. \tag{4.1}$$

An allocation of resources from some date $t_0$ onwards is Pareto efficient if subject only to resource constraints, no individual can be given a higher ex-ante utility (meaning expected utility conditional on date $t_0 - 1$ information) without some other individual obtaining a lower ex-ante utility. The allocations satisfying this condition are first best.

**Step 7 (Requirements for Pareto efficiency)** Any Pareto-efficient allocation of consumption and housing (from $t_0$ onwards) must feature $H_{m,t} = 1$ and have consumption shares $c_{\tau,t}$ that satisfy (2.18) and:

$$\begin{align*}
\text{Risk sharing:} & \quad \frac{c_{y,t}}{E_{t_0-1}c_{y,t}} = \frac{c_{m,t}}{E_{t_0-1}c_{m,t}} = \frac{c_{o,t}}{E_{t_0-1}c_{o,t}}; \\
\text{Consumption smoothing:} & \quad E_t \frac{c_{m,t+1}}{c_{y,t}} = E_t \frac{c_{o,t+1}}{c_{m,t}}; \\
\text{Dynamic efficiency:} & \quad \liminf_{t \to \infty} E_t \frac{c_{y,t+1}}{c_{m,t+1}} \leq \frac{1}{\delta}, \quad \text{and} \quad \liminf_{t \to \infty} E_t \frac{c_{m,t+1}}{c_{o,t+1}} \leq \frac{1}{\delta}, \tag{4.2a,b,c}
\end{align*}$$

for all $t \geq t_0$. Conversely, any allocation satisfying $H_{m,t} = 1$, (2.18), (4.2a), (4.2b), and (4.2c) (with $\limsup_{t \to \infty}$ and strict inequality) is Pareto efficient and maximizes the social welfare function $\mathcal{W}_t$ from (3.4) subject to (4.1) for some sequence of weights $\{\Omega_t\}$ such that $\sup \mathcal{W}_t$ exists. There are infinitely many Pareto-efficient allocations.

**Proof** See appendix A.10.

The requirements for Pareto efficiency are standard. Individuals must share consumption risk in that any shock to consumption is proportionately distributed across everyone alive at a given date (see 4.2a). Individuals with overlapping lifetimes must smooth consumption as much as possible in that their expected rates of consumption growth are equalized (see 4.2b). As the economy has overlapping generations, there is also the requirement of dynamic efficiency. This is equivalent to a long-run upper bound on the ratio of consumption between younger and older individuals (see 4.2c). If this is not satisfied, resources can be shifted from younger to older individuals at each date to leave everyone better off.
The three requirements of Pareto efficiency are silent about questions of distribution. There are infinitely many first-best allocations over which individuals’ utilities will differ starkly.

### 4.2 Financial stability and Pareto efficiency

The three requirements for Pareto efficiency were stated directly in terms of the allocation in Step 7. In the incomplete-markets economy, these conditions can be translated into equivalent tests involving inflation and interest rates to judge the efficiency of a monetary policy.

#### Result 4 (Tests for Pareto efficiency)

The requirements for efficiency (4.2) can be tested with reference to inflation and interest rates as follows:

(i) The risk sharing condition (4.2a) is equivalent to
\[ \frac{c_{m,t}}{E_{t-1}c_{m,t}} = \frac{c_{o,t}}{E_{t-1}c_{o,t}} \text{ for all } t \geq t_0. \]
This holds if and only if there are no fluctuations in financial conditions \( \nu_t \), which is in turn equivalent to perfectly predictable nominal house-price inflation \( \pi_t \):
\[ \nu_t = 0 \text{ with probability } 1, \quad \text{ or equivalently } \pi_t = E_{t-1}\pi_t \text{ with probability } 1. \quad [4.3] \]

(ii) Consumption smoothing (4.2b) holds if and only if the nominal interest rate is sufficiently far above the expected rate of house-price inflation, that is,
\[ i_t = (\beta^{-1} - 1) + \beta^{-1}E_{t}\pi_{t+1} > E_{t}\pi_{t+1} \text{ for all } t \geq t_0. \]
This is equivalent to house-price inflation being perfectly predictable (4.3). A failure of consumption smoothing (4.2b) always leads to
\[ E_{t}c_{m,t+1}/c_{y,t} > E_{t}c_{o,t+1}/c_{m,t}. \]

(iii) The condition (4.2c) for dynamic efficiency holds if and only if the nominal interest rate is above the expected rate of house-price inflation in the long run, that is,
\[ \limsup_{t \to \infty} (i_t - E_{t}\pi_{t+1}) \geq 0. \]
This is implied by house-price inflation being perfectly predictable in the long run (that is, 4.3 holds for large \( t \)).

**Proof** See appendix A.11.

In the incomplete-markets economy, risk-sharing (4.2a) needs to be tested only between the middle-aged and old, that is, between existing homeowners and pensioners. This is because the financial accelerator links the credit received by the young to the financial health of the previous generation of borrowers, the middle aged. Risk sharing between the middle aged and old holds if and only if nominal house-price inflation is perfectly predictable (4.3). Intuitively, any shocks to nominal house prices would fall disproportionately on leveraged homeowners who have fixed nominal debt, which is inconsistent with an equal distribution of risk.

The requirement of consumption smoothing (4.2b) holds if and only if the nominal interest rate is sufficiently far above expected nominal house-price inflation. If this requirement is not met, the investment returns on pensions do not keep pace with the returns of homeowners who are able to finance house purchases with cheap mortgages, and consequently expected consumption growth from middle age to retirement is less than from youth to middle age. Since the difference between
the returns on housing and bonds is a risk premium, the interest rate required for consumption
smoothing prevails if and only if nominal house-price inflation is perfectly predictable.

The condition for dynamic efficiency (4.2c) holds in the incomplete-markets economy if and
only if the nominal mortgage rate exceeds the expected rate of nominal house-price inflation in
the long run. If interest rates were persistently too low relative to the returns earned by leveraged
homeowners, the consumption of the old relative to the middle aged would fall to a point where
transfers from the middle aged to the old could improve the welfare of each generation. Again, since
the difference in returns between housing and bonds is a risk premium, having perfectly predictable
nominal house-price inflation ensures that the dynamic efficiency condition is satisfied.

These findings explain why the equilibrium with financial crises is Pareto inefficient. They also
indicate that an alternative monetary policy regime where nominal house-price inflation is stabilized
would result in an equilibrium which is efficient.

Result 5 (Feasibility and desirability of financial stability) The monetary policy of the cen-
tral bank determines whether the equilibrium of the incomplete-markets economy is Pareto efficient:

(i) Any equilibrium with financial crises (3.9, for some $x > 0$ and some $\epsilon > 0$) is Pareto inefficient,
always violating the risk-sharing and consumption smoothing conditions (4.2a) and (4.2b). For
small $\epsilon > 0$, the equilibrium (3.9) is also dynamically inefficient if $x$ is larger than $\beta^{-1} - 1$.

(ii) Monetary policy can implement an Pareto-efficient allocation of resources by ensuring that
nominal house-price inflation is predictable, that is, $\pi_t = E_{t-1}\pi_t$. Conditional on expected
house-price inflation $E_{t-1}\pi_t$, the required nominal interest rate in equilibrium is $i_{t-1} = (1 +
E_{t-1}\pi_t)/\beta - 1$, which is greater than the nominal interest rate in an equilibrium with financial
crises. By stabilizing house-price inflation, the house price-income ratio $h_t$, new lending relative
to GDP $l_t$, and the value of existing loans relative to the value of houses $d_t$ are equal to the
constants $h^*, l^*, and d^*$:

$$h^* = \frac{\alpha\beta}{\lambda}, \quad l^* = \alpha\beta, \quad and \quad d^* = \frac{\lambda}{\beta},$$

[4.4]

Compared to the equilibrium with financial crises, $h^*$ is less than both $Eh_t$ and $\bar{h}$, and $l^*$ is
less than both $El_t$ and $\bar{l}$.

(iii) The allocation associated with house-price stability is the only first-best allocation that can
be implemented through monetary policy.

Proof See appendix A.12.

An equilibrium with financial crises always fails some, and possibly all, of the requirements for
Pareto efficiency. The unpredictability of house-price inflation violates the sharing of risk between
leveraged homeowners and pensioners. The low level of interest rates compared to expected house-
price inflation drives a wedge between the expected consumption growth of the two groups, and
may even result in a dynamically inefficient equilibrium.
Although markets are incomplete, there is nothing inevitable about the inefficiency of the equilibrium. As stated in Result 5, there exists a monetary policy that supports an efficient allocation of resources. This monetary policy can be interpreted as the central bank pursuing the goal of financial stability by setting interest rates sufficiently high to stabilize house-price inflation. Predictable house prices promote risk sharing across the generations, and higher interest rates raise the expected consumption growth of the old to the level enjoyed by the young and middle aged.

Result 5 shows that financial stability can be expressed in three exactly equivalent ways. First, in avoiding unanticipated swings in nominal house-price inflation. This objective is within the domain of monetary policy because the goal is predictability of nominal house prices, a nominal variable, and it can be achieved using the monetary policy instrument the central bank has available by setting sufficiently high interest rates.

Financial stability can also be expressed in real terms as stabilizing the ratio of house prices to income at a lower level than what occurs during a credit boom and what occurs on average in the equilibrium with financial crises. Finally, financial stability also means avoiding credit booms and busts by stabilizing the ratio of lending to GDP, again at a lower level than what prevails during a credit boom and on average across the boom and bust states of an equilibrium with financial crises. While these ‘real’ interpretations of financial stability are typically seen as outside the domain of monetary policy and being the responsibility of macroprudential or other regulatory policy, the analysis shows that it is possible to view financial instability instead as a monetary phenomenon.

It is important to note that the analysis does not predict that stabilizing nominal house-price inflation would simply cause an extreme amount of volatility in nominal goods prices instead. As financial stability leads to stable ratios of lending and house prices to income, an immediate implication is that the volatility of goods-price inflation would only be the same as the volatility of the real GDP growth rate. Although the financial instability due to unpredictability in nominal house prices is a monetary phenomenon, the financial accelerator effect that played a key role in the analysis predicts nominal house-price fluctuations are associated with movements of the relative price of housing in the same direction.

4.3 Achieving financial stability: Political and institutional challenges

Result 5 demonstrates that monetary policy can achieve a first-best allocation of resources by making financial stability its policy goal. However, inefficient financial instability will occur when political pressure compels the central bank to act in accordance with an unweighted social welfare function.

This raises two important questions. First, since an equilibrium with financial crises violates the conditions for efficiency, it might be wondered how a monetary policy leading to such an equilibrium could ever be associated with a higher value of the unweighted social welfare function than an alternative monetary policy which ensures all the efficiency conditions are met. After understanding this and why financial instability should still be seen as sub-optimal, what political and institutional conditions must be in place for the central bank to be able to pursue efficient financial stability?
Result 6 (Political and institutional requirements for financial stability)  
(i) If in addition to its conventional monetary policy instrument the central bank has access to individual-specific (but not state-contingent) lump-sum taxes and transfers then it chooses a monetary policy with financial stability and predictable nominal house prices for all social welfare weights \( \{\Omega_t\} \) (and hence all values of \( \omega_t \)). Lump-sum taxes and transfers can be used to ensure that moving from financial instability to financial stability is Pareto improving.

(ii) In the absence of lump-sum taxes and transfers, the central bank chooses a monetary policy implying financial stability (predictable nominal house prices) at date \( t \) if and only if \( \omega_t \geq \alpha \), that is, the relative weight on savers \( \omega_t \) is sufficiently high. With democratic equal weights, this occurs if and only if \( \gamma \leq \delta \kappa < 1 \), that is, the population is declining sufficiently fast.

(iii) In the absence of lump-sum taxes and transfers, and where \( \gamma > \delta \kappa \), the central bank chooses a monetary policy implying financial stability at date \( t \) if and only if:

\[
\Omega_{t-2} \geq \frac{\gamma(1 + \gamma)}{\delta \kappa(1 + \delta \kappa)} \left( \frac{N_{m,t}\Omega_{t-1} + N_{y,t}\Omega_t}{N_{m,t} + N_{y,t}} \right),
\]

that is, the weight \( \Omega_{t-2} \) on old savers at date \( t \) is sufficiently larger than the average weight on young and middle-aged borrowers.

**Proof** See appendix A.13.

To understand the first claim, recall Step 7 states the familiar result that a Pareto-efficient allocation maximizes the social welfare function for some weights assigned to each individual, and a corollary is that an allocation violating some of the efficiency conditions cannot maximize the social welfare function for any possible weights. But there is no contradiction with the occurrence of financial crises predicted by Result 3 because in one case a central bank is aiming to maximize social welfare using only interest-rate policy, while in the other case a social planner is maximizing the same function, but has access to a wider set of instruments including individual-specific taxes and transfers.

As an equilibrium with financial crises is never Pareto efficient, it must be the case that subject only to resource constraints, it is feasible to move from such an equilibrium to an efficient allocation where all individuals are weakly better off and some are strictly better off. However, Result 5 states that the central bank can implement only one efficient allocation through monetary policy, namely the equilibrium with financial stability. This is because the central bank has only a single instrument, and cannot independently pursue distributional goals as well as support an efficient allocation of resources. Distributional concerns can then potentially outweigh efficiency concerns when the central bank maximizes a social welfare function.

As described earlier, while financial stability is efficient, achieving it requires systematically tighter interest-rate policy, resulting in less lending relative to the size of the economy, and lower house prices relative to incomes. The political economy analysis of Result 2 indicates these effects will be bad for borrowers, who will therefore lose from moves towards financial stability. While transfers from the savers who gain to the borrowers who lose could in principle make everyone
better off overall, the central bank lacks access to the lump-sum taxes and transfers needed to make this happen. Taking away the punch bowl is good for efficiency, but will leave borrowers disgruntled as the central bank cannot compensate them for the higher interest rates they will face. Inefficient financial instability can therefore persist because there are too many individuals with a vested interest in maintaining cheap credit.

Assuming it is not feasible to endow the central bank with additional policy instruments to make lump-sum transfers to those who lose from a change to monetary policy, funded by lump-sum taxes on the winners, the only other way a central bank will choose financial stability when subject to democratic pressure is if savers are sufficiently numerous. Given that the life cycle determines who are the borrowers and the savers in the economy studied here, a saver-friendly monetary policy of financial stability requires enough old people to support it, which occurs when the population is declining sufficiently fast. In some countries, demographic factors might give sufficient political power to older people to sustain tight monetary policy and avoid financial instability.

In the absence of lump-sum taxes and transfers and without favourable demographics, the third claim in Result 6 is that the central bank chooses a monetary policy delivering efficient financial stability when it maximizes a social welfare function that is weighted in favour of savers. This is analogous to the famous ‘conservative central banker’ result (Rogoff, 1985) that an inefficient inflation bias can be eliminated if the central banker’s preferences attach a greater importance to price stability and a lesser importance to unemployment than the ‘average’ person in the economy. Here, efficient financial stability requires the central bank to care more than the ‘average’ person in the economy about the negative effects of low interest rates on pensions relative to the positive effects on mortgage affordability. The required central banker is ‘conservative’ in the sense of defending creditor interests rather than those of debtors.

The intuition for the conservative central banker is straightforward. In the earlier analysis, inefficient financial instability was the result of political pressure for low interest rates to please borrowers. If the central bank could choose monetary policy to maximize a welfare function that puts a higher weight on savers, who would directly lose from low interest rates, then it would be able to raise interest rates. Of course, borrowers would be unhappy about this, so the argument reaffirms the case for central bank independence. Pursuing an efficient monetary policy will not be popular, thus the central bank needs to be insulated from politics.

5 Endogenizing incomplete markets

So far, the paper has explored how systematically loose monetary policy can create the conditions for financial crises to occur, and how the political economy of monetary policy explains why there will be pressure on central banks to adopt such risky monetary policies. The analysis has been done in the context of an economy with incomplete markets, where all lending must take the form of nominal debt contracts, and where housing services can only be received through homeownership. However, this is unsatisfactory because it exogenously assumes that trade cannot ever take place in other markets.
This section begins by considering a frictionless complete-markets (Arrow-Debreu) benchmark. In that world, monetary policy has no real effects and financial crises will never occur, which demonstrates that incomplete markets is an essential part of the story. The next step is to allow for additional markets: alternative forms of housing finance to mortgages, and a rental market for houses, but where these additional markets are subject to frictions. Trade in those additional markets may or may not take place depending on whether the gains from trade outweigh the frictions. Crucially, whether these markets are active or not is endogenous to the conduct of monetary policy. It is shown that the main conclusions are robust to endogenizing the incompleteness of markets.

5.1 Frictionless complete markets

The model is the same as that of section 2 except all individuals can trade securities with payoffs contingent on any state of the world (Arrow-Debreu securities). Individuals trade in contingent securities markets sequentially during their lives, excluding participation by individuals before birth.\footnote{This turns out to be without loss of generality here.}

Formally, let $A_{\tau,t}$ denote the per-person portfolio of securities making payoffs (denominated in terms of goods without loss of generality) at time $t$ to age $\tau$ individuals conditional on the realization of a specific state of the world (this portfolio being chosen at time $t-1$). The prices of securities (in terms of goods) relative to the probabilities of the future states (conditional on the current state) are $K_{t+1}$, so the cost of a portfolio $A_{t+1}$ is $E_t[K_{t+1}A_{t+1}]$ at time $t$. The new budget constraints, first-order conditions, and market-clearing conditions can be found in the proof of the result below, which characterizes the unique equilibrium for real variables in the complete-markets economy.

Step 8 (Complete markets) With complete financial markets, there is a unique equilibrium for all real variables that is independent of monetary policy. The equilibrium is $c_{y,t} = c^*_y$, $c_{m,t} = c^*_m$, $c_{o,t} = c^*_o$ and $h_t = h^*$ for all $t$, where the constants $c^*_y$, $c^*_m$, and $c^*_o$ are from (2.42) and $h^*$ from (4.4). The equilibrium is Pareto efficient.

The nominal interest rate and house-price inflation continue to satisfy (2.25), and the housing risk premium $\xi_t$ is given by the same formula (2.26) as found in Step 3. A unique equilibrium for nominal variables can be implemented using an interest-rate feedback rule as described in Step 5.

Proof See appendix A.14.

With complete markets, monetary policy has no impact on the real equilibrium of the economy. While monetary policy can still affect the risk premium of real assets over nominal bonds, this is irrelevant to any real decisions because individuals can conduct the trade they desire in financial markets by buying or selling packages of contingent securities. Hence, irrespective of monetary policy, the equilibrium of the economy is Pareto efficient because individuals can directly achieve full risk sharing and consumption smoothing by choosing appropriate long or short positions in each contingent security.\footnote{It turns out that the equilibrium is ex-ante efficient even in respect of newly born individuals who did not participate in financial markets before they were born. This is the sense in which excluding participation by individuals} The outcome is the same as the ‘financial stability’ equilibrium of the
incomplete-markets economy where the central bank stabilizes nominal house-price inflation (see Result 5). Financial crises cannot occur here with complete markets because the mechanism of the central bank holding down real borrowing costs owing to political pressure does not operate.

5.2 Additional markets with frictions

Moving away from the abstract notion of complete contingent securities markets, now consider an extension of the model of section 2 where there are two specific additional markets subject to frictions: a rental market for houses, and a market for equity shares in houses.

The lifetime utility function (2.2) is replaced by:

\[ U_t = \log C_{y,t} + \delta E_t [\log C_{m,t+1} + \Theta(H_{m,t+1} + H_{r,t+1})] + \delta^2 E_t \log C_{o,t+2}, \tag{5.1} \]

where the new variable \( H_{r,t} \) denotes housing services acquired through the rental market rather than homeownership. Here, rented housing is fundamentally the same as owner-occupied housing, so utility depends on the sum \( H_{m,t} + H_{r,t} \). The nominal rent is \( Z_t \), which is paid at date \( t \) for renting one unit of housing during time period \( t \). The supply of houses for rent at time \( t \) is equal to the number of housing units \( H_{o,t} \) held by the old as investment properties.

The friction in the rental market is contract enforceability combined with asymmetric information. A fraction \( \chi_r \) of rental payments will be unenforceable. Renters are of two types: those that always pay (fraction \( 1 - \chi_r \)) and those that always default (fraction \( \chi_r \)). Individual renters know their type, but landlords cannot distinguish the two types in advance (the defaulting type will mimic the repaying type when contracts are written). For simplicity, both types pool consumption risk.

There is also a market for housing equity shares, which is a form of financing for borrowers where payments are proportional to the value of a house. Formally, the seller of one unit of housing equity at date \( t \) makes a nominal payment to the buyer equal to the value of a house \( V_{t+1} \) at date \( t + 1 \). The net housing equity share positions of the young and the middle aged at the end of period \( t \) are denoted by \( e_{y,t} \) and \( e_{m,t} \) respectively, where a positive value indicates a purchase and a negative value a sale. The nominal price of a unit of housing equity is \( S_t \) when it is sold.

The friction in the market for housing equity shares is also contract enforceability with asymmetric information. A fraction \( \chi_e \) of payments to holders of equity shares will be unenforceable. Analogous to the assumptions for renters, there are two types of borrowers (fractions \( 1 - \chi_e \) and \( \chi_e \) of all borrowers) who know their type, but investors cannot distinguish them ex ante.

before they are born is without loss of generality. The complete-markets equilibrium must also avoid dynamic inefficiency to be Pareto efficient. Even though the model has overlapping generations of individuals, the equilibrium with complete markets is always dynamically efficient because of the existence of housing as a physical asset that does not depreciate.
The budget identities (2.6) of the young, middle aged, and old are replaced by:

\[ C_{y,t} + \frac{V_t H_{m,t+1}}{P_t} + \frac{Q_t B_{y,t}}{P_t} + \frac{S_t e_{y,t}}{P_t} = 0; \quad [5.2a] \]

\[ C_{m,t} + \frac{Z_t H_{o,t}}{P_t} + \frac{Q_t B_{m,t}}{P_t} + \frac{V_t H_{o,t+1}}{P_t} + \frac{S_t e_{m,t}}{P_t} = y_{m,t} + \frac{\gamma V_{t+1}}{P_{t+1}} + \frac{B_{y,t-1}}{P_t} + \frac{V_t e_{y,t-1}}{P_t} + \frac{D_t}{\gamma P_t}; \quad [5.2b] \]

and

\[ C_{o,t} = \frac{B_{m,t-1}}{P_t} + \frac{Z_t H_{o,t}}{P_t} + \frac{\gamma V_t H_{o,t}}{P_t} + \frac{V_t e_{m,t-1}}{P_t} - \frac{D_t}{P_t}, \quad [5.2c] \]

where \( D_t \) denotes the sum of defaults on rental contracts and housing equity shares per investor, which is subtracted from the budget identity of the old. Defaults \( (D_t/\gamma \text{ per person}) \) are added to a single budget identity for the middle aged because the repaying and defaulting types pool consumption risk. The first-order condition for maximizing lifetime utility (5.1) with respect to house purchases \( H_{m,t+1} \) subject to (5.2) is:

\[ \frac{V_t}{P_t C_{y,t}} = \delta E_t \left[ \Theta'(H_{m,t+1} + H_{r,t+1}) + \frac{\gamma V_{t+1}}{P_{t+1} C_{m,t+1}} \right], \quad [5.3] \]

which replaces equation (2.8). The first-order conditions with respect to bond holdings \( B_{r,t} \) remain (2.9) as before.

Considering respectively an individual who is a repaying type of renter and a repaying type of borrower, the first-order conditions for maximizing lifetime expected utility (5.1) with respect to \( H_{r,t} \) and \( e_{y,t} \) subject to the budget identities (5.2) are:

\[ \frac{Z_t}{P_t C_{m,t}} = \Theta'(H_{m,t} + H_{r,t}), \quad \text{and} \quad \frac{S_t}{P_t C_{y,t}} = \delta E_t \left[ \frac{V_{t+1}}{P_{t+1} C_{m,t+1}} \right], \quad [5.4] \]

which are derived taking defaults \( D_t \) as given because the individual knows he is a repaying type. The defaulting types mimic the choices of \( H_{r,t} \) and \( e_{y,t} \) implied by (5.4), so all young individuals choose the same \( e_{y,t} \) and all middle-aged individuals choose the same \( H_{r,t} \).

Given the default and asymmetric information frictions, investors’ losses \( D_t \) at date \( t \) are:

\[ D_t = \chi_t Z_t H_{o,t} + \chi e V_t e_{m,t-1}. \quad [5.5] \]

While investors cannot distinguish individuals’ types, they know the fractions of repaying and defaulting types in the population, and they therefore take account of (5.5) when choosing \( H_{o,t} \) and \( e_{m,t} \). Given non-negativity constraints on \( H_{o,t} \) and \( e_{m,t} \), the conditions for maximizing lifetime utility (5.1) with respect to \( H_{o,t+1} \) and \( e_{m,t} \) subject to (5.2) and (5.5) are as follows:

\[ H_{o,t+1} \geq 0, \quad \text{and} \quad \frac{V_t}{P_t C_{m,t}} \geq \delta E_t \left[ \frac{(1 - \chi t) Z_{t+1} + \gamma V_{t+1}}{P_{t+1} C_{o,t+1}} \right] \quad \text{with equality if } H_{o,t+1} > 0; \quad [5.6] \]

\[ e_{m,t} \geq 0, \quad \text{and} \quad \frac{S_t}{P_t C_{m,t}} \geq \delta (1 - \chi) E_t \left[ \frac{V_{t+1}}{P_{t+1} C_{o,t+1}} \right] \quad \text{with equality if } e_{m,t} > 0. \quad [5.7] \]
Housing-market clearing (2.12) is replaced by the first equation below, and two new market-clearing conditions are added for the rental and equity share markets:

\[ N_{y,t} H_{m,t+1} + N_{m,t} H_{o,t+1} = H_t; \] \[ N_{m,t} H_{t,t} = N_{o,t} H_{o,t} \] \[ N_{y,t} e_{y,t} + N_{m,t} e_{m,t} = 0. \]

All other aspects of the model from section 2 are unchanged.

For simplicity, the following restriction is imposed so that the strengths \( \chi_r \) and \( \chi_e \) of the rental and equity share frictions are both indexed by a common parameter \( \chi \) (with \( 0 \leq \chi \leq 1 \)):

\[ \chi_r = \chi, \quad \text{and} \quad \chi_e = (1 - \beta)\chi, \] \[ \text{where } \beta \text{ is as defined in (2.25).} \]

Define a variable \( \sigma_t \) as follows:

\[ \sigma_t = \frac{V_t H_{o,t+1} + S_t e_{m,t}}{Q_t B_{m,t} + V_t H_{o,t+1} + S_t e_{m,t}}, \]

which denotes the fraction of total savings held as rental housing and housing equity shares.

**Result 7 (Equilibrium with additional markets)** Conditional on a particular monetary policy, and hence on a particular path of nominal house-price inflation:

(i) In the presence of frictions \( \chi > 0 \), there is no trade in the additional markets \( H_{o,t+1} = 0 \) and \( e_{m,t} = 0 \), and thus \( \sigma_t = 0 \) if and only if:

\[ \xi_t \leq \varphi, \quad \text{where } \varphi = \frac{(1 - \beta)\chi}{1 - (1 - \beta)\chi} \quad \text{(with } 0 < \varphi \leq \beta^{-1} - 1). \]

(ii) When \( \xi_t > \varphi \), the positive equilibrium value of \( \sigma_t \) is given by the unique solution of the equation:

\[ E_t \left[ \frac{1 - (1 + \varphi)(1 - \nu_{t+1})}{1 - (1 - \sigma_t)(1 - (1 + \varphi)(1 - \nu_{t+1}))} \right] = 0, \]

which satisfies \( \sigma_t < 1 \). Any combination of \( H_{o,t+1} \) and \( e_{m,t} \) such that (5.12) gives the solution for \( \sigma_t \) is consistent with equilibrium.

(iii) Using the \( \sigma_t \) that solves (5.14), the equilibrium is as given in (2.35) and (2.41) except that \( \nu_t \) is replaced by:

\[ \eta_t = \sigma_{t-1}(1 - \beta)\chi + (1 - \sigma_{t-1})\nu_t, \]

and the formula for \( l_t \) in (2.35) is now \( l_t = \alpha\beta(1 - \sigma_t)/(1 - \alpha\nu_t) \).

(iv) With no frictions \( (\chi = 0) \), the equilibrium is the same as with complete markets in Step 8.
Observe that in the absence of frictions ($\chi = 0$), the equilibrium is independent of monetary policy and coincides with the case of complete markets analysed in Step 8. In the setting considered here, a perfect rental market and a perfect market for housing equity shares therefore exhaust all opportunities for frictionless trade between individuals. But the main content of Result 7 is in deriving the equilibrium of the economy when these additional markets are not frictionless.

When frictions are present ($\chi > 0$), one possible outcome is that the additional markets are inactive. Whether or not this happens turns out to depend on monetary policy in a simple way. As long as the central bank does not set the nominal interest rate too low relative to expected nominal house-price inflation, the rental and equity share markets will be dormant, in which case the equilibrium of the economy is the same as found in Result 1. Thus, there is a range of monetary policies and a range of frictions where the earlier analysis is entirely unaffected by the existence of the additional markets.

However, there is a threshold for the nominal interest rate given expected house-price inflation below which savers will begin to shift away from nominal assets to real assets. This is equivalent to the upper bound $\varphi$ in (5.13) for the housing risk premium $\xi_t$. Monetary policies that deliver low returns on bonds and high expected returns on housing will encourage savers to shift their pensions into investments with exposure to the housing market, whether that is directly buying houses to rent out, or financial innovation in lending that allows investors to take equity stakes in houses partially owned and lived in by others.

When the additional markets are active, the equilibrium share $\sigma_t$ of saving allocated to investments in rental housing or buying equity shares is determined by the equation in (5.14). In the setting here, the two additional markets are essentially isomorphic because the precise combination of purchases of rental housing and equity shares is not pinned down. Since the frictions affecting the two markets are of equal size (the parameter restriction in 5.11), it does not matter whether savers obtain their exposure to the housing market through ‘buy-to-let’ or financial innovation in lending to homebuyers.

Having found the portfolio share $\sigma_t$, it is straightforward to obtain the equilibrium values of all other variables as explained in Result 7. Note that there is now some limit on the size of the real effects of monetary policy, which depends on the extent of the frictions. If $\sigma_t$ is pushed up close to one, savers hold almost no nominal assets and any further real effects of monetary policy would be negligible.

### 5.3 Monetary policy in the presence of additional markets

This section explores the robustness of the earlier results about the political economy of monetary policy to the presence of the additional markets, and the alternative assumption that politics forces the central bank to adopt policies in the interests of the median individual rather than maximizing the mean welfare of all individuals.
Result 8 (Monetary policy with additional markets)  

(i) If social welfare is the objective function of the central bank and the weight $\omega$ on savers is less than $\alpha$ then the financial crisis equilibrium (3.9) occurs with:

$$x = \min \left\{ \frac{\alpha - \omega}{(1 - \alpha)\omega}, \varphi \right\},$$

and the additional markets are inactive in equilibrium ($\sigma_t = 0$).

(ii) If the utility of the median individual is the goal of the central bank then the financial crisis equilibrium (3.9) occurs with $x = \varphi$ when $\gamma > 2/(1 + \sqrt{5}) \approx 0.6$, and the additional markets are inactive in equilibrium ($\sigma_t = 0$).

(iii) The presence of the additional markets implies that any equilibrium must be dynamically efficient. However, risk sharing and consumption smoothing always fail when the additional markets are active ($\sigma_t > 0$), so any such equilibrium must be Pareto inefficient. Monetary policy retains its ability to achieve an efficient allocation by stabilizing nominal house-price inflation, in which case the additional markets would be inactive ($\sigma_t = 0$).

Proof See appendix A.16.

The first finding is that financial crises continue to occur even with the possibility of trade in the additional markets. The only change is that the size of the credit boom preceding the crisis, as measured by $x$, might be smaller. If the value of $x$ found in Result 3 is no greater than $\varphi$ then the additional markets make no difference whatsoever. If $x$ would have been greater than $\varphi$ without the additional markets then $x$ becomes exactly equal to $\varphi$ here. In all cases, the additional markets remain inactive in equilibrium. Nonetheless, the possibility of using the additional markets puts a limit on how large a credit boom the central bank can engineer by lowering interest rates because savers would substitute from nominal bonds to real assets if interest rates were reduced too far.

The second finding in Result 8 concerns the robustness of the analysis to a different assumption about the political pressure faced by the central bank. The earlier assumption was that this pressure would push the central bank to adopt a monetary policy that delivers the highest unweighted mean utility of the individuals currently alive, which could be interpreted as a simple representation of ‘democratic’ political pressure. However, the median voter theorem suggests it is the utility of the median individual that would become the central bank’s goal if it were subject to political pressure in a democracy. The median voter theorem is applicable here because when considering monetary policy at a specific date, all individuals’ interests exactly align with one of two groups: ‘borrowers’ or ‘savers’, and policy will be chosen to please the more numerous group.

With the young and the middle aged in the group of borrowers, the median individual will be a borrower except when demographics are such that the age distribution of the population is very heavily skewed towards the old. When monetary policy is set entirely in the interests of a borrower as the median individual, interest rates are lowered to create as large a credit boom as possible.
subject to savers not substituting away from nominal bonds to real assets (that is, \( x = \varphi \)). As before, occasional financial crises will be the result of this policy.\(^{23}\)

The third finding relates to whether the additional markets have any implications for the efficiency or inefficiency of the equilibrium. The presence of the additional markets means the economy can never be dynamically inefficient. If it were, savers would always switch from bonds to investments in housing to enjoy capital gains even if frictions were so severe that they collected little in rent.\(^{24}\) It could also be imagined that the risk sharing requirement for efficiency might hold if real assets were entirely to replace nominal bonds in savers’ portfolios. This never occurs in equilibrium, but the share of nominal assets can be negligible, in which case the equilibrium would be very close to satisfying risk sharing. However, an equilibrium where the additional markets are active will never be efficient (or even arbitrarily close to an efficient allocation) because the consumption smoothing condition is always violated. This condition fails to hold as the frictions in the additional markets drive a wedge between the return received by investors and the contractual obligations of renters and those with equity loans. None of these findings has any bearing on the ability of monetary policy to achieve an efficient allocation of resources by adopting financial stability as its goal because in that case there would be no incentive for anyone to use the additional markets.

6 Concluding remarks

This paper revisits an old debate about whether monetary policy should be influenced by financial stability concerns. The current consensus view is that monetary policy should focus on stabilizing inflation rates for goods and services, but should not concern itself with asset-price ‘bubbles’ or ‘excessive’ levels of debt. To the extent that these are seen as problems for policymakers to address, the solutions are believed to lie in the domain of regulation or macroprudential policy. This paper challenges the consensus view by presenting a simple theoretical mechanism based on incomplete markets through which the monetary policy actions of the central bank can have long-lasting effects on real interest rates and the level and volatility of asset prices.

In the theory presented here, loose monetary policy creates the conditions for excessive growth in lending, booms in asset prices, and ultimately financial crises: rare events where asset prices drop dramatically and result in painful deleveraging. The central bank is responsible for these problems in the sense that tighter monetary policy could have prevented them. However, the theory also explains why, for political economy reasons, the central bank will find tough action difficult: taking away the punch bowl will not be popular.

How should the challenges for monetary policy highlighted by this paper be addressed? One analogy is with the old problem of the ‘inflation bias’. There, price stability helps an economy op-

\(^{23}\)Although making the utility of the median individual the goal of policy has stronger political-economy foundations, the inefficiency of the resulting equilibrium is less surprising because some individuals receive a zero weight in the objective function for policy. When social welfare is the goal of policy, the equilibrium is Pareto inefficient in spite of all individuals receiving some positive weight.

\(^{24}\)This is a version of the standard argument that adding a land-like asset to an overlapping-generations economy rules out a dynamically inefficient equilibrium (Tirole, 1985).
erate more efficiently, but policymakers face pressure to reduce unemployment by trying to exploit a Phillips curve, which results in excessively high inflation. The inflation bias has been conquered through a combination of conservative central bankers, central bank independence, and the institutional framework provided by inflation targeting. Here, policymakers face pressure for low interest rates even though this results in inefficient fluctuations in asset prices and lending: a ‘financial instability’ bias. Overcoming this problem reaffirms the need for central bank independence and conservative central bankers: but now ‘conservative’ in the sense of standing up for the interests of savers rather than ignoring unemployment. More systematically, the results of the paper suggest a case for embedding financial stability concerns into the monetary policy framework.

References


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A Appendices

A.1 Proof of Step 1

Substituting the budget identity of the old from (2.19b) into the bond Euler equation of the middle-aged from (2.19d):

\[
\frac{1}{c_{m,t}} = \delta E_t \left[ \frac{1 + r_{t+1}}{(1 + g_{t+1})d_{t+1}h_{t+1}} \right],
\]

and then substituting the accounting identity for the burden of debt from (2.19a) and cancelling terms:

\[
\frac{1}{c_{m,t}} = \delta E_t \left[ \frac{(1 + r_{t+1})(1 + g_{t+1})h_{t+1}}{(1 + g_{t+1})h_{t+1}(1 + r_{t+1})l_t} \right] = \frac{\delta}{l_t}.
\]

Rearranging this equation confirms the link between \(l_t\) and \(c_{m,t}\) given in equation (2.21), which is also valid in the case where \(l_t = 0\) and \(c_{m,t} = 0\). This completes the proof.

A.2 Proof of Step 2

Let \(\kappa_t\) be defined as the fraction of loans used to finance consumption:

\[
\kappa_t \equiv \frac{c_{y,t}}{l_t}. \tag{A.2.1}
\]

Since \(c_{y,t}\), \(h_t\), and \(l_t\) must be non-negative, the definition above and the budget identity for the young in (2.19b) imply that \(\kappa_t\) must satisfy \(0 \leq \kappa_t \leq 1\). Using equations (2.19b) and (A.2.1):

\[
c_{y,t} = \kappa_t l_t, \quad \text{and} \quad h_t = (1 - \kappa_t)l_t. \tag{A.2.2}
\]

The ratio of house prices \(h_t\) to consumption of the young \(c_{y,t}\) is therefore:

\[
\frac{h_t}{c_{y,t}} = \frac{1}{\kappa_t} - 1. \tag{A.2.3}
\]

Since equation (2.21) of Step 1 implies \(c_{m,t} = l_t/\delta\), by using (A.2.2) it follows that the ratio of house prices to the consumption of the middle-aged is:

\[
\frac{h_t}{c_{m,t}} = \delta (1 - \kappa_t). \tag{A.2.4}
\]

Note that as \(\kappa_{t+1}\) is bounded above by 1, the ratio \(h_t/c_{m,t}\) must also be bounded above, even if \(c_{m,t}\) approaches zero.

By substituting equations (A.2.3) and (A.2.4) into the housing Euler equation (2.19c), the variable \(\kappa_t\) must satisfy the following expectational difference equation:

\[
\frac{1}{\kappa_t} - 1 = \delta \theta + \delta E_t \left[ \delta (1 - \kappa_{t+1}) \right].
\]

Given that \(0 \leq \kappa_{t+1} \leq 1\), the right-hand side always remains bounded. Using (A.2.3), it follows that the ratio \(h_t/c_{y,t}\) must always remain bounded even if \(c_{y,t}\) approaches zero. By dividing both sides of the equation above by \(\delta^2\) and rearranging:

\[
E_t \kappa_{t+1} = \left( 1 + \frac{1}{\delta^2} + \frac{\theta}{\delta} \right) - \frac{1}{\delta^2 \kappa_t}, \tag{A.2.5}
\]

and the expectational difference equation can be written as follows in terms of a function \(F(\kappa)\):

\[
E_t \kappa_{t+1} = F(\kappa_t), \quad \text{where} \quad F(\kappa) \equiv \left( 1 + \frac{1}{\delta^2} + \frac{\theta}{\delta} \right) - \frac{1}{\delta^2 \kappa}. \tag{A.2.6}
\]

Steady states of the difference equation (A.2.6) are roots of the quadratic equation \(G(\kappa) = 0\), where \(G(\kappa)\) is given by:

\[
G(\kappa) \equiv \delta^2 \kappa^2 - (1 + \delta^2 + \delta \theta)\kappa + 1, \quad \text{since} \quad F(\kappa) = \kappa - \frac{G(\kappa)}{\delta^2 \kappa}. \tag{A.2.7}
\]

Observe that \(G(0) = 1 > 0\) and \(G(1) = -\delta \theta < 0\), so the quadratic equation has one root in the unit interval
denoted by \( \kappa \). The product of the roots is \( 1/\delta^2 \), so the other root is \( (\delta^2 \kappa)^{-1} \), which must be greater than 1 since \( G(1) < 0 \).

Now consider possible solutions of the expectational difference equation (A.2.6) for \( \kappa_t \). Note the properties of the function \( F(\kappa) \):

\[
F'(\kappa) = \frac{1}{\delta^2 \kappa^2}, \quad F''(\kappa) = -\frac{2}{\delta^2 \kappa^3}, \quad \text{and} \quad \lim_{\kappa \to 0} F(\kappa) = -\infty, \tag{A.2.8}
\]

that is, \( F(\kappa) \) is a strictly increasing and strictly concave function with an asymptote at \( \kappa = 0 \). Given that there are two steady states (the solutions of \( G(\kappa) = 0 \)), there are two intersections of \( F(\kappa) \) with the 45° line at \( \kappa \) and \( (\delta^2 \kappa)^{-1} \). Since \( \kappa \) is smaller than \( (\delta^2 \kappa)^{-1} \) and given the concavity of \( F(\kappa) \), it must be the case that \( F'(\kappa) > 1 \) and \( F'((\delta^2 \kappa)^{-1}) < 1 \). This shows that the steady state at \( \kappa \) is unstable, while \( (\delta^2 \kappa)^{-1} \) is stable.

The equilibrium values of \( \kappa_t \) must always satisfy \( 0 \leq \kappa_t \leq 1 \), and the steady state \( \kappa \) is known to lie in this region. The other steady state \( (\delta^2 \kappa)^{-1} \) is outside the admissible range because \( (\delta^2 \kappa)^{-1} > 1 \). Now let \( \varepsilon_t \equiv \kappa_t - E_{t-1} \kappa_t \) denote the innovation to \( \kappa_t \). The expectational difference equation (A.2.6) can be rewritten as a stochastic difference equation in terms of \( \varepsilon_t \):

\[
\kappa_{t+1} = F(\kappa_t) + \varepsilon_{t+1}, \quad \text{with} \quad E_t \varepsilon_{t+1} = 0, \tag{A.2.9}
\]

where \( \varepsilon_t \) must be a martingale difference process. Now take any \( \kappa_t \) in the range \( 0 \leq \kappa_t \leq 1 \) with \( \kappa_t \neq \kappa \). The first case to consider is \( \kappa_t < \kappa \). As \( F(\kappa) \) is a concave function, \( F(\kappa) \) satisfies the following bound for any \( 0 \leq \kappa \leq \kappa \):

\[
F(\kappa) \leq \kappa + F'(1)(\kappa - \kappa), \tag{A.2.10}
\]

which uses \( F(\kappa) = \kappa \). It is known that \( F'(1) > 1 \).

Since \( E_t \varepsilon_{t+1} = 0 \), there must always be positive probability realizations of \( \varepsilon_{t+1} \) with \( \varepsilon_{t+1} \leq 0 \) irrespective of past realizations of \( \varepsilon_t \). Using (A.2.9), it follows that there must be positive probability realizations of \( \kappa_{t+1} \) such that \( \kappa_{t+1} \leq F(\kappa_t) \). Together with (A.2.10) and \( \kappa_t < \kappa \) it follows that \( \kappa_{t+1} \leq \kappa + F'(1)(\kappa_t - \kappa) \) and hence \( \kappa_{t+1} - \kappa \leq F'(1)(\kappa_t - \kappa) \). With \( F'(1) > 1 \), all these positive probability realizations satisfy \( \kappa_{t+1} < \kappa \). Similar reasoning then implies there are positive probability realizations of \( \kappa_{t+2} \) with \( \kappa_{t+2} \leq \kappa + F'(1)(\kappa_{t+1} - \kappa) \) and hence \( \kappa_{t+2} < \kappa \). Multiplying both sides the earlier inequality by \( F'(1) \) implies \( F'(1)(\kappa_{t+1} - \kappa) \leq (F'(1))^2(\kappa_t - \kappa) \), and putting this together with the inequality for \( \kappa_{t+2} \) leads to \( \kappa_{t+2} - \kappa \leq (F'(1))^2(\kappa_t - \kappa) \). Proceeding by induction, for any finite \( t \geq 1 \) there must be positive probability realizations of \( \kappa_{t+\ell} \) with:

\[
\kappa_{t+\ell} - \kappa \leq (F'(1))^{\ell}(\kappa_t - \kappa). \tag{A.2.11}
\]

Since \( F'(1) > 1 \), the term \((F'(1))^\ell\) grows without bound as \( \ell \) increases. Hence for any \( \kappa_t < \kappa \), the inequality above shows that will be positive probability realizations of \( \kappa_{t+\ell} \) with \( \kappa_{t+\ell} < 0 \) for some finite \( \ell \), which is outside the admissible range. This rules out any equilibrium with \( \kappa_t < \kappa \).

The second case to consider is \( \kappa_t > \kappa \). Using the concavity property of \( F(\kappa) \), for any \( \kappa \leq \kappa \leq 1 \):

\[
F(\kappa) \geq \kappa + F'(1)(\kappa - \kappa), \quad \text{where} \quad F = \frac{F(1) - \kappa}{1 - \kappa} > 1. \tag{A.2.11}
\]

The first inequality follows because the concave function \( F(\kappa) \) intersects the straight line \( \kappa + F'(1)(\kappa - \kappa) \) at both \( \kappa = 1 \) and \( \kappa = \kappa \) (using \( F(\kappa) = \kappa \)). The bound on the constant \( F \) follows from \( F'(1) > 1 \), which must be case since the only fixed point of \( F(\kappa) \) other than \( \kappa \) is strictly greater than one.

Using the martingale difference property of \( \varepsilon_{t+1} \) \((E_t \varepsilon_{t+1} = 0) \), there must be positive probability realizations of \( \varepsilon_{t+1} \) with \( \varepsilon_{t+1} \geq 0 \) irrespective of past realizations of \( \varepsilon_t \). Equation (A.2.9) and the inequality (A.2.11) then imply \( \kappa_{t+1} \geq F(\kappa_t) \geq \kappa + F'(1)(\kappa_t - \kappa) \) as \( \kappa_t > \kappa \), and hence \( \kappa_{t+1} - \kappa \geq F'(1)(\kappa_t - \kappa) \). Following the same method as the case of \( \kappa_t < \kappa \), the argument proceeds by induction to deduce that for any finite \( \ell \), there must be positive probability realizations of \( \kappa_{t+\ell} \) with:

\[
\kappa_{t+\ell} - \kappa \geq (F'(1))^\ell(\kappa_t - \kappa). \tag{A.2.11}
\]

The term \((F'(1))^\ell\) grows without bound as \( \ell \) increases because \( F > 1 \). Hence for any \( \kappa_t > \kappa \), the inequality shows that there will be positive probability realizations of \( \kappa_{t+\ell} \) with \( \kappa_{t+\ell} > 1 \) for some finite \( \ell \), which is outside the admissible range and therefore rules out any equilibrium with \( \kappa_t > \kappa \).
Finally, if \( \kappa_t = \kappa \) and the realization of \( \varepsilon_{t+1} \) is non-zero then \( \kappa_{t+1} \neq \kappa \), and subsequent values of \( \kappa_t \) would fall outside the admissible range with positive probability. This leaves only \( \kappa_t = \kappa \) and \( \varepsilon_{t+1} = 0 \), in which case \( \kappa_{t+1} = \kappa \). Similar reasoning then shows \( \varepsilon_t = 0 \) for all \( t \), in which case \( \kappa_t = \kappa \) for all \( t \). This demonstrates that the only equilibrium of the expectational difference equation (A.2.5) that satisfies \( 0 \leq \kappa_t \leq 1 \) at all times is the (unstable) steady-state solution \( \kappa_t = \kappa \). Substituting this solution into (A.2.2) immediately confirms the claims in (2.22).

Since \( \kappa \) is the smaller root of the quadratic equation \( G(\kappa) = 0 \) in (A.2.7), and as the product of both roots is \( 1/\delta^2 \), the quadratic formula for the larger root implies:

\[
\frac{1}{\delta^2 \kappa} = \frac{(1 + \delta^2 + \delta \theta) + \sqrt{(1 + \delta^2 + \delta \theta)^2 - 4\delta^2}}{2\delta^2}.
\]

Note that:

\[
(1 + \delta^2 + \delta \theta)^2 - 4\delta^2 = (1 + \delta^2 + \delta \theta)^2 - (2\delta)^2 = (1 - 2\delta + \delta^2 + \delta \theta)(1 + 2\delta + \delta^2 + \delta \theta) = ((1 - \delta)^2 + \delta \theta)((1 + \delta)^2 + \delta \theta),
\]

by making use of the formula for the difference of two squares. Using the equation above together with (A.2.12) yields the formula for \( \kappa \) in (2.23). It has already been shown that \( \kappa \) must satisfy \( 0 < \kappa < 1 \).

Differentiating the function \( G(\kappa) \) from (A.2.7) with respect to the parameters \( \delta \) and \( \theta \), and with respect to \( \kappa \) itself, all evaluated at the solution \( \kappa = \kappa \):

\[
\frac{\partial G(\kappa)}{\partial \delta} = -\kappa(\theta + 2\delta(1 - \kappa)) < 0, \quad \frac{\partial G(\kappa)}{\partial \theta} = -\delta \kappa < 0, \quad \text{and} \quad \frac{\partial G(\kappa)}{\partial \kappa} < 0,
\]

where the first inequality makes use of \( 0 < \kappa < 1 \) and the final inequality follows from \( G''(\kappa) = 2\delta^2 > 0 \), and \( \kappa \) being the smaller root of \( G(\kappa) = 0 \). Since the value of \( \kappa \) satisfies the equation \( G(\kappa) = 0 \), the effects on \( \kappa \) of changing the parameters \( \delta \) and \( \theta \) can be obtained by total differentiation:

\[
\frac{\partial \kappa}{\partial \delta} = -\frac{\partial G(\kappa)}{\partial \delta} \bigg|_{\delta} < 0, \quad \text{and} \quad \frac{\partial \kappa}{\partial \theta} = -\frac{\partial G(\kappa)}{\partial \theta} \bigg|_{\theta} < 0.
\]

This confirms the claims in the proposition and completes the proof.

### A.3 Proof of Step 3

Substituting the ex-post real return on nominal bonds from (2.19e) into the bond Euler equation of the young in (2.19d):

\[
\frac{1}{c_{y,t}} = \delta E_t \left[ \frac{1 + i_t}{(1 + \pi_{t+1})(1 + g_{t+1})c_{m,t+1}} \right].
\]

The accounting identity for inflation in (2.19a) can be used to write \( (1 + \pi_{t+1})(1 + g_{t+1}) = (1 + \pi_{t+1})h_t/h_{t+1} \). Substituting this into the equation above and rearranging:

\[
\frac{h_t}{c_{y,t}} = \delta E_t \left[ \left( \frac{1 + i_t}{1 + \pi_{t+1}} \right) \frac{h_{t+1}}{c_{m,t+1}} \right].
\]  

[A.3.1]

Step 1 shows that \( l_t = \delta c_{y,t} \), and Step 2 shows that \( h_t = (1 - k)l_t \) and \( c_{y,t} = kl_t \) for a constant \( k \) satisfying \( 0 < k < 1 \). Note that the proofs of these results are valid even in cases where some or all of these variables are zero. It follows that:

\[
\frac{h_t}{c_{y,t}} = \frac{(1 - k)l_t}{kl_t} = \frac{1 - \kappa}{\kappa}, \quad \text{and} \quad \frac{h_{t+1}}{c_{m,t+1}} = \frac{(1 - k)l_{t+1}}{c_{m,t+1}} = \frac{\delta(1 - \kappa)c_{m,t+1}}{c_{m,t+1}} = \delta(1 - \kappa)
\]

Substituting the constant values of these ratios into (A.3.1):

\[
\frac{1 - \kappa}{\kappa} = \delta E_t \left[ \delta(1 - \kappa) \left( \frac{1 + i_t}{1 + \pi_{t+1}} \right) \right].
\]
Multiplying both sides by the non-zero finite constant \( \kappa/(1 - \kappa) \) and using the definition of \( \beta = \delta^2 \kappa \) from (2.25) leads to:

\[
1 = \beta \mathbb{E}_t \left[ \frac{1 + i_t}{1 + \pi_{t+1}} \right],
\]

and rearranging this equation yields the solution for \( i_t \) in (2.25). If \( \pi_{t+1} = -1 \) with positive probability then the expectation \( \mathbb{E}_t \left[ (1 + \pi_{t+1})^{-1} \right] \) is not finite. The right-hand side of (A.3.2) is infinite for any \( i_t > -1 \), and hence cannot be equal to the left-hand side. If \( i_t = -1 \) then the right-hand side of (A.3.2) is zero, and thus cannot equal the left-hand side. This implies that there is no equilibrium when \( \pi_{t+1} = -1 \) with positive probability.

The constant term \( \kappa \) is the smaller root of the quadratic equation \( G(\kappa) = 0 \) defined in the proof of Step 2 (see equation A.2.7). The product of the two roots of the quadratic is \( 1/\delta^2 \), so \( \beta^{-1} = (\delta^2 \kappa)^{-1} \) is the larger root of \( G(\kappa) = 0 \). Since \( G(1) = -\delta \theta < 0 \) and \( G''(\kappa) = 2\delta^2 > 0 \), it follows that \( \beta^{-1} > 1 \). This confirms that \( 0 < \beta < 1 \). A tighter bound on \( \beta \) can be derived by using (A.2.7) to observe that:

\[
G \left( \frac{2\delta + \theta}{2\delta} \right) = \frac{\delta^2 (2\delta + \theta)^2}{4\delta^2} - \frac{(1 + \delta^2 + \delta \theta)(2\delta + \theta)}{2\delta} + 1 = \frac{\delta(2\delta + \theta)^2 - 2(1 + \delta^2 + \delta \theta)(2\delta + \theta) + 4\delta}{4\delta} = \frac{4\delta^3 + 4\delta^2 \theta - \delta \theta^2 - 4\delta - 4\delta^3 - 4\delta^2 \theta - 2\delta - 2\theta^2 \theta + 4\delta}{4\delta} = -\frac{\theta(\delta \theta + 2(1 + \delta^2))}{4\delta} < 0.
\]

As \( G((2\delta + \theta)/2\delta) < 0 \), \( \beta^{-1} \) must be greater than \((2\delta + \theta)/2\delta\), and therefore \( \beta < 2\delta/(2\delta + \theta) \).

Differentiating the function \( G(\kappa) \) from (A.2.7) with respect to the parameters \( \delta, \theta \), and with respect to \( \kappa \) itself, evaluated at \( \kappa = \beta^{-1} \):

\[
\frac{\partial G(\beta^{-1})}{\partial \delta} = \beta^{-2}(2\delta - (2\delta + \theta)\beta) > 0, \quad \frac{\partial G(\beta^{-1})}{\partial \theta} = -\beta^{-1} \delta < 0, \quad \text{and} \quad \frac{\partial G(\beta^{-1})}{\partial \kappa} > 0,
\]

where the first inequality is derived from \( \beta < 2\delta/(2\delta + \theta) \), and the final inequality follows from \( G''(\kappa) = 2\delta^2 > 0 \) and \( \kappa = \beta^{-1} \) being the larger root of \( G(\kappa) = 0 \). Using these results, the effects of \( \delta \) and \( \theta \) on \( \beta \) can be obtained by total differentiation of \( G(\beta^{-1}) = 0 \):

\[
\frac{\partial \beta}{\partial \delta} = \beta^2 \frac{\partial G(\beta^{-1})}{\partial \delta}, \quad \frac{\partial \beta}{\partial \theta} = \beta \frac{\partial G(\beta^{-1})}{\partial \theta}, \quad \frac{\partial \beta}{\partial \kappa} < 0.
\]

This confirms that \( \beta \) is increasing in \( \delta \) and decreasing in \( \theta \).

The ex-post nominal return on bonds follows immediately from (2.10) and (2.17):

\[
R_t = \frac{1}{1 + \pi_{t-1}} - 1 = i_{t-1},
\]

confirming the formula in (2.26). The ex-post nominal return on housing \( \hat{R}_t \) can be stated as follows by using (2.15) and (2.16):

\[
1 + \hat{R}_t = 1 + \frac{V_t - V_{t-1}}{V_{t-1}} + (\gamma - 1) \frac{V_t}{V_{t-1}} + \frac{Z_t}{V_t} = \gamma \frac{V_t}{V_{t-1}} \left( 1 + \frac{Z_t}{\gamma V_t} \right). \tag{A.3.3}
\]

Making use of the formula for imputed rent \( Z_t \) in (2.16), the definitions in (2.14), and equations (2.1), (2.4), and (2.12):

\[
\frac{Z_t}{\gamma V_t} = \frac{P_tC_m,t \Theta'(H_{m,t})}{\gamma V_t} = \left( \frac{P_t Y_t}{V_t H_t} \right) \left( \frac{H_t C_{m,t}}{\gamma Y_t} \right) \Theta'(1) = \theta \left( \frac{P_t Y_t}{V_t H_t} \right) \left( \frac{I_{m,t} C_{m,t}}{Y_t} \right) = \frac{\theta c_{m,t}}{h_t}, \tag{A.3.4}
\]

noting that \( H_{m,t} = 1 \) in equilibrium, and using the definition of \( \theta \) from (2.19c). Taking the results from (2.21) and (2.22), it follows that:

\[
\frac{\theta c_{m,t}}{h_t} = \frac{\theta c_{m,t}}{(1 - \kappa) h_t} = \frac{\theta c_{m,t}}{\delta(1 - \kappa) c_{m,t}} = \frac{\theta}{\delta(1 - \kappa)},
\]

and substituting this into (A.3.4) implies:

\[
1 + \frac{Z_t}{\gamma V_t} = 1 + \frac{\theta}{\delta(1 - \kappa)} = \frac{\delta^2 \kappa(1 - \kappa) + \delta \theta \kappa}{\delta^2 \kappa(1 - \kappa)}. \tag{A.3.5}
\]
It is shown in the proof of Step 2 that \( \kappa \) is a root of the quadratic equation in (A.2.7), and hence:
\[
\delta \theta \kappa = 1 + \delta^2 \kappa^2 - (1 + \delta^2) \kappa.
\]
Using this formula together with (A.3.5) leads to:
\[
1 + \frac{Z_t}{\gamma \nu_t} = \frac{\delta^2 \kappa - \delta^2 \kappa^2 + 1 + \delta^2 \kappa^2 - \kappa - \delta^2 \kappa}{\delta^2 \kappa (1 - \kappa)} = \frac{1 - \kappa}{\delta^2 \kappa (1 - \kappa)} = \frac{1}{\beta}.
\]
Taking the definition of \( \beta \) from (A.3.2). Noting that the definition of \( \pi_t \) in (2.15) implies \( 1 + \pi_t = \frac{\gamma \nu_t}{V_t} \), substitution of the equation above into (A.3.3) confirms the expression for \( \hat{\delta}_t \) in (2.26).

Substituting the expressions for \( R_{t+1} \) and \( \hat{R}_{t+1} \) from (A.3.3) into the definition of the expected excess return \( \xi_t \) from (2.17):
\[
\xi_t = \frac{E_t \left[ \frac{1 + \pi_{t+1}}{\beta} \right]}{1 + i_t} - 1.
\]
The next step is to substitute the equilibrium interest rate \( i_t \) from (2.25) into the above:
\[
\xi_t = \frac{E_t [1 + \pi_{t+1}]}{\beta E_t [(1 + \pi_{t+1})^{-1}]} - 1 = E_t [1 + \pi_{t+1}] E_t [(1 + \pi_{t+1})^{-1}] - 1,
\]
which confirms the formula given in (2.26).

Since \( (1 + \pi_{t+1})^{-1} \) is a strictly convex function of \( 1 + \pi_{t+1} \) everywhere, Jensen’s inequality implies \( E_t [(1 + \pi_{t+1})^{-1}] \geq (E_t [1 + \pi_{t+1}])^{-1} \), and \( E_t [(1 + \pi_{t+1})^{-1}] > (E_t [1 + \pi_{t+1}])^{-1} \) if \( \pi_{t+1} \) has a non-degenerate probability distribution. It follows that \( \xi_t > 0 \) with a non-degenerate distribution. A mean-preserving spread of \( \pi_{t+1} \) leaves \( E_t [1 + \pi_{t+1}] \) unchanged while increasing \( E_t [(1 + \pi_{t+1})^{-1}] \) owing to \( (1 + \pi_{t+1})^{-1} \) being a convex function of \( \pi_{t+1} \), which implies \( \xi_t \) increases. If \( \pi_{t+1} \) is perfectly predictable then \( E_t [(1 + \pi_{t+1})^{-1}] = (E_t [1 + \pi_{t+1}])^{-1} \) and it follows that \( \xi_t = 0 \), completing the proof.

### A.4 Proof of Step 4

Net worth \( n_t = 1 + (1 - d_t) h_t \) from (2.28) is equal to \( n_t = A(\nu_t) \) in equilibrium, where \( A(\nu) \) is the function implicitly defined in (2.32) (compare with 2.29), and \( \nu_t \) denotes the measure of financial conditions defined in (2.31). Collecting terms in \( A(\nu) \) on the left-hand side of equation (2.32):
\[
\left( 1 - \delta \frac{1 - \kappa}{1 + \delta} \lambda \beta - \delta \frac{1 - \kappa}{1 + \delta} \lambda \beta \frac{1}{1 + \delta} \right) A(\nu) = 1.
\]
Since \( 0 < \delta < \infty, 0 < \kappa < 1, \) and financial conditions \( \nu \) satisfy \( -\infty < \nu \leq 1 \), the term in parentheses above is strictly positive, so there exists a unique solution for \( A(\nu) \):
\[
A(\nu) = \frac{1}{1 - \delta (1 - \kappa) \lambda \beta - \delta (1 - \kappa) \lambda \beta \frac{1}{1 + \delta} \beta \nu},
\]
which is positive for all admissible values of \( \nu \), confirming the property stated in (2.33). Dividing numerator and denominator of the above by the denominator of the expression for \( \alpha \) in (2.34) allows the function \( A(\nu) \) to be written in terms of \( \alpha \):
\[
A(\nu) = \frac{\alpha \beta (1 + \delta)}{\beta (1 - \nu)},
\]
where (2.24) is used to eliminate the coefficient \( \lambda \). Using \( n_t = A(\nu_t) \) and the equation for lending \( l_t \) in terms of net worth from (2.28), the equilibrium value of \( l_t \) given in (2.35) is obtained. Similarly, using the equation for housing values \( h_t \) in terms of net worth from (2.29), the expression for \( h_t \) in (2.35) follows with reference to the definition of \( \lambda \) from (2.24). Since \( A(\nu) \) is non-negative for all possible values of \( \nu \), net worth \( n_t = A(\nu_t) \) is unambiguously non-negative. Using the definition of \( n_t \) from (2.28), it can be seen that the no-default condition (2.20) always holds.

The coefficient \( \alpha \) appearing in the solutions (2.35) is taken from equation (2.34). Multiplying numerator
and denominator by $\beta/\lambda$ and substituting the expressions for $\lambda$ and $\beta$ from (2.22) and (2.25):

$$
\alpha = \frac{\delta(1-\kappa)}{1 + \delta} = \frac{\delta(1-\kappa)}{1 + \delta} + \frac{\delta(1-\kappa)}{1 + \delta} + \left(\frac{1+\delta-\delta(1-\kappa)}{1 + \delta}\right) \delta^2 \kappa(1 - \kappa) .
$$

Cancelling $\delta(1-\kappa)/(1+\delta)$ from numerator and denominator and simplifying yields an alternative expression for $\alpha$:

$$
\alpha = \frac{1}{1 + \delta \kappa (1 + \delta - \delta \kappa)} = \frac{1}{1 + \delta \kappa + \delta^2 \kappa^2} ,
$$

which confirms the formula for $\alpha$ given in (2.35). Since $\delta > 0$ and $\kappa > 0$, it follows that $\alpha < 1$. Using (2.23), note also:

$$
\delta \kappa = \frac{2 \delta}{2 \delta + (1-\delta)^2 + \delta \theta + \sqrt{(1-\delta)^2 + \delta \theta)((1+\delta)^2 + \delta \theta)} < 1 ,
$$

which uses $1 + \delta^2 = 2 \delta + (1-\delta)^2$. This implies that $1 + \delta \kappa + \delta^2 \kappa^2 < 3$, and hence $\alpha < 1/3$, confirming the bounds on $\alpha$ in (2.35).

Next, the definition of financial conditions $\nu_t$ in (2.31) implies:

$$
1 - \alpha \nu_t = 1 - \alpha \left(1 - \beta \frac{1 + i_{t-1}}{1 + \pi_t}\right) = \frac{(1 - \alpha)(1 + \pi_t) + \alpha \beta (1 + i_{t-1})}{1 + \pi_t} , \quad [A.4.1]
$$

Substituting the equilibrium value of $\mu_t$ from (2.35) into the second accounting identity from (2.19a):

$$
\frac{1 + \pi_t}{1 + \omega_t} = \frac{(1 + g_t)(1 - \alpha \nu_{t-1})}{1 - \alpha \nu_t} .
$$

Using the expression for $1 - \alpha \nu_t$ from (A.4.1) in the above:

$$
\frac{1 + \pi_t}{1 + \omega_t} = \frac{(1 - \alpha)(1 + \pi_{t-1}) + \alpha \beta (1 + i_{t-2})}{1 + \pi_t} = \frac{(1 + \pi_t)(1 + g_t)((1 - \alpha)(1 + \pi_{t-1}) + \alpha \beta (1 + i_{t-2}))}{1 + \pi_{t-1}} .
$$

By cancelling the common term $1 + \pi_t$ from both sides and rearranging, the equation for goods-price inflation $\omega_t$ in (2.36) is confirmed.

Finally, as Step 2 shows that $\kappa$ is decreasing in $\theta$, it follows immediately that $\alpha$ is increasing in housing preference $\theta$. With reference to (A.2.7) in the proof of Step 2, $\kappa$ is the smaller root of the quadratic equation $\delta^2 \kappa^2 - (1 + \delta^2 + \delta \theta) \kappa + 1 = 0$. Equivalently, $\delta \kappa$ is the smaller root of the quadratic equation $Q(\phi) = 0$, where $Q(\phi)$ is:

$$
Q(\phi) = \phi^2 - (\delta + \delta^{-1} + \theta) \phi + 1 .
$$

Note the following:

$$
\frac{\partial Q(\delta \kappa)}{\partial (\delta + \delta^{-1})} = -\delta \kappa < 0 , \quad \text{and} \quad \frac{\partial Q(\delta \kappa)}{\partial \phi} = 2 \delta \kappa - (\delta + \delta^{-1} + \theta) < 0 ,
$$

where the former follows from $\delta \kappa > 0$ and the latter from $Q''(\phi) = 2 > 0$ and $\phi = \delta \kappa$ being the smaller root of $Q(\phi) = 0$. By differentiating the equation $Q(\delta \kappa) = 0$, the two results above imply:

$$
\frac{\partial (\delta \kappa)}{\partial (\delta + \delta^{-1})} = -\frac{\partial Q(\delta \kappa)}{\partial \delta} < 0 . \quad [A.4.2]
$$

Now consider the derivatives of the expression $\delta + \delta^{-1}$ for any $\delta > 0$:

$$
\frac{\partial (\delta + \delta^{-1})}{\partial \delta} = \frac{\delta^2 - 1}{\delta^2} , \quad \text{and} \quad \frac{\partial^2 (\delta + \delta^{-1})}{\partial \delta^2} = \frac{2}{\delta^3} > 0 ,
$$

which shows that $\delta + \delta^{-1}$ is decreasing for $\delta < 1$ and increasing for $\delta > 1$, that is, $\delta + \delta^{-1}$ is increasing in the distance of $\delta$ from 1. As $\alpha = 1/(1 + (\delta \kappa) + (\delta \kappa)^2)$ is decreasing in $\delta \kappa$, this result together with (A.4.2) implies that $\alpha$ is increasing in the distance of $\delta$ from 1, completing the proof.
A.5 Proof of Result 1

Take as given a state-contingent path of nominal house-price inflation $\pi_t$. According to Step 3, the equilibrium conditions imply that the nominal interest rate must satisfy (2.25). Substituting the equilibrium value of $i_{t-1}$ into the definition of financial conditions $\nu_t$ from (2.31) implies that financial conditions must be given by (2.37) in equilibrium. According to Step 3, no equilibrium exists in the case where $\pi_t = -1$ with positive probability. Using (2.31), this would correspond to a realization of financial conditions with $\nu_t = -\infty$, therefore $-\infty < \nu_t \leq 1$ with probability one is necessary for the existence of an equilibrium.

Now take the stochastic process for financial conditions $\nu_t$ implied by (2.37). Since $E_{t-1}\nu_t = 0$ follows immediately, financial conditions must be a martingale difference sequence in equilibrium. Rearranging equation (2.37) implies:

$$\left(1 + \pi_{t+1}\right)E_t \left(1 + \pi_{t+1}\right)^{-1} - 1 = \frac{\nu_{t+1}}{1 - \nu_{t+1}}.$$ 

By taking expectations conditional on date $t$ information and using the equilibrium housing risk premium from (2.26), the expression for $\xi_t$ in (2.38) is confirmed. Since $\nu_{t+1}/(1 - \nu_{t+1})$ is a strictly convex function of $\nu_{t+1}$, Jensen’s inequality implies for any non-degenerate distribution of $\nu_{t+1}$ conditional on what is known at date $t$:

$$E_t \left[ \frac{\nu_{t+1}}{1 - \nu_{t+1}} \right] > \frac{E_t \nu_{t+1}}{1 - E_t \nu_{t+1}} = 0,$$

which uses $E_t \nu_{t+1} = 0$. The risk premium $\xi_t$ is therefore strictly positive for any $\nu_{t+1}$ that is not perfectly predictable and zero otherwise. Moreover, the strict convexity of $\nu_{t+1}/(1 - \nu_{t+1})$ also means that any spread of $\nu_{t+1}$ around its conditional mean $E_t \nu_{t+1} = 0$ increases $\xi_t$.

Now consider the ex-post real returns $r_{t+1}$ and $\hat{r}_{t+1}$ on nominal bonds and housing. By using the first accounting identity in (2.19a) and the expressions for $d_{t+1}$, $h_{t+1}$, and $l_t$ from (2.31) and (2.35):

$$\frac{1 + r_{t+1}}{1 + \nu_{t+1}} = \frac{\lambda}{1 - \nu_{t+1}} \frac{\beta}{1 - \alpha \nu_{t+1}} = \frac{(1 - \alpha \nu_t)(1 - \nu_{t+1})}{\beta(1 - \alpha \nu_{t+1})},$$

which confirms the claim in (2.38). Using the formula for the real return on housing $\hat{r}_{t+1}$ from (2.16), the expression for $\hat{R}_{t+1}$ from (2.26), the second accounting identity from (2.19a), and the equilibrium value of $h_t$ from (2.35):

$$\frac{1 + \hat{r}_{t+1}}{1 + \nu_{t+1}} = \frac{1 + \hat{R}_{t+1}}{(1 + \alpha \nu_{t+1})(1 + \nu_{t+1})} = \frac{1 + \pi_{t+1}}{\beta(1 + \alpha \nu_{t+1})(1 + \nu_{t+1})} = \frac{h_{t+1}}{\beta h_t} \frac{\lambda}{1 - \nu_{t+1}} \frac{\beta}{1 - \alpha \nu_{t+1}} = \frac{(1 - \alpha \nu_t)}{\beta (1 - \alpha \nu_{t+1})},$$

corning the remaining claim in (2.38). Taking expectations of the expressions for $r_{t+1}$ and $\hat{r}_{t+1}$ in (2.38) conditional on information available at date $t$:

$$E_t \left[ \frac{1 + r_{t+1}}{1 + \nu_{t+1}} \right] = \frac{1 - \alpha \nu_t}{\beta} E_t \left[ \frac{1 - \nu_{t+1}}{1 - \alpha \nu_{t+1}} \right], \text{ and } E_t \left[ \frac{1 + \hat{r}_{t+1}}{1 + \nu_{t+1}} \right] = \frac{1 - \alpha \nu_t}{\beta} E_t \left[ \frac{1}{1 - \alpha \nu_{t+1}} \right], \quad [A.5.1]$$

since $\nu_t$ is known at date $t$. As $(1 - \nu_{t+1})/(1 - \alpha \nu_{t+1}) = 1 - (1 - \alpha)\nu_{t+1}/(1 - \alpha \nu_{t+1})$ is a strictly concave function of $\nu_{t+1}$, a spread of $\nu_{t+1}$ around $E_t \nu_{t+1} = 0$ decreases $E_t[(1 - \nu_{t+1})/(1 - \alpha \nu_{t+1})]$ and also $E_t[(1 + r_{t+1})/(1 + \nu_{t+1})]$ given that $(1 - \alpha \nu_t)/\beta$ is always positive. Similarly, since $1/(1 - \alpha \nu_{t+1})$ is a strictly convex function of $\nu_{t+1}$, a spread of $\nu_{t+1}$ around its zero conditional mean increases $E_t[1/(1 - \alpha \nu_{t+1})]$ and therefore also $E_t[(1 + \hat{r}_{t+1})/(1 + \nu_{t+1})]$ using the equation above. Moreover, for any non-degenerate conditional distribution of $\nu_{t+1}$, Jensen’s inequality and $E_t \nu_{t+1} = 0$ imply:

$$E_t \left[ \frac{1 - \nu_{t+1}}{1 - \alpha \nu_{t+1}} \right] < \frac{1}{1 - \alpha E_t \nu_{t+1}} = 1, \text{ and } E_t \left[ \frac{1}{1 - \alpha \nu_{t+1}} \right] > \frac{1}{1 - \alpha E_t \nu_{t+1}} = 1.$$ 

As $(1 - \alpha \nu_t)/\beta$ is always positive, substituting these inequalities into (A.5.1) implies:

$$E_t \left[ \frac{1 + r_{t+1}}{1 + \nu_{t+1}} \right] < \frac{1 - \alpha \nu_t}{\beta}, \text{ and } E_t \left[ \frac{1 + \hat{r}_{t+1}}{1 + \nu_{t+1}} \right] > \frac{1 - \alpha \nu_t}{\beta}.$$
Taking unconditional expectations of both sides and applying the law of iterated expectations:

\[ \mathbb{E} \left[ \frac{1 + r_{t+1}}{1 + g_{t+1}} \right] < \frac{1 - \alpha \mathbb{E} \nu_t}{\beta} = \frac{1}{\beta}, \quad \text{and} \quad \mathbb{E} \left[ \frac{1 + \hat{r}_{t+1}}{1 + g_{t+1}} \right] > \frac{1 - \alpha \mathbb{E} \nu_t}{\beta} = \frac{1}{\beta}, \]

which therefore confirm the claims in (2.39) using \( \mathbb{E} \nu_t = 0 \). If there were no uncertainty so that \( \nu_{t+1} = 0 \) with probability one then the expectations on the right-hand sides of the equations in (A.5.1) would disappear and taking unconditional expectations would imply that the claims in (2.39) hold with equality instead.

Next, summing the bond and housing returns from (2.38) multiplied by \( \alpha \) and \( 1 - \alpha \) respectively:

\[ \alpha \frac{1 + r_{t+1}}{1 + g_{t+1}} + (1 - \alpha) \frac{1 + \hat{r}_{t+1}}{1 + g_{t+1}} = \frac{\alpha(1 - \alpha \nu_t)(1 - \nu_{t+1}) + (1 - \alpha)(1 - \alpha \nu_t)}{\beta(1 - \alpha \nu_{t+1})} = \frac{1 - \alpha \nu_t}{\beta}, \quad \text{[A.5.2]} \]

and taking unconditional expectations of both sides yields:

\[ \alpha \mathbb{E} \left[ \frac{1 + r_{t+1}}{1 + g_{t+1}} \right] + (1 - \alpha) \mathbb{E} \left[ \frac{1 + \hat{r}_{t+1}}{1 + g_{t+1}} \right] = \mathbb{E} \left[ \frac{1 - \alpha \nu_t}{\beta} \right] = \frac{1 - \alpha \mathbb{E} \nu_t}{\beta} = \frac{1}{\beta}. \]

This claim holds for any non-degenerate or degenerate probability distribution of financial conditions. Multiplying both sides of (A.5.2) by \( 1 + g_{t+1} \):

\[ \alpha(1 + r_{t+1}) + (1 - \alpha)(1 + \hat{r}_{t+1}) = \frac{(1 - \alpha \nu_t)(1 + g_{t+1})}{\beta}. \]

Taking expectations of both sides conditional on information available at date \( t \):

\[ \mathbb{E} \left[ (1 + E_t r_{t+1}) + (1 - \alpha)(1 + E_t \hat{r}_{t+1}) \right] = \mathbb{E}_t \left[ \frac{(1 - \alpha \nu_t)(1 + g_{t+1})}{\beta} \right] = \frac{(1 - \alpha \nu_t)}{\beta}(1 + E_t g_{t+1}), \]

noting that \( \nu_t \) is known at date \( t \). By dividing both sides by \( 1 + E_t g_{t+1} \) and using the definitions of the ex-ante real returns \( \nu_t \) and \( \hat{\nu}_t \) on bonds and houses:

\[ \frac{\alpha}{1 + E_t g_{t+1}} \frac{1 + \nu_t}{\beta} + (1 - \alpha) \frac{1 + \hat{\nu}_t}{1 + E_t g_{t+1}} = \frac{1 - \alpha \nu_t}{\beta}, \]

and by taking unconditional expectations of both sides of this equation and using \( \mathbb{E} \nu_t = 0 \):

\[ \alpha \mathbb{E} \left[ \frac{1 + \nu_t}{1 + E_t g_{t+1}} \right] + (1 - \alpha) \mathbb{E} \left[ \frac{1 + \hat{\nu}_t}{1 + E_t g_{t+1}} \right] = \frac{1 - \alpha \mathbb{E} \nu_t}{\beta} = \frac{1}{\beta}. \]

This confirms the result stated in (4.40).

Using the formula for \( \beta \) from (2.25) and the equilibrium value of \( l_t \) from (2.35):

\[ l_t = \frac{\alpha \delta^2 \kappa}{1 - \alpha \nu_t}. \]

**Step 1** established that \( c_{m,t} = l_t / \delta \), which immediately leads to the expression for \( c_{m,t} \) in (2.41) using the equation for \( l_t \) above. Similarly, **Step 2** showed that \( c_{y,t} = k l_t \), and substituting for \( l_t \) from the above equation confirms the expression for \( c_{y,t} \) given in (2.41). Using the budget identity for \( c_{o,t} \) from (2.19b), and the expressions for \( d_t \) and \( h_t \) from (2.31) and (2.35):

\[ c_{o,t} = d_t h_t = \frac{\lambda(1 - \nu_t)}{\beta} \frac{\alpha \beta}{\lambda(1 - \alpha \nu_t)} = \frac{\alpha(1 - \nu_t)}{1 - \alpha \nu_t}, \]

which confirms the formula for \( c_{o,t} \) in (2.41). Given that \( \mathbb{E}_{t-1} \nu_t = 0 \), a degenerate probability distribution of financial conditions \( \nu_t \) implies that \( \nu_t = 0 \) with probability one, in which case the equilibrium levels of consumption would be \( c_{y,t} = c^*_y = \alpha \delta^2 \kappa, \ c_{m,t} = c^*_m = \alpha \delta \kappa, \) and \( c_{o,t} = c^*_o = \alpha \).

Observe that \( 1/(1 - \alpha \nu_t) \) is a strictly convex function of \( \nu_t \), and \( \alpha(1 - \nu_t)/\alpha \nu_t = 1/(1 - \alpha) \) is a strictly concave function of \( \nu_t \). Referring to the solutions in (2.41), this implies \( c_{y,t} \) and \( c_{m,t} \) are strictly convex functions of \( \nu_t \), and \( c_{o,t} \) is a strictly concave function of \( \nu_t \). For any non-degenerate probability distribution of \( \nu_t \), Jensen’s inequality and \( \mathbb{E} \nu_t = 0 \) imply that \( \mathbb{E} c_{y,t} > c^*_y, \mathbb{E} c_{m,t} > c^*_m, \) and \( \mathbb{E} c_{o,t} < c^*_o \), confirming the claims in (2.42). Similarly, the strict convexity and concavity of these functions of \( \nu_t \) implies that \( \mathbb{E} c_{y,t} \) and \( \mathbb{E} c_{m,t} \) increase with a spread of \( \nu_t \), while \( \mathbb{E} c_{o,t} \) decreases. This completes the proof.
A.6 Proof of Step 5

Suppose that the interest-rate rule (2.44) is used for some \( \xi > 1 \), an exogenous martingale difference sequence \( \mu_t (E_{t-1} \mu_t = 0) \) satisfying \(-1 < \mu_t < \infty \) with probability one, and an exogenous predictable sequence \( \pi_t^* (\pi_t^* = E_{t-1} \pi_t^*) \) satisfying \(-1 < \pi_t^* < \infty \) with probability one and not converging to \(-1 \) or \( \infty \) as \( t \to \infty \). The equilibrium selection criterion \( \pi_t < \infty \) with probability one and no positive-probability realizations of \( \pi_t \) converging to \(-1 \) or \( \infty \) is imposed.

Equilibrium requires that the nominal interest rate \( i_t \) is related to the probability distribution of \( \pi_{t+1} \) according to (2.25). Combining equations (2.25) and (2.44) to eliminate \( i_t \):

\[
\frac{1}{\beta E_t \left[ (1 + \pi_{t+1})^{-1} \right]} - 1 = \frac{1 + \pi_t}{\beta E_t \left[ (1 + \pi_{t+1})^{-1} \right]} \max \left\{ 1 - \zeta \left( \frac{1 + \pi_t - \pi_t^*}{1 + \pi_t} \right) + \mu_t \right\} - 1.
\]

[A.6.1]

Make the following definition of \( \Pi_t \) in terms of \( \pi_t, \pi_t^* \), and \( \mu_t \):

\[
\Pi_t = \frac{1 + \pi_t - \pi_t^*}{1 + \pi_t} + \mu_t, \quad \text{noting} \quad (1 + \pi_t)^{-1} = \frac{(1 - \Pi_t + \mu_t)E_{t-1} \left[ (1 + \mu_t)^{-1} \right]}{1 + \pi_t}.
\]

[A.6.2]

Substituting the definition above into equation (A.6.1) to write it in terms of \( \Pi_t \):

\[
\frac{1}{\beta E_t \left[ (1 - \Pi_{t+1} + \pi_{t+1})E_t \left[ (1 + \mu_t)^{-1} \right] \right]} \equiv \frac{1 + \pi_t^*}{\beta E_t \left[ (1 + \mu_t)^{-1} \right]} \max \left\{ 1 - \zeta \Pi_t, 0 \right\}.
\]

and since \( \pi_t^* \) is known at date \( t \), it can be removed from the conditional expectation operator on the left-hand side. By cancelling the positive and finite terms \( \beta, 1 + \pi_t^* \), and \( E_t \left[ (1 + \mu_t)^{-1} \right] \) from both sides, the equation is equivalent to:

\[
E_t \left[ 1 - \Pi_{t+1} + \mu_{t+1} \right] = \max \{ 1 - \zeta \Pi_t, 0 \}.
\]

Using \( E_t \mu_{t+1} = 0 \) because \( \mu_t \) is a martingale difference sequence, the equation can be written as follows:

\[
E_t \Pi_{t+1} = 1 - \max \{ 1 - \zeta \Pi_t, 0 \} = \min \{ \zeta \Pi_t, 1 \},
\]

which is an expectational difference equation in \( \Pi_t \):

\[
E_t \Pi_{t+1} = \mathcal{H}(\Pi_t), \quad \text{where} \quad \mathcal{H}(\Pi) \equiv \min \{ \zeta \Pi, 1 \}.
\]

[A.6.3]

The function \( \mathcal{H}(\Pi) \) is weakly increasing and weakly concave in \( \Pi \).

Any solution of (A.6.3) must have that \( \varepsilon_t = \Pi_t - E_{t-1} \Pi_t \) is a martingale difference sequence \( (E_{t-1} \varepsilon_t = 0) \), hence a solution must always satisfy the following stochastic difference equation for some martingale difference sequence \( \varepsilon_t \):

\[
\Pi_t = \mathcal{H}(\Pi_{t-1}) + \varepsilon_t, \quad \text{where} \quad E_{t-1} \varepsilon_t = 0,
\]

[A.6.4] which follows from (A.6.3) because \( E_t \Pi_{t+1} = \Pi_{t+1} - \varepsilon_{t+1} \). House-price inflation \( \pi_t \) must satisfy \(-1 < \pi_t \leq \infty \), recalling that there can be no equilibrium with \( \pi_t = -1 \) according to Step 3. The equilibrium selection criteria also rule out \( \pi_t \) being infinite, or \( \pi_t \) approaching \(-1 \) or \( \infty \) as \( t \to \infty \). Using (A.6.2), these restrictions are equivalent to \(-\infty < \Pi_t < 1 + \mu_t \), \( \Pi_t \) does not approach \(-\infty \) as \( t \to \infty \), and the distance between \( \Pi_t \) and \( 1 + \mu_t \) does not tend to zero as \( t \) becomes large.

Now consider the possibility of an equilibrium with \( \Pi_t > 0 \) at some date \( t \). Since \( E_t \varepsilon_{t+1} = 0 \), there must be positive probability realizations of \( \varepsilon_{t+1} \) with \( \varepsilon_{t+1} \geq 0 \), and hence (A.6.4) implies there must be positive-probability realizations of \( \Pi_{t+1} \) such that \( \Pi_{t+1} \geq \mathcal{H}(\Pi_{t+1}) \). As \( \mathcal{H}(\Pi) \) is weakly increasing, this implies \( \mathcal{H}(\Pi_{t+1}) \geq \mathcal{H}(\Pi_t) \). Similar reasoning reveals there are positive-probability realizations of \( \varepsilon_{t+2} \) with \( \varepsilon_{t+2} \geq 0 \), and therefore \( \Pi_{t+2} \geq \mathcal{H}(\Pi_{t+1}) \geq \mathcal{H}(\Pi_t) \). Proceeding iteratively, for any finite \( \ell > 0 \) there exist positive-probability realizations of \( \varepsilon_{t+1}, \ldots, \varepsilon_{t+\ell} \) such that:

\[
\Pi_{t+\ell} \geq \mathcal{H}^{[\ell]}(\Pi_t),
\]

[A.6.5] where \( \mathcal{H}^{[\ell]}(\Pi) \) denotes composition of the function \( \mathcal{H}(\Pi) \) with itself \( \ell \) times. If \( \mathcal{H}^{[\ell]}(\Pi_t) < 1 \) then the
definition of the function $\mathcal{H}(II)$ in (A.6.3) implies:

$$\mathcal{H}^{[\ell]}(\Pi_t) = \zeta^\ell \Pi_t.$$ 

Therefore, given $\zeta > 1$, with $\Pi_t > 0$, it follows that $\mathcal{H}^{[\ell]}(\Pi_t) = 1$ for finite $\ell > -(\log \Pi_t)/(\log \zeta)$. Hence, using (A.6.5) and the finite $\ell$, there must exist positive-probability realizations of $\Pi_{t+\ell}$ such that $\Pi_{t+\ell} \geq 1$. Since $\zeta > 1$, by using (A.6.3) and (A.6.4), such realizations must feature $\Pi_{t+\ell+1} = 1 + \varepsilon_{t+\ell+1}$ for some $\varepsilon_{t+\ell+1}$ with $E_{t+\ell} \varepsilon_{t+\ell+1} = 0$. In order to satisfy $\Pi_{t+\ell+1} < 1 + \mu_{t+\ell+1}$, the stochastic process $\varepsilon_{t+\ell+1}$ must be such that $\varepsilon_{t+\ell+1} < \mu_{t+\ell+1}$ for all realizations of $\varepsilon_{t+\ell+1}$. But $E_{t+\ell} \mu_{t+\ell+1} = 0$ implies $E_{t+\ell} [\mu_{t+\ell+1} - \varepsilon_{t+\ell+1}] = 0$, so there is no stochastic process $\varepsilon_{t+\ell+1}$ that ensures $\Pi_t < 1 + \mu_t$ with probability one. It follows that there cannot be an equilibrium with $\Pi_t > 0$ for any $t$.

Now consider the possibility of an equilibrium with $\Pi_t < 0$ at some date $t$. For any $\ell \geq 1$, taking expectations of equation (A.6.3) at date $t + \ell$ conditional on period-$t$ information:

$$E_{t} E_{t+\ell} \Pi_{t+\ell+1} = E_{t} \mathcal{H}(\Pi_{t+\ell}) \leq \mathcal{H}(E_{t} \Pi_{t+\ell}),$$

where the upper bound is deduced by applying Jensen’s inequality to the weakly concave function $\mathcal{H}(II)$. The law of iterated expectations then implies:

$$E_{t} \Pi_{t+\ell+1} \leq \mathcal{H}(E_{t} \Pi_{t+\ell}),$$

which holds for all $\ell \geq 0$ (the case of $\ell = 0$ where $E_t \Pi_t = \Pi_t$ is trivial). Since $E_t \Pi_{t+\ell} \leq \mathcal{H}(E_t \Pi_{t+\ell-1})$ and $E_t \Pi_{t+\ell-1} \leq \mathcal{H}(E_t \Pi_{t+\ell-1})$ for any $\ell \geq 2$, and as $\mathcal{H}(E_t \Pi_{t+\ell-1}) \leq \mathcal{H}(E_t \Pi_{t+\ell-1})$ given that $\mathcal{H}(II)$ is a weakly increasing function, it follows that $E_t \Pi_{t+\ell} \leq \mathcal{H}^{[2]}(E_t \Pi_{t+\ell-2})$. This generalizes to the following result for all finite $\ell \geq 1$:

$$E_t \Pi_{t+\ell} \leq \mathcal{H}^{[\ell]}(\Pi_t),$$

where $\mathcal{H}^{[\ell]}(II)$ denotes the composition of the function $\mathcal{H}(II)$ with itself $\ell$ times. Using the definition of $\mathcal{H}(II)$ in (A.6.3), for any $\Pi_t < 0$:

$$\mathcal{H}^{[\ell]}(\Pi_t) = \zeta^\ell \Pi_t.$$ 

Together with (A.6.6), it follows that $E_t \Pi_{t+\ell} \leq \zeta^\ell \Pi_t$ for all $\ell \geq 1$. Since there must always be positive-probability realizations of $\Pi_{t+\ell}$ with $\Pi_{t+\ell} \leq E_t \Pi_{t+\ell}$ for each $\ell$, it follows that $\Pi_{t+\ell} \leq \zeta^\ell \Pi_t$ with positive probability. With $\zeta > 1$ and $\Pi_t < 0$, this means that $\Pi_{t+\ell}$ approaches $-\infty$ as $\ell \to \infty$ with positive probability, but that would be inconsistent with the equilibrium conditions. This demonstrates there is no equilibrium with $\Pi_t < 0$ at some date $t$.

After ruling out $\Pi_t > 0$ and $\Pi_t < 0$ for each $t$, the only remaining possibility is $\Pi_t = 0$ at all dates $t$ (implying $\varepsilon_t = 0$ for all $t$). This can be seen immediately to satisfy equation (A.6.3), so $\Pi_t = 0$ is the unique equilibrium. Substituting the unique equilibrium into (A.6.2) implies:

$$(1 + \pi_t)^{-1} = \frac{(1 + \mu_t)E_{t-1}[(1 + \mu_t)^{-1}]}{1 + \pi_t},$$

and thus $E_{t-1}[(1 + \pi_t)^{-1}] = \frac{E_{t-1}[(1 + \mu_t)^{-1}]}{1 + \pi_t}$. This leads to:

$$E_{t-1}[(1 + \pi_t)^{-1}] - (1 + \pi_t)^{-1} = \frac{E_{t-1}[(1 + \mu_t)^{-1}]}{1 + \pi_t} - \frac{[1 + \mu_t]E_{t-1}[(1 + \mu_t)^{-1}]}{1 + \pi_t} = -\mu_t,$$

and comparison with (2.37) immediately confirms the claim that the unique equilibrium has financial conditions given by $\nu_t = -\mu_t$. By taking the reciprocal of the first equation in (A.6.7) and using $\pi_t = E_{t-1} - \pi_t$:

$$\pi_t = \frac{(1 + \pi_t^*) (1 + \mu_t)}{E_{t-1}[(1 + \mu_t)^{-1}]} - 1,$$

hence $E_{t-1} \pi_t = (1 + \pi_t^*) - 1 = \pi_t^*$,

which verifies that $E_{t-1} \pi_t = \pi_t^*$ in the unique equilibrium. The state-contingent path of $\pi_t$ is obtained by rearranging equation (2.37) for $\nu_t$:

$$(1 + \pi_t)E_{t-1}[(1 + \pi_t)^{-1}] = 1 + \frac{\nu_t}{1 - \nu_t},$$

and also $E_{t-1}[(1 + \pi_t)^{-1}] = \frac{1 + E_{t-1} \left[\frac{\nu_t}{1 - \nu_t}\right]}{1 + E_{t-1} \pi_t}$.
where the latter follows by taking conditional expectations of the former and dividing both sides by $1 + E_{t-1}\pi_t$. By substituting the second equation above into the first to eliminate $E_{t-1}[(1 + \pi_t)^{-1}]$ and rearranging terms:

$$
\pi_t = \left( 1 + E_{t-1}\pi_t \right) \left( \frac{1 + \frac{\nu_t}{1-\pi_t}}{1 + E_{t-1} \left[ \frac{\nu_t}{1-\pi_t} \right]} \right) - 1 = E_{t-1}\pi_t + \left( 1 + E_{t-1}\pi_t \right) \left( \frac{1 + \frac{\nu_t}{1-\pi_t}}{1 + E_{t-1} \left[ \frac{\nu_t}{1-\pi_t} \right]} - 1 \right).
$$

Simplifying this expression confirms that $\pi_t$ is given by (2.43) with $E_{t-1}\pi_t = \pi_t^*$ and $\nu_t = -\mu_t$.

Now suppose that $(d_t, l_t, h_t, c_{y,t}, c_{m,t}, c_{o,t}, r_t, \pi_t, \bar{\pi}_t, i_t)$ is an equilibrium of the economy under some alternative interest-rate feedback rule:

$$
i_t = \mathcal{F}(d_t, l_t, h_t, c_{y,t}, c_{m,t}, c_{o,t}, r_t, \pi_t, \bar{\pi}_t).
$$

The equilibrium values denoted with a tilde are functions of the exogenous variables. Since the results in Step 3, Step 4 and Result 1 were derived without reference to the particular monetary policy rule in place, the equilibrium must be given by (2.25), (2.31), (2.35), (2.36), (2.38), and (2.41) for some martingale difference sequence $\hat{\nu}_t$ satisfying (2.37). Result 1 has shown that $-\infty < \hat{\nu}_t \leq 1$ with probability one is a necessary condition for the existence of an equilibrium, so this must hold. The equilibrium selection criteria are $\bar{\pi}_t < \infty$ with probability one, and no positive probability realizations of $\pi_t$ converging to $-1$ or $\infty$ as $t \to \infty$. Given these requirements, equation (2.43) implies it must be the case that $-\infty < \pi_t \leq 1$ with probability one, $-1 < E_{t-1}\pi_t < \infty$ with probability one, and $E_{t-1}\pi_t$ does not converge to $-1$ or $\infty$ as $t \to \infty$.

Taking the equilibrium values with the properties given above, let $\pi_t^* = E_{t-1}\bar{\pi}_t$ and $\mu_t = -\hat{\nu}_t$, where these are functions of the exogenous variables. Now take any $\zeta > 1$, and consider using the monetary policy feedback rule (2.44) with these choices of $\mu_t$ and $\pi_t^*$. With the properties of $\hat{\nu}_t$ and $E_{t-1}\bar{\pi}_t$, the variables $\mu_t$ and $\pi_t^*$ satisfy the conditions imposed earlier, and the unique equilibrium is $\nu_t = -\mu_t$ and $E_{t-1}\pi_t = \pi_t^*$. It follows that $E_{t-1}\bar{\pi}_t = E_{t-1}\pi_t$ and $\nu_t = \hat{\nu}_t$, and hence $\pi_t = \bar{\pi}_t$ using (2.43). As $\pi_t$ and $\nu_t$ are the same as $\pi_t$ and $\bar{\nu}_t$, equations (2.25), (2.35), (2.36), (2.38), and (2.41) imply that the values of $d_t$, $l_t$, $h_t$, $c_{y,t}$, $c_{m,t}$, $c_{o,t}$, $r_t$, $\pi_t$, $\bar{\pi}_t$, and $i_t$ are the same as $d_t$, $l_t$, $h_t$, $c_{y,t}$, $c_{m,t}$, $c_{o,t}$, $r_t$, $\pi_t$, $\bar{\pi}_t$, and $i_t$. Any equilibrium that can be implemented using (A.6.8) can therefore equally well be implemented using the feedback rule (2.44) for some valid $\mu_t$ and $\pi_t^*$. This completes the proof.

### A.7 Proof of Result 2

Using the definition of $c_{\tau,t}$ in (2.14):

$$
\log C_{\tau,t} = \log c_{\tau,t} + \log Y_t - \log N_{\tau,t} = \log c_{\tau,t} + \text{t.i.p.},
$$

for each $\tau \in \{y, m, o\}$, where $Y_t$ is real GDP and $N_{\tau,t}$ is the population of age-$\tau$ individuals at date $t$. According to (2.1) and (2.3), both population and real GDP are exogenous variables and are thus included in the terms labelled ‘t.i.p.’ in the equation above (terms independent of monetary policy). Using the exogenous supply of houses (2.4) and the housing-market clearing condition (2.12):

$$
\Theta(H_{m,t}) = \Theta(1) = \text{t.i.p.},
$$

for all $t$. Substituting (A.7.1) and (A.7.2) into (2.2) and the continuation utilities from (3.1):

$$
\mathcal{U}_{y,t} = \log c_{y,t} + \delta E_t \log c_{m,t+1} + \delta^2 E_t \log c_{o,t+2} + \text{t.i.p.},
$$

$$
\mathcal{U}_{m,t} = \log c_{m,t} + \delta E_t \log c_{o,t+1} + \text{t.i.p.},
$$

and

$$
\mathcal{U}_{o,t} = \log c_{o,t} + \text{t.i.p.}
$$

[7.3]

For any monetary policy, the equilibrium values of $c_{\tau,t}$ are given by the expressions in (2.41):

$$
\log c_{y,t} = \log (\alpha \delta^2 \kappa^2) - \log (1 - \alpha \nu_t) = -\log (1 - \alpha \nu_t) + \text{t.i.p.};
$$

$$
\log c_{m,t} = \log (\alpha \delta \kappa) - \log (1 - \alpha \nu_t) = -\log (1 - \alpha \nu_t) + \text{t.i.p.};
$$

$$
\log c_{o,t} = \log \alpha + \log (1-\nu_t) - \log (1 - \alpha \nu_t) = \log (1 - \nu_t) - \log (1 - \alpha \nu_t) + \text{t.i.p.},
$$

where financial conditions $\nu_t$ are defined in (2.31). Using the formulas for $U_{b,t}$ and $U_{s,t}$ from (3.3):

$$
E_{t-1}c_{y,t} = U_{b,t} + \text{t.i.p.},
$$

$$
E_{t-1}c_{m,t} = U_{b,t} + \text{t.i.p.},
$$

and

$$
E_{t-1}c_{o,t} = U_{s,t} + \text{t.i.p.},
$$

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and substituting these into the expected values of the continuation utilities from (A.7.3) leads to the results in (3.1) after applying the law of iterated expectations.

Now define the following functions $B(\nu)$ and $S(\nu)$:

$$B(\nu) = -\log(1 - \alpha \nu), \quad \text{and} \quad S(\nu) = \log(1 - \nu) - \log(1 - \alpha \nu),$$  \hspace{1cm} \text{[A.7.4]}

where the range of $\nu$ is $-\infty < \nu < 1$, and $\alpha$ is a parameter satisfying $0 < \alpha < 1$. Comparing (3.3) and (A.7.4):

$$U_{b,t} = E_{t-1}B(\nu_t), \quad \text{and} \quad U_{s,t} = E_{t-1}S(\nu_t),$$  \hspace{1cm} \text{[A.7.5]}

where $\nu_t$ is the equilibrium value of financial conditions from (2.37), which is a random variable. Differentiating twice the functions in (A.7.4):

$$B'(\nu) = \frac{\alpha}{1 - \alpha \nu}, \quad \text{and} \quad B''(\nu) = \frac{\alpha^2}{(1 - \alpha \nu)^2} > 0, \quad \text{and}$$

$$S'(\nu) = \frac{\alpha}{1 - \nu} - \frac{1}{1 - \nu}, \quad S''(\nu) = \frac{\alpha^2}{(1 - \nu)^2} - \frac{1}{(1 - \nu)^2} = -\frac{(1 - \alpha)((1 - \alpha) + 2\alpha(1 - \nu))}{(1 - \nu)(1 - \alpha \nu)} < 0,$$

recalling that $-\infty < \nu < 1$. This demonstrates that $B(\nu)$ is a strictly convex function of $\nu$ and $S(\nu)$ is a strictly concave function of $\nu$. Now take any non-degenerate random variable $\nu_t$. Using Jensen’s inequality, the convexity and concavity of $B(\nu)$ and $S(\nu)$ imply:

$$E_{t-1}B(\nu_t) > B(E_{t-1}\nu_t), \quad \text{and} \quad E_{t-1}S(\nu_t) < S(E_{t-1}\nu_t).$$

Given that (2.45) requires $E_{t-1}\nu_t = 0$, it follows from the above and (A.7.4) that $E_{t-1}B(\nu_t) > B(0) = 0$ and $E_{t-1}S(\nu_t) < S(0) = 0$. Hence, from (A.7.5), it must be the case that $U_{b,t} > 0 > U_{s,t}$ for any non-degenerate probability distribution of financial conditions $\nu_t$. If the probability distribution of $\nu_t$ is degenerate then $\nu_t = 0$ with probability one, and it immediately follows that $U_{b,t} = 0 = U_{s,t}$. The convexity of $B(\nu)$ and the concavity of $S(\nu)$ in (A.7.4) also imply that a mean-preserving spread of $\nu_t$ raises $U_{b,t}$ and lowers $U_{s,t}$. This completes the proof.

### A.8 Proof of Step 6

(i) Result 1 shows that the equilibrium conditions (2.19a)-(2.19c) imply that (2.41) holds for some sequence of financial conditions $\nu_t$ satisfying (2.45). Furthermore, Step 5 shows that monetary policy can be set to achieve any probability distribution of $\nu_t$ satisfying (2.45). Substituting the lifetime and continuation utility functions from (2.2) and (3.1) in the social welfare function (3.4):

$$\mathcal{W}_{t_0} = N_{y,t_0} - 2\Omega_{t_0} - 2E_{t_0-1} \log C_{o,t_0} + N_{y,t_0-1}\Omega_{t_0-1}E_{t_0-1} \left[ \log C_{m,t_0} + \Theta(H_{m,t_0}) + \delta \log C_{o,t_0+1} \right]$$

$$+ \sum_{t=t_0}^{\infty} N_{y,t}\Omega_{t}E_{t_0-1} \left[ \log C_{y,t} + \delta \log C_{m,t+1} + \delta \Theta(H_{m,t+1}) + \delta^2 \log C_{o,t+2} \right],$$  \hspace{1cm} \text{[A.8.1]}

which uses the law of iterated expectations, $N_{o,t_0} = N_{y,t_0-2}$, $N_{m,t_0} = N_{y,t_0-1}$, and that the weights $\Omega_t$ are not random variables conditional on date $t_0 - 1$ information (see 3.4). A housing allocation must satisfy the market clearing condition (and resource constraint) (2.12). Together with the housing stock (2.4), this implies it must be the case that $H_{m,t} = 1$ for all $t$. Knowing the endogenous variables $c_{y,t}$, $c_{m,t}$, and $c_{o,t}$ together with the exogenous variables provides enough information to determine the consumption allocation and thus evaluate the welfare function $\mathcal{W}_{t_0}$ in (3.4). It follows that the constraints in the Ramsey problem (3.5) are equivalent to (2.41) for some sequence of probability distributions $\{\nu_t\}$ satisfying (2.45).

(ii) Define the following sequence $\Gamma_t$:

$$\Gamma_{t_0-2} = \delta^{-2}N_{y,t_0-2}\Omega_{t_0-2}, \quad \Gamma_{t_0-1} = \delta^{-1}N_{y,t_0-1}\Omega_{t_0-1}, \quad \text{and} \quad \Gamma_t = N_{y,t}\Omega_t \quad \text{for all} \quad t \geq t_0,$$  \hspace{1cm} \text{[A.8.2]}

and note that $\Gamma_t = E_{t_0-1}\Gamma_t > 0$ for all $t \geq t_0 - 2$ given the properties of $N_{y,t}$ and $\Omega_t$. The two stated conditions are given below in terms of $\Gamma_t$:

$$\sum_{t=t_0}^{\infty} \Gamma_t < \infty, \quad \text{and} \quad \sum_{t=t_0}^{\infty} t\Gamma_t < \infty.$$  \hspace{1cm} \text{[A.8.3]}
Using the definition of \( \Gamma_t \) from (A.8.2), the social welfare function (A.8.1) can be written as follows:

\[
W_{t_0} = \sum_{t=t_0}^{\infty} E_{t_0-1} \left[ \Gamma_t \log C_{y,t} + \delta \Gamma_{t-1} \log C_{m,t} + \Theta(H_{m,t}) + \delta^2 \Gamma_{t-2} \log C_{o,t} \right].
\]  

[A.8.4]

Substitute the housing-market equilibrium requirement \( H_{m,t} = 1 \) into the above to obtain:

\[
W_{t_0} = \sum_{t=t_0}^{\infty} E_{t_0-1} \left[ \Gamma_t \log C_{y,t} + \delta \Gamma_{t-1} \log C_{m,t} + \delta^2 \Gamma_{t-2} \log C_{o,t} \right] + \delta \Theta(1) \sum_{t=t_0-1}^{\infty} \Gamma_t,
\]  

[A.8.5]

and observe that the second term is finite only if \( \sum_{t=t_0}^{\infty} \Gamma_t < \infty \). This confirms that the first condition in (A.8.3) is necessary for the social welfare function (3.4) to be well defined.

A consumption allocation must satisfy the market clearing condition (and resource constraint) (2.11), and the allocation must be non-negative for all generations. Given that \( N_{m,t}/N_{y,t} = \gamma^{-1} \) and \( N_{m,t}/N_{o,t} = \gamma \) according to (2.1), and \( Y_t = N_{m,t} y_{m,t} \) given the expression for real GDP \( Y_t \) in (2.3), it follows that \( C_{y,t} \leq \gamma^{-1} y_{m,t} \cdot C_{m,t} \leq y_{m,t} \), and \( C_{o,t} \leq \gamma y_{m,t} \). Equations (2.1), (2.3), and (2.5) imply that \( y_{m,t}/y_{m,t-1} = (1 + g_t)/\gamma \), where \( g_t \) is the real GDP growth rate and \( \gamma - 1 \) is the population growth rate. According to the assumption in (2.5), \( g_t \) bounded above by \( \bar{y} \), and hence by iterating the earlier equation backward from \( t \) to \( t_0 \):

\[
y_{m,t} = y_{m,t_0} \prod_{t=t_0}^{t_0-1} (1 + g_{t_0-t}) \leq y_{m,t_0} \left( 1 + \bar{y} \right)^{t-t_0+1}.
\]

Together with the bounds on \( C_{y,t} \), \( C_{m,t} \), and \( C_{o,t} \), this implies the following bound on the social welfare function in (A.8.5):

\[
W_{t_0} \leq (\log(1 + \bar{y}) - \log \gamma) \sum_{t=t_0}^{\infty} (t - t_0 + 1)(\Gamma_t + \delta \Gamma_{t-1} + \delta^2 \Gamma_{t-2}) + (\log y_{m,t_0-1} - \log \gamma) \sum_{t=t_0}^{\infty} \Gamma_t
\]

\[
+ \delta(\log y_{m,t_0-1} + \Theta(1)) \sum_{t=t_0}^{\infty} \Gamma_{t-1} + \delta^2(\log y_{m,t_0-1} + \log \gamma) \sum_{t=t_0}^{\infty} \Gamma_{t-2}.
\]

Using the inequality above, convergence of \( \sum_{t=t_0}^{\infty} \Gamma_t \) and \( \sum_{t=t_0}^{\infty} t \Gamma_t \) guarantees that \( W_{t_0} < \infty \). Therefore, the second condition in (A.8.3) is sufficient for the social welfare function to be well defined. It has already been established that the first condition (implied by the second) is a necessary condition.

(iii) Using the law of iterated expectations and \( N_{y,t_0-2} = N_{o,t_0} \) and \( N_{y,t_0-1} = N_{m,t_0} \), the social welfare function from (3.4) can be written as:

\[
W_{t_0} = E_{t_0-1} \left[ N_{y,t_0-2} \Omega_{t_0-2} E_{t_0-1} W_{o,t_0} + N_{y,t_0-1} \Omega_{t_0-1} E_{t_0-1} W_{m,t_0} + \sum_{t=t_0}^{\infty} N_{y,t} \Omega_t E_{t-1} W_{y,t} \right],
\]

noting from (3.4) that \( N_{y,t} \) and \( \Omega_t \) are measurable with respect to period \( t_0-1 \) information for all \( t \geq t_0-2 \). Substituting the results from (3.2) into the above equation:

\[
W_{t_0} = E_{t_0-1} \left[ \sum_{t=t_0}^{\infty} N_{y,t} \Omega_t (U_{b,t} + \delta U_{b,t+1} + \delta^2 U_{s,t+2}) + N_{y,t_0-1} \Omega_{t_0-1} (U_{b,t_0} + \delta U_{s,t_0+1}) \right]
\]

\[
+ E_{t_0-1} [N_{y,t_0-2} \Omega_{t_0-2} U_{s,t_0}] + \text{t.i.p.} = E_{t_0-1} \left[ (N_{y,t_0} \Omega_{t_0} + N_{y,t_0-1} \Omega_{t_0-1}) U_{b,t_0} + N_{y,t_0-2} \Omega_{t_0-2} U_{s,t_0} \right]
\]

\[
+ E_{t_0-1} [(N_{y,t_0+1} \Omega_{t_0+1} + \delta N_{y,t_0} \Omega_{t_0}) U_{b,t_0+1} + \delta N_{y,t_0-1} \Omega_{t_0-1} U_{s,t_0+1}]
\]

\[
+ E_{t_0-1} \left[ \sum_{t=t_0+2}^{\infty} \{ (N_{y,t} \Omega_t + \delta N_{y,t-1} \Omega_{t-1}) U_{b,t} + \delta^2 N_{y,t-2} \Omega_{t-2} U_{s,t} \} \right] + \text{t.i.p.,}
\]

where the final expression groups together terms in \( U_{b,t} \) and \( U_{s,t} \) at the same date, and t.i.p. denotes terms independent of monetary policy. Comparing this to the expressions for \( W_{t_0} \) and \( W_t \) in (3.6a) confirms the formula for \( W_{t_0} \) with reference to the definitions of \( \omega_t \) in (3.6b) and \( \Delta_t \) in the text.
(iv) Substituting the expressions for $U_{b,t}$ and $U_{s,t}$ from (3.3) into $W_t$ from (3.6a):

$$W_t = \omega_t E_{t-1} [\log(1 - \nu_t) - \log(1 - \alpha \nu_t)] - (1 - \omega_t) E_{t-1} [\log(1 - \alpha \nu_t)]$$

$$= E_{t-1} [\omega_t \log(1 - \nu_t) - \log(1 - \alpha \nu_t)],$$

which confirms the formula for $W_t$ in (3.7). Observe that $W_t$ depends only on financial conditions $\nu_t$ with the same time index $t$. The only constraint on $\nu_t$ is $E_{t-1} \nu_t = 0$ from (2.45), which does not depend on financial conditions at any other date.

(v) Let $\ell$ denote the number of time periods ahead of $t_0$ for which monetary policy is committed to a state-contingent path of financial conditions $\{\nu_t\}_{t=t_0}^{t_0+\ell}$ (the case of full commitment corresponds to $\ell = \infty$). When $\ell < \infty$, the subsequent path of financial conditions $\{\nu_t\}_{t=t_0+\ell+1}^{\infty}$ is assumed to depend only on exogenous variables. Using (3.6a), the social welfare function can be written as follows:

$$\mathcal{W}_t = \sum_{t=t_0}^{t_0+\ell} \Delta_t E_{t_0-1} W_t + \sum_{t=t_0+\ell+1}^{\infty} \Delta_t E_{t_0-1} W_t = \sum_{t=t_0}^{t_0+\ell} \Delta_t E_{t_0-1} W_t + \text{t.i.p.},$$

where the second term is independent of the $t_0$ choice of monetary policy because for $t > t_0 + \ell$, $W_t$ depends only on $\nu_t$, which is a function of exogenous variables only.

The Ramsey problem is to obtain the highest value of $\mathcal{W}_t$ in (A.8.6) by choosing the sequence of probability distribution $\{\nu_t\}_{t=t_0}^{t_0+\ell}$ subject to $E_{t-1} \nu_t = 0$ for all $t = t_0, \ldots, t_0 + \ell$. Since $\Delta_t$ is positive for all $t$ and (A.8.6) is additive in $W_t$, for any $t = t_0, \ldots, t_0 + \ell$, the solution for $\nu_t$ must be the solution of the problem sup$_{\nu_t} W_t$ subject to (2.45). The function $W_t$ depends only on the relative welfare weight $\omega$ and the parameter $\alpha$, and the constraints $E_{t-1} \nu_t = 0$ and $-\infty < \nu_t < 1$ do not depend on any other variables. This completes the proof.

## A.9 Proof of Result 3

The solution of the Ramsey problem

Step 6 shows that the social-welfare-maximizing monetary policy at date $t$ implies a probability distribution of $\nu_t$ that solves the following problem:

$$\sup_{\nu_t} E_{t-1} [\omega_t \log(1 - \nu_t) - \log(1 - \alpha \nu_t)], \quad \text{subject to } -\infty < \nu_t < 1 \text{ and } E_{t-1} \nu_t = 0, \quad [A.9.1]$$

where $\omega_t$ is the non-stochastic welfare weight defined in (3.6a) (with $0 < \omega_t < 1$) and $\alpha$ is the constant from (2.35) (with $0 < \alpha < 1$). Dropping the time subscript $t$ and defining a function $W(\nu)$, the problem can be stated equivalently as:

$$\sup_{\nu \in (-\infty, 1)} E W(\nu), \quad \text{subject to } E \nu = 0, \quad \text{where } W(\nu) = \omega \log(1 - \nu) - \log(1 - \alpha \nu), \quad [A.9.2]$$

for given values of $\omega$ and $\alpha$.

First consider the properties of the function $W(\nu)$ defined in (A.9.2). Its first derivative is:

$$W'(\nu) = \frac{\alpha}{1 - \alpha \nu} - \frac{\omega}{1 - \nu}. \quad [A.9.3]$$

Since $1 - \nu > 0$ and $1 - \alpha \nu > 0$ for all values of $\nu$ in the admissible range, $W'(\nu)$ is positive if and only if $\alpha(1 - \nu) - \omega(1 - \alpha \nu) > 0$. Rearranging this inequality shows that $W$ is increasing in $\nu$ if $\nu < (1 - \alpha^{-1})/(1 - \omega)$ and decreasing if $\nu > (1 - \alpha^{-1})/(1 - \omega)$. The second derivative of $W(\nu)$ is:

$$W''(\nu) = \frac{\alpha^2}{(1 - \alpha \nu)^2} - \frac{\omega}{(1 - \nu)^2}, \quad [A.9.4]$$

which is positive if and only if $\alpha^2(1 - \alpha \nu)^2 - \omega(1 - \nu)^2 > 0$. Since both $1 - \alpha \nu$ and $1 - \nu$ must be positive, this inequality is equivalent to $\alpha(1 - \alpha \nu) > \sqrt{\omega}(1 - \nu)$, which can be rearranged to show that $W(\nu)$ is a convex function for $\nu < (1 - \alpha^{-1})/(1 - \sqrt{\omega})$ and concave for $\nu > (1 - \alpha^{-1})/(1 - \sqrt{\omega})$. Noting that $(1 - \alpha^{-1})/(1 - \omega) - (1 - \alpha^{-1})/(1 - \sqrt{\omega}) = (\alpha^{-1} - 1)(\sqrt{\omega}/(1 - \sqrt{\omega}) - \omega/(1 - \omega)) > 0$ since $0 < \alpha < 1$.
and $0 < \omega < 1$, it can be seen that the function $W(\nu)$ is first increasing and convex, then increasing and concave, and finally decreasing and concave. Finally, observe that $\lim_{\nu \to 1} W(\nu) = -\infty$ because $\omega > 0$.

Formally, the problem (A.9.2) is the choice of a probability distribution function $G(\nu)$ for $\nu$ on support $(-\infty, 1)$ subject to the constraint that the mean of the distribution is equal to zero:

$$\sup_{\nu \to -\infty} \int_{\nu \to -\infty} W(\nu) dG(\nu) \text{ s.t. } \int_{\nu \to -\infty} \nu dG(\nu) = 0. \quad [A.9.5]$$

The problem is analysed by setting up a Lagrangian as follows:

$$\mathcal{L} = \int_{\nu \to -\infty} W(\nu) dG(\nu) - \vartheta \int_{\nu \to -\infty} \nu dG(\nu), \quad [A.9.6]$$

where $\vartheta$ is the Lagrangian multiplier. For any $\nu$ that receives positive density or mass in the probability distribution, the first-order condition of the Lagrangian (A.9.6) is:

$$W'(\nu) = \vartheta. \quad [A.9.7]$$

Given the behaviour of the second derivative $W''(\nu)$ in (A.9.4), the function $W'(\nu)$ is first increasing and then decreasing. Since the right-hand side of (A.9.7) does not depend on $\nu$, there can be at most two values of $\nu$ that satisfy the first-order condition. For probability distributions with finite support, this means attention can be restricted to distributions with two point masses or degenerate distributions. A degenerate distribution must feature $\nu = 0$ with probability one to satisfy the constraint in (A.9.5). Note that $\nu = 1$ cannot satisfy (A.9.7) in any case because (A.9.3) implies that $\lim_{\nu \to -\infty} W'(\nu) = -\infty$.

Consider the case of a distribution of $\nu$ with two mass points, $\nu = \nu$ with probability $\epsilon$, and $\nu = \nu$ with probability $1 - \epsilon$ for some $0 < \epsilon < 1$ and $\nu$ and $\nu$ satisfying $-\infty < \nu < \nu < 1$. The Lagrangian (A.9.6) can be written as follows for this simpler problem:

$$\mathcal{L} = \epsilon W(\nu) + (1 - \epsilon) W(\nu) - \vartheta (\epsilon \nu + (1 - \epsilon) \nu), \quad [A.9.8]$$

with the full set of first-order conditions and constraints characterizing $\nu$, $\nu$, $\epsilon$, and $\vartheta$ being:

$$W'(\nu) = \frac{W'(\nu) - W'(\nu)}{\nu - \nu} = \nu'(\nu) = \vartheta, \text{ and } \epsilon \nu + (1 - \epsilon) \nu = 0. \quad [A.9.9]$$

Suppose these equations hold. Since $W(\nu)$ is a continuously differentiable function for $-\infty < \nu < 1$, the mean value theorem (using A.9.9) implies there is a $\nu'$ such that $\nu < \nu' < \nu$ and $W'(\nu') = \vartheta$. This would mean there is a third value of $\nu$ satisfying the first-order condition (A.9.7). However, given the properties of the second derivative $W''(\nu)$, it has already been established that there are at most two solutions of $W'(\nu) = \vartheta$. This argument rules out a probability distribution with finite support and two mass points being the solution of the constrained maximization problem (A.9.5).

There are two remaining cases. First, a degenerate distribution with $\nu = 0$, in which case the value of the objective function would be $W(0)$. In the second case, a distribution that has unbounded support, in particular, the limiting case where an arbitrarily large negative value of $\nu$ receives some vanishingly small probability. This case has a mass point at $\nu = \nu$ and some vanishingly small mass attached to a value of $\nu = \nu$, where $\nu \to -\infty$. Since the mean value of $\nu$ must be 0, the only feasible values of $\nu$ in this case must satisfy $0 < \nu < 1$.

Suppose $\nu = \nu$ occurs with probability $1 - \epsilon$ and some arbitrarily large negative $\nu = \nu(\epsilon) \to -\infty$ with probability $\epsilon$ as $\epsilon \to 0$. Since it is known the supremum (A.9.5) is not attained for any positive $\epsilon$, the limiting case where $\epsilon \to 0$ is considered. For a given value of $\epsilon$, the two mass points $\nu(\epsilon)$ and $\nu$ must satisfy the following equation for the distribution of $\nu$ to have a zero mean:

$$\epsilon \nu(\epsilon) + (1 - \epsilon) \nu = 0, \text{ and hence } \nu(\epsilon) = -\left(\frac{1}{\epsilon} - 1\right) \nu, \quad [A.9.10]$$

observing that $\nu(\epsilon) < 0$ tends to $-\infty$ as $\epsilon \to 0$. Using the expression for $\nu(\epsilon)$ above:

$$1 - \nu(\epsilon) = \frac{(1 - \epsilon) \nu + \epsilon}{\epsilon}, \text{ and } 1 - \alpha \nu(\epsilon) = \frac{(1 - \epsilon) \alpha \nu + \epsilon}{\epsilon},$$

and substituting these into the welfare function from (A.9.2):

$$W(\nu(\epsilon)) = \omega \log \left((1 - \epsilon) \nu + \epsilon\right) - \log \left((1 - \epsilon) \alpha \nu + \epsilon\right) + (1 - \omega) \log \epsilon. \quad [A.9.11]$$
Note the following limit as \( \epsilon \) becomes small:

\[
\lim_{\epsilon \to 0} \epsilon \log \epsilon = 0, \quad \text{and hence} \quad \lim_{\epsilon \to 0} \epsilon W(\nu(\epsilon)) = 0, \]

where the latter follows from the former because the first two terms in (A.9.11) have a finite limit as \( \epsilon \) approaches zero. The expected value of the welfare function under this distribution of \( \nu \) for a given value of \( \epsilon \) is:

\[
E_{W}(\nu) = (1 - \epsilon)W(\nu) + \epsilon W(\nu(\epsilon)).
\]

Since \( \nu \) remains the same as \( \epsilon \to 0 \), combining the equation above with (A.9.12) implies:

\[
\lim_{\epsilon \to 0} E_{W}(\nu) = W(\nu), \quad \text{[A.9.13]}
\]

and this applies for any choice of \( \nu \) satisfying \( 0 < \nu < 1 \).

Now assume that \( \omega < \alpha \), which implies \( 1 - \alpha^{-1}\omega > 0 \). Using (A.9.3), the unconstrained maximum of the function \( W(\nu) \) is at \( \nu = (1 - \alpha^{-1}\omega)/(1 - \omega) \), which lies between 0 and 1. Let \( \nu \) be set to this value of \( \nu \), and note that since \( W(\nu) \) is decreasing for \( \nu \) below this point:

\[
\nu = \frac{1 - \alpha^{-1}\omega}{1 - \omega}, \quad \text{with} \quad W(\nu) > W(0). \quad \text{[A.9.14]}
\]

Using the unbounded distribution of \( \nu \) as \( \epsilon \to 0 \), equation (A.9.13) shows that the limiting value of expected welfare \( E_{W}(\nu) \) is \( W(\nu) \), which is valid for the \( \nu \) in (A.9.14) because it satisfies \( 0 < \nu < 1 \). Since \( W(\nu) \) is the unconstrained maximum of \( W(\nu) \) over \(-\infty < \nu < 1\), this implies the supremum of expected welfare \( E_{W}(\nu) \) subject to the constraint \( E\nu = 0 \) is \( W(\nu) \). It is possible to obtain an expected welfare \( E_{W}(\nu) \) arbitrarily close to \( W(\nu) \) by selecting a sufficiently small probability \( \epsilon > 0 \). Note that the value of the Lagrangian multiplier \( \theta \) is 0, which can be seen from (A.9.9) using \( W'(\nu) = 0 \). This is consistent with \( \lim_{\epsilon \to 0} W'(\nu(\epsilon)) = 0 \) and \( \lim_{\epsilon \to 0}(W(\nu) - W(\nu(\epsilon)))/(\nu - \nu(\epsilon)) = 0 \).

Therefore, using the result above and (A.9.10), at each date \( t \) where \( \omega_{t} < \alpha \), the probability distribution of \( \nu_{t} \) is:

\[
\nu_{t} = \begin{cases} 
1 - \omega_{t} & \text{with probability } 1 - \epsilon_{t} \\
\frac{1 - \omega_{t}}{1 - \alpha\omega_{t}} & \text{with probability } \epsilon_{t}
\end{cases}, \quad \text{[A.9.15]}
\]

for small \( \epsilon_{t} > 0 \). Define \( x_{t} \) as follows:

\[
x_{t} = \frac{\alpha - \omega_{t}}{1 - \alpha\omega_{t}}, \quad \text{noting that} \quad \frac{x_{t}}{1 + x_{t}} = \frac{1 - \omega_{t}}{1 - \alpha\omega_{t}}. \quad \text{[A.9.16]}
\]

and given that \( 0 < \omega_{t} < \alpha \) and \( 0 < \alpha < 1 \), \( x_{t} \) satisfies \( 0 < x_{t} < \infty \). Using (A.9.15) and (A.9.16), the probability distribution (3.9) is confirmed. Observe that the equilibrium is not unique in the sense that the exogenous states of the world in which the two regimes occur are not pinned down. What is uniquely determined is the value of \( x_{t} \) and that the probability of the large negative realization of \( \nu_{t} \) should be small.

With equal democratic weights, \( \omega_{t} \) is given by the formula in (3.8). Using the expression for \( \alpha \) in (2.35), the condition \( \omega_{t} < \alpha \) is equivalent to:

\[
\frac{1}{1 + \gamma + \gamma^{2}} < \frac{1}{1 + \delta \kappa + (\delta \kappa)^{2}},
\]

and hence to \( \gamma > \delta \kappa \). Since (2.35) states that \( \alpha > 1/3 \), it follows that \( \delta \kappa < 1 \).

**Features of a financial-crisis equilibrium**

Using the probability distribution of \( \nu_{t} \) from (3.9), the high realization of \( \nu_{t} \) implies \((1 - \nu_{t})^{-1} = 1 + x_{t}\) and the low realization of \( \nu_{t} \) implies \((1 - \nu_{t})^{-1} = \epsilon_{t}(1 + x_{t})/(x_{t} + \epsilon_{t})\). The implied housing risk premium can be obtained from the formula in (2.38):

\[
\xi_{t-1} = \lim_{\epsilon_{t} \to 0} \left( (1 - \epsilon_{t})(1 + x_{t}) + \frac{\epsilon_{t}^{2}(1 + x_{t})}{x_{t} + \epsilon_{t}} \right) - 1 = x_{t}. \quad \text{[A.9.17]}
\]

Substituting this result into (2.27) yields the expression for \( i_{t-1} \). Using the results in Step 4 together with the probability distribution (3.9) it follows that there are two states for the variables \( h_{t} \) and \( l_{t} \). In the
boom, \( h_t \) and \( l_t \) are:

\[
\bar{h} = \frac{\alpha \beta (1 + x)}{\lambda (1 + (1 - \alpha) x)}, \quad \text{and} \quad \bar{l} = \frac{\alpha \beta (1 + x)}{1 + (1 - \alpha) x},
\]

both of which are increasing in \( x \). In a crisis, the values of \( h_t \) and \( l_t \) depend on the probability \( \epsilon \), and these become arbitrarily small as \( \epsilon \) tends to zero (\( \bar{h} \to 0 \) and \( \bar{l} \to 0 \) as \( \epsilon \to 0 \)).

**Pareto-improving policy changes**

Now consider the question of whether it is possible to make a Pareto-improving change of monetary policy at date \( t \) starting from some initial monetary policy. To address this question, first write the expected payoffs \( U_{b,t} \) and \( U_{s,t} \) from (3.3) as \( \bar{U}_{b,t} = E_{t-1} B(\nu_t) \) and \( U_{s,t} = E_{t-1} S(\nu_t) \) in terms of two function \( B(\nu) \) and \( S(\nu) \):

\[
B(\nu) = -\log(1 - \alpha \nu), \quad \text{and} \quad S(\nu) = \log(1 - \nu) - \log(1 - \alpha \nu). \tag{A.9.18}
\]

Take an arbitrary non-degenerate random variable \( \tilde{\nu}_t \) (with support between \( -\infty \) and 1, and \( E_{t-1} \tilde{\nu}_t = 0 \)) that is the outcome of the existing policy stance adopted at date \( t \), and let \( \tilde{U}_{b,t} = E_{t-1} B(\tilde{\nu}_t) \) and \( \tilde{U}_{s,t} = E_{t-1} S(\tilde{\nu}_t) \) denote the expected payoffs under this policy. Using the formulas for the continuation expected utilities (3.1), it follows that a Pareto improvement to policy at date \( t \) can be found if there exists a random variable \( \nu_t \) on support \((-\infty, 1) \) with \( E_{t-1} \nu_t = 0 \) where \( U_{s,t} \) is higher than \( \tilde{U}_{s,t} \) while \( U_{b,t} \) is no lower than \( \tilde{U}_{b,t} \). Whether such a Pareto improvement is possible can be investigated by solving the following constrained optimization problem:

\[
\sup_{\nu_t} U_{s,t}, \quad \text{subject to} \quad -\infty < \nu_t < 1, \quad E_{t-1} \nu_t = 0, \quad \text{and} \quad U_{b,t} \geq \tilde{U}_{b,t}. \tag{A.9.19}
\]

Dropping time subscripts and using the functions defined in (A.9.18):

\[
\sup_{\nu \in (-\infty, 1)} ES(\nu), \quad \text{subject to} \quad E\nu = 0 \quad \text{and} \quad E B(\nu) \geq \tilde{U},
\]

and formally this problem is the choice of a probability distribution function \( G(\nu) \) for \( \nu \) on support \((-\infty, 1) \) subject to two constraints:

\[
\sup_{\nu \rightarrow -\infty} \int_{\nu \rightarrow -\infty}^{1} S(\nu) dG(\nu) \quad \text{subject to} \quad \int_{\nu \rightarrow -\infty}^{1} \nu dG(\nu) = 0 \quad \text{and} \quad \int_{\nu \rightarrow -\infty}^{1} B(\nu) dG(\nu) \geq \tilde{U}. \tag{A.9.20}
\]

The Lagrangian for this problem is:

\[
\Lambda = \int_{\nu \rightarrow -\infty}^{1} S(\nu) dG(\nu) - \kappa \int_{\nu \rightarrow -\infty}^{1} \nu dG(\nu) + \nu \left\{ \int_{\nu \rightarrow -\infty}^{1} B(\nu) dG(\nu) - \tilde{U} \right\}, \tag{A.9.21}
\]

where \( \kappa \) and \( \nu \) are the Lagrangian multipliers on the zero-mean and minimum utility constraints respectively.

**Result 2** demonstrates that \( B(\nu) \) is a strictly convex function, so by Jensen’s inequality, \( E B(\tilde{\nu}) > B(E\tilde{\nu}) = B(0) = 0 \) because \( \tilde{\nu} \) has a non-degenerate distribution, \( E\tilde{\nu} = 0 \), and \( B(0) = 0 \) using (A.9.18). Now suppose that the minimum utility constraint is not binding (\( \nu = 0 \)), so the solution of (A.9.20) is the same when only the zero-mean constraint is imposed. For any non-degenerate distribution of \( \nu \) with \( E\nu = 0 \), since **Result 2** demonstrates that \( S(\nu) \) is a strictly concave function, Jensen’s inequality implies \( E S(\nu) < S(E\nu) = S(0) = 0 \), where the final equality follows from (A.9.18). As a degenerate distribution with \( \nu = 0 \) in all states of the world satisfies the zero mean constraint and yields \( E S(\nu) = S(0) = 0 \), it follows that \( E S(\nu) \) is maximized subject to the constraint \( E\nu = 0 \) by the constant \( \nu = 0 \). But this implies \( E B(\nu) = B(0) = 0 < E B(\tilde{\nu}) \), which violates the minimum utility constraint in (A.9.20). It follows that the minimum utility constraint must be binding (with multiplier \( \nu > 0 \) in A.9.21).

Using the results of Step 6, the function \( W(\nu) \) in (A.9.2) is \( W(\nu) = (1 - \omega) B(\nu) + \omega S(\nu) \) in terms of the functions \( B(\nu) \) and \( S(\nu) \) from (A.9.18). Hence, by using equations (A.9.6) and (A.9.21):

\[
\mathcal{L} = \omega \Lambda + (1 - \omega) \tilde{U}, \quad \text{where} \quad \omega = \frac{1}{1 + \nu} \quad \text{and} \quad \vartheta = \frac{\kappa}{1 + \nu}, \tag{A.9.22}
\]

with the expressions for \( \omega \) and \( \vartheta \) giving the values of these terms from (A.9.2) and (A.9.6) for which the equation in (A.9.22) linking \( \mathcal{L} \) and \( \Lambda \) holds. Since \( \nu \) must be positive, the implied value of \( \omega \) satisfies
0 < \omega < 1 as was assumed earlier. With the Lagrangian \mathcal{L} being a positive multiple of \Lambda plus a constant, the probability distribution of \nu must be as derived earlier conditional on a value of \omega. The only difference here is that \omega is an endogenous variable that depends on the Lagrangian multiplier \nu, which is pinned down by the binding minimum utility constraint.

As demonstrated earlier, the probability distribution of \nu is either the degenerate case \nu = 0 or (A.9.15) for small \epsilon > 0. Using the argument above, the case of \nu = 0 would violate the minimum utility constraint. The other case occurs when \omega < \alpha, so it is conjectured that this bound on \omega holds. Calculating borrowers’ expected payoff under the probability distribution in (A.9.15):

\[ \mathbb{E}B(\nu) = -(1 - \epsilon) \log \left( 1 - \frac{\alpha(1 - \epsilon - 1) - 1}{\omega} \right) - \epsilon \log \left( 1 + \frac{(1 - \epsilon)(1 - \alpha) - 1}{\omega} \right) \]

\[ = (1 - \epsilon) \log \left( \frac{1 - \omega}{1 - \alpha} \right) + \epsilon \log \left( \frac{1 - \omega}{\omega + \epsilon(1 - \alpha)} \right) + \epsilon \log \epsilon. \]

Taking the limit as \epsilon \to 0 and using (A.9.12):

\[ \lim_{\epsilon \to 0} \mathbb{E}B(\nu) = \log \frac{1 - \omega}{1 - \alpha}, \]

and hence the binding minimum utility constraint from (A.9.20) is the following equation:

\[ \log \frac{1 - \omega}{1 - \alpha} = \bar{U}_b = \mathbb{E}B(\bar{\nu}). \] [A.9.23]

Note from (A.9.18) that \mathcal{B}(\nu) < \log(1 - \alpha)^{-1} since \nu < 1, and hence \bar{U}_b < \log(1 - \alpha)^{-1}. Given \bar{U}_b, the equation above can be solved for \omega and the implied value of the Lagrangian multiplier \nu using (A.9.22):

\[ \omega = \alpha - (1 - \alpha)(\exp(\bar{U}_b) - 1), \quad \text{and} \quad \nu = \frac{(1 - \alpha) \exp(\bar{U}_b)}{1 - (1 - \alpha) \exp(\bar{U}_b)}. \]

These satisfy 0 < \omega < \alpha < 1 and \nu > 0 since \( (1 - \alpha) \exp(\bar{U}_b) < 1 \), which confirm the earlier claims that the minimum utility constraint is binding and that the implied value of \omega satisfies \omega < \alpha. From (A.9.16), it follows that the distribution (3.9) with \( x_t = (\exp(\bar{U}_{b,t}) - 1)/(1 - (1 - \alpha) \exp(\bar{U}_{b,t})) \) delivers the highest expected payoff \( \mathbb{E}S(\nu) \) for savers conditional on not reducing the expected payoff of borrowers. Therefore, from any initial monetary policy where the probability distribution of financial conditions at date \( t \) is non-degenerate, there is a Pareto-improving change to monetary policy at date \( t \) that leads to an equilibrium with the probability distribution (3.9) for \( \nu_t \).

To see the effect of the monetary policy change on the housing risk premium \( \xi \), define a transformation of the random variable \( \bar{\nu} \) as follows:

\[ \tilde{\xi} = (1 - \nu)^{-1} - 1, \quad \text{where} \quad \tilde{\xi} = \mathbb{E}\tilde{\xi}, \] [A.9.24]

which uses the result in (2.38). The original housing risk premium \( \tilde{\xi} \) is simply the expectation of the random variable \( \tilde{\xi} \), which has a non-degenerate distribution because \( \nu \) has. Now define the following function \( \mathcal{B}(\xi) \) in terms of the function \( \mathcal{B}(\nu) \) from (A.9.18):

\[ \mathcal{B}(\xi) = \mathcal{B} \left( \frac{\xi}{1 + \xi} \right) = -\log \left( 1 - \frac{\alpha \xi}{1 + \xi} \right) = \log(1 + \xi) - \log(1 + (1 - \alpha)\xi), \] [A.9.25]

and the first and second derivatives of this function are:

\[ \mathcal{B}'(\xi) = \frac{\alpha}{(1 + \xi)(1 + (1 - \alpha)\xi)} > 0, \quad \text{and} \quad \mathcal{B}''(\xi) = -\frac{\alpha(\alpha + 2(1 - \alpha)(1 + \xi))}{(1 + \xi)(1 + (1 - \alpha)\xi)^2} < 0. \]

Thus, \( \mathcal{B}(\xi) \) is a strictly increasing and strictly concave function. Rearranging the definition of \( \tilde{\xi} \) in (A.9.24), it follows from (A.9.25) that:

\[ \bar{\nu} = \frac{\tilde{\xi}}{1 + \tilde{\xi}}, \quad \text{hence} \quad \mathcal{B}(\bar{\nu}) = \mathcal{B}(\tilde{\xi}) \quad \text{and} \quad \mathbb{E}B(\bar{\nu}) = \mathbb{E}\mathcal{B}(\tilde{\xi}). \]

Since \( \mathcal{B}(\xi) \) is a strictly concave function and \( \tilde{\xi} \) has a non-degenerate distribution, Jensen’s inequality implies:

\[ \mathbb{E}B(\bar{\nu}) = \mathbb{E}\mathcal{B}(\tilde{\xi}) < \mathcal{B}(\mathbb{E}\tilde{\xi}) = \mathcal{B}(\tilde{\xi}), \] [A.9.26]
where the final equality uses (A.9.24). The link between $\omega$ and $x$ in (A.9.16) implies $\omega = \alpha/(1 + (1 - \alpha)x)$, and together with the binding minimum utility constraint (A.9.23):
\[
\mathbb{E}B(\tilde{\nu}) = \log \frac{1 - \frac{\alpha}{1+(1-\alpha)x}}{1 - \alpha} = \log(1 + x) - \log(1 + (1 - \alpha)x) = \mathfrak{B}(x),
\]
which uses the expression for $\mathfrak{B}(\xi)$ in (A.9.25). According to (A.9.17), the new risk premium $\xi$ is equal to $x$. Hence, the equation above combined with (A.9.26) implies:
\[
\mathfrak{B}(\xi) = \mathbb{E}B(\tilde{\nu}) < \mathfrak{B}(\tilde{\xi}),
\]
and since $\mathfrak{B}(\xi)$ is a strictly increasing function, this demonstrates that $\xi < \tilde{\xi}$. The Pareto-improving policy change that leads to the equilibrium where financial crises occur actually lowers the housing risk premium. This completes the proof.

A.10 Proof of Step 7

Necessary conditions for Pareto efficiency

Using the definitions in (2.14), the first resource constraint in (4.1) implies that the consumption shares $c_{t,t}$ must satisfy (2.18). Since housing utility is received by the middle-aged (see 2.2), it must be the case that $H_{m,t}$ is at the highest value consistent with the second resource constraint in (4.1), which by using (2.4) is $H_{m,t} = 1$ for all $t \geq t_0$.

Consider the three generations alive at any date $t \geq t_0$, and consider an adjustment to the consumption allocation $C_{y,t}$, $C_{m,t}$, and $C_{o,t}$ for date $t$ only. The only restriction on the allocation is the first resource constraint in (4.1). Using the utility function (2.2) and maximizing the expected utility (from the standpoint of just these two generations at dates $t$ and $t + 1$) implies:
\[
\frac{C_{y,t}}{E_{t_0-1}C_{y,t}} = \frac{C_{m,t}}{E_{t_0-1}C_{m,t}} = \frac{C_{o,t}}{E_{t_0-1}C_{o,t}}.
\]

The equations require consumption risk sharing (from the standpoint of $t_0 - 1$) among all generations alive at date $t$. Using the definition of $c_{t,t}$ in (2.14), this necessary condition is equivalent to (4.2a).

Now consider the two generations alive at both dates $t$ and $t + 1$ for some $t \geq t_0$. As of date $t$, these two generations are the young and middle-aged. Consider an adjustment to the consumption allocations of just these two generations at dates $t$ and $t + 1$, namely $C_{y,t}$, $C_{m,t}$, $C_{m,t+1}$, and $C_{o,t+1}$, with all other consumption levels held constant. The only restrictions on the allocation are the resource constraints at dates $t$ and $t + 1$. Using the utility function (2.2) and maximizing the expected utility of one generation subject to achieving given levels of expected utility for the other two generations requires that $C_{y,t}$, $C_{m,t}$, and $C_{o,t}$ are proportional to variables measurable with respect to $t_0 - 1$ information. This implies the following necessary condition for Pareto efficiency:
\[
\frac{C_{m,t+1}}{C_{y,t}} = \frac{C_{o,t+1}}{C_{m,t}}.
\]

Taking expectations of both sides conditional on date $t$ information implies that equating expected consumption growth rates among overlapping generations is a necessary condition for Pareto efficiency:
\[
\mathbb{E}_t \frac{C_{m,t+1}}{C_{y,t}} = \mathbb{E}_t \frac{C_{o,t+1}}{C_{m,t}}.
\]

The equation requires an equal degree of consumption smoothing across all overlapping generations. Using the definition of $c_{t,t}$ in (2.14), this necessary condition is equivalent to (4.2b).

Finally, consider modifying a given allocation $\{C_{\tau,t}\}$ through the following system of perpetual transfers. For each $t$ from $t_0$ onwards, the consumption $C_{m,t}$ of each middle-aged person is reduced by a non-negative amount $T_t$ and the resources are transferred to the current old and equally divided among them. Given the first resource constraint in (4.1) and the relative sizes of the age groups in (2.1), each old person’s consumption $C_{o,t}$ is increased by $N_{m,t}T_t/N_{o,t} = \gamma T_t$ at time $t$. Since the consumption of the middle-aged
must remain non-negative, feasibility requires that the transfers are bounded above by $C_{m,t}$. Introducing this system of transfers at time $t_0$ benefits the initial generation of old if $T_{t_0} > 0$.

Assuming the elements of the sequence $\{T_t\}$ are sufficiently small, differentiation of the lifetime expected utility function (2.2) shows that for the system of transfers not to make the initial and subsequent generations of middle-aged worse off, it is necessary that the following holds for all $t \geq t_0$:

$$\frac{T_t}{C_{m,t}} \leq \delta E_t \left[ \frac{T_{t+1}}{C_{o,t+1}} \right]. \quad [A.10.1]$$

For each $t > t_0$, let $\tau_t$ denote the transfer at time $t$ relative to middle-aged consumption $T_t/C_{m,t}$ divided by the relative transfer $T_{t-1}/C_{m,t-1}$ at time $t-1$:

$$\tau_t = \frac{T_tC_{m,t-1}}{T_{t-1}C_{m,t}}. \quad [A.10.2]$$

Using this definition and $C_{m,t+1}/C_{o,t+1} = (c_{m,t+1}/c_{o,t+1})/\gamma$ deduced from (2.1) and (2.14), the requirement (A.10.1) that the transfer scheme does not make the middle-aged worse off is equivalent to:

$$E_t \left[ \tau_{t+1} \frac{c_{m,t+1}}{c_{o,t+1}} \right] \geq \frac{1}{\delta}, \quad [A.10.3]$$

for all $t \geq t_0$.

Now suppose that the ratio $c_{m,t+1}/c_{o,t+1}$ does not satisfy the condition in (4.2c), so that:

$$\liminf_{t \to \infty} E_t \left[ \frac{c_{m,t+1}}{c_{o,t+1}} \right] > \frac{1}{\delta}. \quad [A.10.4]$$

Construct a sequence $\{\tau_t\}$ for all $t > t_0$ as follows:

$$\tau_t = \frac{1}{\delta E_{t-1}} \frac{c_{m,t}}{c_{o,t}}. \quad [A.10.5]$$

noting that the non-negativity of the consumption allocation implies $\tau_t \geq 0$ for all $t > t_0$. With this sequence of $\tau_t$:

$$E_t \left[ \tau_{t+1} \frac{c_{m,t+1}}{c_{o,t+1}} \right] = \frac{1}{\delta},$$

for all $t \geq t_0$, so the condition in (A.10.3) holds exactly. Since $\delta E_t[c_{m,t+1}/c_{o,t+1}]$ is a non-negative sequence for all $t$, it follows that:

$$\limsup_{t \to \infty} \frac{1}{\delta E_t} \frac{c_{m,t+1}}{c_{o,t+1}} = \frac{1}{\liminf_{t \to \infty} \delta E_t} \frac{c_{m,t+1}}{c_{o,t+1}}.$$ 

Together with (A.10.4) and (A.10.5), this implies that $\limsup_{t \to \infty} \tau_t < 1$. The definition of $\tau_t$ in (A.10.2) shows that the implied sequence of transfers $\{T_t\}$ can be obtained from $\{\tau_t\}$ and $T_{t_0}$ as follows:

$$\frac{T_t}{C_{m,t}} = T_{t_0} \prod_{\ell=1}^{t-t_0} \tau_{t_0+\ell}.$$ 

With $\limsup_{t \to \infty} \tau_t < 1$, the equation above demonstrates that $\lim_{t \to \infty} T_t/C_{m,t} = 0$ for any initial $T_{t_0} > 0$, so the sequence of transfers can be made arbitrarily small by reducing the initial value $T_{t_0}$. It follows that there exists a feasible sequence of transfers $\{T_t\}$ that makes the initial generation of old better off without making any other generation worse off, hence the allocation cannot be Pareto efficient if (A.10.4) holds.

Exactly analogous arguments can be used to construct a Pareto-improving sequence of transfers between the young and the middle-aged if $\liminf_{t \to \infty} E_t[c_{y,t+1}/c_{m,t+1}] > 1/\delta$. This demonstrates that the conditions in (4.2c) are necessary for an allocation to be Pareto efficient. Given the nature of the Pareto-improving system of transfers when (4.2c) fails to hold, the condition can be interpreted as requiring dynamic efficiency.

The social planner’s problem

Now consider an allocation chosen to maximize the social welfare function (3.4) subject to the resource constraints (4.1). A constrained maximum exists when the welfare function is bounded, and Step 6 gives
sufficient conditions for this. Take a sequence of weights \( \Omega_t = E_{t_0-1} \Omega_t > 0 \) for \( t \geq t_0 - 2 \) for which there is a well-defined solution to the maximization problem.

Define the following sequence \( \Gamma_t \):

\[
\Gamma_{t_0-2} = \delta^{-2}N_{y,t_0-2} \Omega_{t_0-2}, \quad \Gamma_{t_0-1} = \delta^{-1}N_{y,t_0-1} \Omega_{t_0-1}, \quad \text{and} \quad \Gamma_t = N_{y,t} \Omega_t \quad \text{for all} \quad t \geq t_0,
\]

and note that \( \Gamma_t = E_{t_0-1} \Gamma_t > 0 \) for all \( t \geq t_0 - 2 \) given the properties of \( N_{y,t} \) and \( \Omega_t \). Using the proof of Step 6, the social welfare function (3.4) is given by the expression in (A.8.4) with the same definition of \( \Gamma_t \), and hence the maximum of the social welfare function subject to the resource constraints in (4.1) is found by setting up the following Lagrangian:

\[
L_{t_0} = \sum_{t=t_0}^{\infty} E_{t_0-1} \left[ \Gamma_t \log C_{y,t} + \delta \Gamma_{t_1} - 1 \left( \log C_{m,t} + \Theta(H_{m,t}) \right) + \delta^2 \Gamma_{t_2} \log C_{o,t} \right]
\]

\[
+ \sum_{t=t_0}^{\infty} E_{t_0-1} \left[ \Phi_t \left\{ Y_t - N_{y,t} C_{y,t} - N_{m,t} C_{m,t} - N_{o,t} C_{o,t} \right\} + \Upsilon_t \left\{ H_{t-1} - N_{y,t-1} H_{m,t} \right\} \right],
\]

where \( \Phi_t \) and \( \Upsilon_t \) are respectively the date-\( t \) state-contingent Lagrangian multipliers on the consumption goods resource constraint and the housing resource constraint from (4.1). The first-order conditions for a maximum of the welfare function are as follows for all \( t \geq t_0 \):

\[
\frac{\Gamma_t}{N_{y,t} C_{y,t}} = \frac{\delta \Gamma_{t_1} - 1}{N_{m,t} C_{m,t}} = \frac{\delta^2 \Gamma_{t_2} - 1}{N_{o,t} C_{o,t}} = \Phi_t; \quad \text{and} \quad \delta \Gamma_{t_1} - 1 \Theta'(H_{m,t}) = N_{y,t-1} \Upsilon_t.
\]

The objective function (3.4) is globally concave and the constraints (4.1) are linear, so the first-order conditions (A.10.7) are necessary and sufficient for a constrained maximum of the social welfare function, assuming one exists.

The housing constraint in (4.1) together with housing supply (2.4), the definition of \( \theta \) in (2.8), and the first-order condition (A.10.7b) implies:

\[
H_{m,t} = 1, \quad \text{and} \quad \Upsilon_t = \delta \theta \Gamma_{t_1} - 1 \frac{N_{y,t-1} \Upsilon_t}{N_{y,t-1}}.
\]

The definitions in (2.14) and the consumption goods resource constraint in (4.1) imply that the consumption shares \( c_{y,t} \) must satisfy (2.18). Using the definitions in (2.14) and (A.10.6), the first-order conditions (A.10.7a) are equivalent to:

\[
c_{y,t} = \frac{\Gamma_t}{\Phi_t Y_t}, \quad c_{m,t} = \frac{\delta \Gamma_{t_1} - 1}{\Phi_t Y_t}, \quad \text{and} \quad c_{o,t} = \frac{\delta^2 \Gamma_{t_2} - 1}{\Phi_t Y_t}.
\]

Since the consumption ratios must satisfy (2.18), it follows from the above that the Lagrangian multiplier \( \Phi_t \) is equal to:

\[
\Phi_t = \frac{\Gamma_t + \delta \Gamma_{t_1} - 1 + \delta^2 \Gamma_{t_2} - 1}{Y_t}.
\]

With \( \Gamma_t = E_{t_0-1} \Gamma_t \), it follows from (A.10.9) that for all \( t \geq t_0 \):

\[
\frac{c_{y,t}}{E_{t_0-1} c_{y,t}} = \frac{c_{m,t}}{E_{t_0-1} c_{m,t}} = \frac{c_{o,t}}{E_{t_0-1} c_{o,t}} = \frac{(\Phi_t Y_t)^{-1}}{E_{t_0-1} \left((\Phi_t Y_t)^{-1}\right)},
\]

which yields the necessary condition for efficiency in (4.2a). As (A.10.9) holds for all \( t \geq t_0 \), it must be the case that:

\[
E_t \frac{c_{y,t+1}}{c_{y,t}} = \delta E_t \left[ \frac{\Phi_t Y_t}{\Phi_{t+1} Y_{t+1}} \right] = \frac{1}{\delta} \frac{c_{o,t+1}}{c_{m,t}}
\]

which is the necessary condition for efficiency in (4.2b). Finally, since \( \Gamma_t = E_{t_0-1} \Gamma_t \) for all \( t \geq t_0 - 2 \), the equations in (A.10.9) can be used to deduce the following for all \( t \geq t_0 \):

\[
E_t \frac{c_{y,t+1}}{c_{y,t}} = \frac{1}{\delta} \frac{\Gamma_{t+1}}{\Gamma_t}, \quad \text{and} \quad E_t \frac{c_{m,t+1}}{c_{o,t+1}} = \frac{1}{\delta} \frac{\Gamma_{t+1}}{\Gamma_{t-1}}.
\]

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Convergence of the first series in (A.8.3) is a necessary condition for existence of a maximum of the social welfare function according to Step 6. With \( \sum_{t=0}^{\infty} \Gamma_t < \infty \), the ratio test therefore yields the following necessary condition:

\[
\liminf_{t \to \infty} \frac{\Gamma_t}{\Gamma_{t-1}} \leq 1. \tag{A.10.12}
\]

The final necessary condition for efficiency (4.2c) is then implied by (A.10.11) and (A.10.12).

**Sufficient conditions for Pareto efficiency**

Suppose a consumption and housing allocation satisfies \( H_{m,t} = 1 \), (2.18), (4.2a), (4.2b), and (4.2c) (with \( \limsup_{t \to \infty} \) and strict inequality), for all \( t \geq t_0 \). This allocation is shown to be efficient by demonstrating that it is the solution to the social welfare maximization problem for a sequence of welfare weights \( \{ \Omega_t \} \) for which the social welfare function has a well-defined constrained maximum. Using (A.10.6), the sequence of weights \( \Omega_t \) can be equivalently specified as a sequence of values of \( \Gamma_t \), with \( \Omega_{t_0-2} = \delta^2 \Gamma_{t_0-2}/N_{y,t_0-2} \), \( \Omega_{t_0-1} = \delta \Gamma_{t_0-1}/N_{y,t_0-1} \), and \( \Omega_t = \Gamma_t/N_{y,t} \) for all \( t \geq t_0 \).

Observe that the first-order conditions (A.10.7) of the social welfare maximization problem are homogeneous of degree zero in \( \Gamma_t \) and hence the values of \( \Omega_t \) implied by (A.10.6) are also strictly positive. Since the consumption allocation satisfies (4.2a) and (4.2b):

\[
\text{Taking expectations conditional on date } t_0 \text{ information for both sides of the first equation and then rearranging yields:}
\]

\[
\frac{\Gamma_{t_0-1}}{\Gamma_{t_0}} = \frac{\delta c_{m,t}}{\delta c_{o,t}} = \frac{c_{o,t}}{c_{m,t}}\frac{E_{t_0-1}c_{m,t}}{E_{t_0-1}c_{o,t}} = \frac{E_{t_0-1}c_{m,t}}{E_{t_0-1}c_{o,t}}.
\tag{A.10.15}
\]

Using the above and similarly rearranging both of the original equations in (A.10.15):

\[
\frac{c_{m,t}}{c_{o,t}} = \frac{E_{t_0-1}c_{m,t}}{E_{t_0-1}c_{o,t}} = \frac{E_{t_0-1}c_{m,t}}{E_{t_0-1}c_{o,t}} = \frac{E_{t_0-1}c_{m,t}}{E_{t_0-1}c_{o,t}}.
\]

and hence \( c_{y,t-1}/c_{m,t-1} = c_{m,t}/c_{o,t} \) holds for any \( t \) greater than \( t_0 \). Therefore, the recursion in (A.10.13) implies:

\[
\Gamma_{t-1} = \delta \frac{c_{y,t-1}}{c_{m,t-1}} \Gamma_{t-2}, \quad \text{and} \quad \Gamma_t = \delta \frac{c_{y,t}}{c_{m,t}} \Gamma_{t-1} = \delta \frac{c_{y,t}}{c_{m,t}} \delta \frac{c_{m,t}}{c_{o,t}} \Gamma_{t-2} = \delta^2 \frac{c_{y,t}}{c_{o,t}} \Gamma_{t-2}. \tag{A.10.16}
\]

Equation (A.10.14) shows that (A.10.16) is also valid for \( t = t_0 \), so (A.10.16) holds for all \( t \geq t_0 \).
Since \( \delta_{c_{y,t}}/c_{m,t} = \Gamma_t/\Gamma_{t-1} \) and \( \delta_{c_{m,t}}/c_{o,t} = \Gamma_{t-1}/\Gamma_{t-2} \) according to the equations in (A.10.16), the conditions in (4.2c) (with \( \lim \sup_{t \to \infty} \) and strict inequality) satisfied by the consumption allocation imply that \( \lim \sup_{t \to \infty} \Gamma_t/\Gamma_{t-1} < 1 \). As \( t/(t-1) \) converges to 1 as \( t \to \infty \):

\[
\lim_{t \to \infty} \frac{t\Gamma_t}{(t-1)\Gamma_{t-1}} = \left( \lim_{t \to \infty} \frac{t}{t-1} \right) \left( \limsup_{t \to \infty} \frac{\Gamma_t}{\Gamma_{t-1}} \right) = \limsup_{t \to \infty} \frac{\Gamma_t}{\Gamma_{t-1}} < 1.
\]

This yields the sufficient condition specified by the ratio test for convergence of the series \( \sum_{t=0}^{\infty} \Gamma_t \) and \( \sum_{t=0}^{\infty} t\Gamma_t \), so (A.8.3) is satisfied. Step 6 has shown that these conditions are sufficient for the social welfare function to be bounded subject to the constraints. The social welfare function therefore has a well-defined constrained maximum.

Now use equation (A.10.10) and the sequence for \( \Gamma_t \) constructed above to define a sequence \( \Phi_t \) for all \( t \geq t_0 \). Observe that

\[
\Phi_t Y_t = \Gamma_t + \delta \Gamma_{t-1} + \delta^2 \Gamma_{t-2} = \frac{\delta^2 c_{y,t}}{c_{o,t}} \Gamma_{t-2} + \frac{\delta^2 c_{m,t}}{c_{o,t}} \Gamma_{t-2} + \frac{\delta^2 c_{o,t}}{c_{o,t}} \Gamma_{t-2} = \frac{\delta^2 \Gamma_{t-2}}{c_{o,t}}, \quad \text{[A.10.17]}
\]

which uses (A.10.10), (A.10.16), and the resource constraint (2.18) on the consumption allocation. Rearranging this equation shows that \( c_{o,t} = \delta^2 \Gamma_{t-2}/(\Phi_t Y_t) \). Combining this with the following implications of (A.10.16):

\[
c_{m,t} = \frac{c_{o,t} \Gamma_{t-1}}{\delta^2 \Gamma_{t-2}}, \quad \text{and} \quad c_{y,t} = \frac{c_{o,t} \Gamma_t}{\delta^2 \Gamma_{t-2}},
\]

it is established that \( c_{m,t} = \delta^2 \Gamma_{t-1}/(\Phi_t Y_t) \) and \( c_{y,t} = \Gamma_t/(\Phi_t Y_t) \). This establishes that the consumption allocation, \( \Gamma_t \), and \( \Phi_t \) satisfy the equations in (A.10.9), which are equivalent to the first-order conditions (A.10.7a). Setting \( \Gamma_t = \delta \Gamma_{t-1}/N_{y,t-1} \) so that (A.10.7b) holds (with \( H_{m,t} = 1 \)), this argument has demonstrated that the consumption allocation is the solution of the social planner’s problem for some valid sequence of social welfare weights, and therefore that the allocation is Pareto efficient.

For any given constant \( c \) satisfying \( 0 < \delta < c < \infty \), consider the following allocation for all \( t \geq t_0 \):

\[
c_{y,t} = \frac{1}{1 + c + c^2}, \quad c_{m,t} = \frac{c}{1 + c + c^2}, \quad c_{o,t} = \frac{c^2}{1 + c + c^2}, \quad \text{and} \quad H_{m,t} = 1.
\]

This satisfies \( H_{m,t} = 1 \) and the resource constraint (2.18). It satisfies (4.2a) because \( c_{y,t}, c_{m,t}, \) and \( c_{o,t} \) are all non-stochastic constants. The condition (4.2b) holds because \( c_{m,t+1}/c_{y,t} = c/c_{m,t+1} \). Finally, since \( c_{y,t+1}/c_{m,t+1} = 1/c = c_{m,t+1}/c_{o,t+1} \) and \( 1/c < 1/\delta \), condition (4.2c) holds (for \( \lim \sup_{t \to \infty} \) with strict inequality). It follows that there are infinitely many Pareto efficient allocations, completing the proof.

### A.11 Proof of Result 4

(i) It is claimed that the risk sharing condition (4.2a) is equivalent to the following for all \( t \geq t_0 \):

\[
\frac{c_{m,t}}{E_{t-1} c_{m,t}} = \frac{c_{o,t}}{E_{t-1} c_{o,t}}. \quad \text{[A.11.1]}
\]

Substituting the expressions in (2.41) for equilibrium consumption in the incomplete-markets economy, the condition in (A.11.1) is equivalent to:

\[
E_{t-1} \left[ \frac{\alpha \nu_k}{(1 - \alpha \nu_l)} \right] = E_{t-1} \left[ \frac{\alpha(1 - \nu_l)}{(1 - \alpha \nu_l)} \right],
\]

which must hold in all states of the world at date \( t \). After cancelling common terms and simplification this becomes:

\[
\frac{1}{E_{t-1} [(1 - \alpha \nu_l)^{-1}]} = \frac{1 - \nu_l}{E_{t-1} [(1 - \nu_l)(1 - \alpha \nu_l)^{-1}]].
\]

This can be rearranged to give an expression for financial conditions \( \nu_l \):

\[
\nu_l = 1 - \frac{E_{t-1} [(1 - \nu_l)(1 - \alpha \nu_l)^{-1}]}{E_{t-1} [(1 - \alpha \nu_l)^{-1}]]. \quad \text{[A.11.2]}
\]
If this holds in all states of the world at date \( t \) then \( \nu_t \) must be measurable with respect to information available at date \( t - 1 \). Combined with \( \mathbb{E}_{t-1} \nu_t = 0 \), this implies \( \nu_t = 0 \) with probability 1. Conversely, \( \nu_t = 0 \) with probability 1 immediately implies that (2.37) holds in all states of the world, which is equivalent to (A.11.1). This confirms the equivalence of (A.11.1) to the first condition given in (4.3). Using (2.43), \( \nu_t = 0 \) with probability 1 implies \( \pi_t = \mathbb{E}_{t-1} \pi_t \) with probability 1. Conversely, if \( \pi_t = \mathbb{E}_{t-1} \pi_t \) in all states of the world then (2.43) implies \( \nu_t/(1 - \nu_t) = \mathbb{E}_{t-1}[\nu_t/(1 - \nu_t)] \), and hence:

\[
\nu_t = \frac{\mathbb{E}_{t-1} \left[ \frac{\nu_t}{1 - \nu_t} \right]}{1 + \mathbb{E}_{t-1} \left[ \frac{\nu_t}{1 - \nu_t} \right]}. 
\]

This shows that \( \nu_t \) is measurable with respect to information available at date \( t - 1 \), and combined with \( \mathbb{E}_{t-1} \nu_t = 0 \), it implies \( \nu_t = 0 \) with probability 1. It follows that the two conditions stated in (4.3) are equivalent, and both are in turn equivalent to (A.11.1).

Now consider the equivalence of (A.11.1) in the incomplete-markets economy to the risk-sharing requirements in (4.2a). Using the results in (2.41), for any \( t \geq t_0 \):

\[
\frac{c_{y,t}}{\mathbb{E}_{t_0-1} c_{y,t}} = \frac{\alpha \delta \kappa^2}{\sqrt{1 - \alpha \nu_t}} \mathbb{E}_{t_0-1} \left[ \frac{\alpha \delta \kappa}{\sqrt{1 - \alpha \nu_t}} \right] = \frac{c_{m,t}}{\mathbb{E}_{t_0-1} c_{m,t}},
\]

and hence the first equality in (4.2a) always holds in the incomplete-markets economy. Suppose the second equality holds. Taking expectations of both sides conditional on informational available at date \( t - 1 \):

\[
\frac{\mathbb{E}_{t-1} c_{m,t}}{\mathbb{E}_{t_0-1} c_{m,t}} = \frac{\mathbb{E}_{t-1} c_{o,t}}{\mathbb{E}_{t_0-1} c_{o,t}},
\]

which follows because \( t_0 - 1 \leq t - 1 \). Dividing both sides of \( c_{m,t}/\mathbb{E}_{t_0-1} c_{m,t} = c_{o,t}/\mathbb{E}_{t_0-1} c_{o,t} \) by the equation above implies the condition in (A.11.1). Conversely, \( \mathbb{E}_{t_0-1} c_{m,t} \) is always positive by Jensen’s inequality (A.11.3).

(ii) Consider the following inequality version of the consumption smoothing condition (4.2b):

\[
\frac{\mathbb{E}_t c_{m,t+1}}{c_{y,t}} \geq \frac{\mathbb{E}_t c_{o,t+1}}{c_{m,t}}. \tag{A.11.3}
\]

Substituting the expressions for consumption in (2.41), the inequality is equivalent to:

\[
\frac{\mathbb{E}_t \left[ \frac{\alpha \delta \kappa}{\sqrt{1 - \alpha \nu_{t+1}}} \right]}{\sqrt{1 - \alpha \nu_{t+1}}} \geq \frac{\mathbb{E}_t \left[ \frac{\alpha (1 - \nu_{t+1})}{\sqrt{1 - \alpha \nu_{t+1}}} \right]}{\sqrt{1 - \alpha \nu_{t+1}}},
\]

by cancelling common positive terms from both sides and simplifying:

\[
\mathbb{E}_t \left[ \frac{1}{1 - \alpha \nu_{t+1}} \right] \geq \mathbb{E}_t \left[ \frac{1 - \nu_{t+1}}{1 - \alpha \nu_{t+1}} \right].
\]

By grouping all the terms together on the left-hand side, the inequality (A.11.3) is equivalent to:

\[
\mathbb{E}_t C(\nu_{t+1}) \geq 0, \quad \text{where} \quad C(\nu) = \frac{\nu}{1 - \alpha \nu}. \tag{A.11.4}
\]

The first and second derivatives of the function \( C(\nu) \) are:

\[
C'(\nu) = \frac{1}{(1 - \alpha \nu)^2}, \quad \text{and} \quad C''(\nu) = \frac{2\alpha}{(1 - \alpha \nu)^3}, \tag{A.11.5}
\]

and since \( 1 - \alpha \nu > 0 \) for all admissible values of \( \nu \), the function \( C(\nu) \) is strictly convex. Jensen’s inequality and \( \mathbb{E}_t \nu_{t+1} = 0 \) then imply \( \mathbb{E}_t C(\nu_{t+1}) \geq C(\mathbb{E}_t \nu_{t+1}) = C(0) \), and hence \( \mathbb{E}_t C(\nu_{t+1}) \geq 0 \) using the formula for \( C(\nu) \). This confirms that the inequality in (A.11.4) always holds, and thus (A.11.3). Given that (A.11.3) always holds, any failure of consumption smoothing (4.2b) must lead to \( \mathbb{E}_t c_{m,t+1}/c_{y,t} > \mathbb{E}_t c_{o,t+1}/c_{m,t} \).
Now consider the following condition for the nominal interest rate and expected house-price inflation:

\[ i_t = (\beta^{-1} - 1) + \beta^{-1}E_t \pi_{t+1}. \]  

[A.11.6]

This is equivalent to \( 1 + i_t = \beta^{-1}E_t[1 + \pi_{t+1}] \), and by using (2.27), it holds if and only if \( \xi_t = 0 \). Since Result 1 has shown that \( \xi_t \) is strictly positive for any non-degenerate distribution of \( \nu_{t+1} \), this means \( \nu_{t+1} \) has a degenerate probability distribution conditional on information available at date \( t \). As \( E_t \nu_{t+1} = 0 \), it must be the case that \( \nu_{t+1} = 0 \) in all states of the world, which is the first of the two equivalent conditions in (4.3).

Taking \( \nu_{t+1} = 0 \) with probability 1, the formula for \( C(\nu) \) from (A.11.4) implies \( E_t C(\nu_{t+1}) = C(0) = 0 \), and hence the inequality from (A.11.4) holds exactly. Using the earlier argument, this is equivalent to (A.11.3) holding with equality, which is the requirement for consumption smoothing in (4.2b). Hence, this is implied by (A.11.6).

Conversely, suppose the requirement (4.2b) for consumption smoothing is met, which is equivalent to the condition in (A.11.4) holding with equality, but that \( \nu_{t+1} \) has a non-degenerate probability distribution. The strict convexity of \( C(\nu) \) established by (A.11.5) and Jensen’s inequality then imply \( E_t C(\nu_{t+1}) > C(E_t \nu_{t+1}) = C(0) = 0 \). This contradiction confirms that \( \nu_{t+1} \) must have a degenerate probability distribution in this case. Using the earlier results, this means that (4.2b) implies (A.11.6), so the equivalence between these conditions is confirmed.

(iii) Note that the expressions for consumption in (2.41) imply:

\[ \frac{c_{y,t+1}}{c_{m,t+1}} = \frac{\alpha \delta^2 \kappa^2}{1 - \alpha \delta^2 \kappa^2} = \delta \kappa, \quad \text{and hence} \quad \liminf_{t \to \infty} E_t \frac{c_{y,t+1}}{c_{m,t+1}} = \delta \kappa. \]

The first requirement for dynamic efficiency in (4.2c) is thus equivalent to \( \delta \kappa \leq 1/\delta \), and hence \( \beta \leq 1 \) using the expression for \( \beta \) from (2.25). Since it is known that \( \beta < 1 \), this means the first requirement in (4.2c) always holds with strict inequality.

Using (2.41) again:

\[ \frac{c_{m,t+1}}{c_{o,t+1}} = \frac{\alpha \delta}{1 - \alpha \delta} = \delta \kappa, \quad \text{and hence} \quad E_t \frac{c_{m,t+1}}{c_{o,t+1}} = \delta \kappa E_t \left[ (1 - \nu_{t+1})^{-1} \right]. \]

[A.11.7]

Hence, by using the expression for the housing risk premium \( \xi_t \) from (2.38):

\[ E_t \frac{c_{m,t+1}}{c_{o,t+1}} = \delta \kappa (1 + \xi_t). \]

Substituting the definition of \( \xi_t \) from (2.17) and making use of the results in (2.26):

\[ E_t \frac{c_{m,t+1}}{c_{o,t+1}} = \frac{\delta \kappa}{\beta} \left( \frac{1 + E_t \pi_{t+1}}{1 + i_t} \right) = \frac{1}{\delta} \left( \frac{1 + E_t \pi_{t+1}}{1 + i_t} \right), \]

where \( \beta = \delta \kappa \) from (2.25) has been used. It follows immediately that the second requirement for dynamic efficiency in (4.2c) is equivalent to:

\[ \liminf_{t \to \infty} \frac{1}{\delta} \left( \frac{1 + E_t \pi_{t+1}}{1 + i_t} \right) \leq 1. \]

Since \( 0 < \delta < \infty \), this condition can be stated as follows:

\[ \liminf_{t \to \infty} (E_t \pi_{t+1} - i_t) \left( \frac{1}{1 + i_t} \right) \leq 0. \]

From (2.25), \( 1/(1 + i_t) = \beta E_t[(1 + \pi_{t+1})^{-1}] \). The equilibrium conditions rule out \( \pi_{t+1} \) approaching \(-1\) or \( \infty \), which implies that \( 1/(1 + i_t) \) cannot approach 0 or \( \infty \). The second requirement for dynamic efficiency in (4.2c) therefore holds if and only if:

\[ \liminf_{t \to \infty} (E_t \pi_{t+1} - i_t) \leq 0, \]

which is equivalent to \( \limsup_{t \to \infty} (i_t - E_t \pi_{t+1}) \geq 0 \), confirming the claim.

Suppose house-price inflation is predictable in the long run. Using (2.31) and (2.41), it follows that
\[ \lim_{t \to \infty} c_{m,t+1} = \alpha \delta \kappa \text{ and } \lim_{t \to \infty} c_{o,t+1} = \alpha. \] Consequently, \[ \lim_{t \to \infty} E_t c_{m,t+1}/c_{o,t+1} = \alpha \delta \kappa/\alpha = \delta \kappa. \] Since \( \beta = \delta \kappa < 1 \), this directly implies that the second requirement in (4.2c) holds. This completes the proof.

**A.12 Proof of Result 5**

(i) According to Result 4, the conditions (4.2a) and (4.2b) for risk sharing and consumption smoothing are both equivalent in the incomplete-markets economy to nominal house-price inflation being perfectly predictable. Since the probability distribution of \( \nu_t \) in (3.9) is non-degenerate for any \( \epsilon_t > 0 \) and \( x_t > 0 \), a financial crisis equilibrium violates risk sharing and consumption smoothing. These are requirements for Pareto efficiency according to Step 7, so a financial crisis equilibrium is never Pareto efficient.

**Result 3** shows that for a given \( x_{t+1} \), the probability distribution in (3.9) implies \( \xi_t = x_{t+1} \) for small \( \epsilon_t \). Together with the definition of \( \xi_t \) in (2.17) and the results in (2.26):

\[ x_{t+1} = \frac{1}{\beta} \left( 1 - \frac{i_t - E_t \pi_{t+1}}{1 + i_t} \right) - 1. \]

According to Result 4, the dynamic efficiency condition (4.2c) is equivalent to \( \lim sup_{t \to \infty}(i_t - E_t \pi_{t+1}) \geq 0 \), hence this fails to hold if:

\[ \lim_{t \to \infty} x_t > \frac{1}{\beta} - 1. \]

Since (4.2c) is necessary for dynamic efficiency, this confirms the claim in the proposition. With \( x_t = (\alpha - \omega_t)/((1 - \alpha) \omega_t) \), the financial crisis equilibrium is dynamically inefficient for \( \alpha \) sufficiently close to one or \( \omega_t \) sufficiently low.

(ii) Monetary policy can implement an equilibrium where \( \nu_t = 0 \) in all states of the world, which is equivalent to nominal house-price inflation being perfectly predictable (\( \pi_t = E_{t-1} \pi_t \) in all states of the world). Using Step 7 and Result 4, the equilibrium in this case satisfies all the sufficient conditions for Pareto efficiency.

Using Step 3, perfectly predictable nominal house-price inflation (\( \pi_t = E_{t-1} \pi_t \)) implies a zero housing risk premium \( \xi_t = 0 \). Substituting this into (2.27) implies \( i_{t-1} = (1 + E_{t-1} \pi_t)/\beta - 1 \), confirming the claim. The equilibrium nominal interest rate is greater than the value of \( i_{t-1} \) from (3.11) for the equilibrium with financial crises because \( x_t > 0 \) there.

With \( \nu_t = 0 \) in all states of the world, equations (2.31) and (2.35) imply that \( d_t = d^*, h_t = h^* \), and \( l_t = l^* \), where \( d^*, h^* \), and \( l^* \) are as given in (4.4). Direct comparison with (3.10) shows that \( h^* < \bar{h} \) and \( l^* < \bar{l} \) because \( x \) is positive in (3.10). Since the expressions for \( h_t \) and \( l_t \) in (2.35) are strictly convex functions of \( \nu_t \) and coincide with \( h^* \) and \( l^* \) from (4.4) when \( \nu_t = E_t \nu_t = 0 \), and as \( \nu_t \) has a non-degenerate probability distribution in an equilibrium with financial crises, Jensen’s inequality implies that \( E h_t > h^* \) and \( E l_t > l^* \), thus confirming these claims.

(iii) Since the necessary conditions for efficiency require the predictability of house-price inflation, the financial stability equilibrium is the only Pareto-efficient allocation that can be implemented through monetary policy. This completes the proof.

**A.13 Proof of Result 6**

(i) By choosing a monetary policy where nominal house-price inflation is predictable, the consumption-GDP ratios \( c_{x,t} \) of each generation would be non-stochastic. Together with access to lump-sum taxes and transfers, the policymaker can choose any allocation satisfying the resource constraint (2.18). This gives the policymaker the same degree of control as a social planner, and means a higher value of the social welfare function is attainable. Moreover, any solution of the social planner’s problem must be Pareto efficient, so the monetary policy instrument still needs to be used to ensure nominal house-price inflation is predictable (the lump-sum taxes and transfers are not state contingent, so efficiency cannot be achieved without the use of monetary policy).
With complete financial markets, the budget identities (2.6) are replaced by the following:

A.14 Proof of Step 8

The logic in the proof of Result 3 demonstrates that the solution is either a degenerate distribution \( \nu = 0 \) in all states of the world which leads to \( \mathbb{E}W(\nu) = W(0) \), or an unbounded distribution where \( \mathbb{E}W(\nu) = W(\overline{\nu}) \) and \( \overline{\nu} \) must satisfy \( 0 < \overline{\nu} < 1 \). Using the expression for the first derivative in (A.9.3), \( W(\nu) \) is strictly decreasing for all \( \nu > (1 - \alpha^{-1})/(1 - \omega) \). As \( \alpha \leq \omega < 1 \), it follows that \( (1 - \alpha^{-1})/(1 - \omega) < 0 \), hence \( W(\nu) \) is decreasing for all \( 0 \leq \nu < 1 \). This implies \( W(\overline{\nu}) < W(0) \) for any value of \( \overline{\nu} \) satisfying \( 0 < \overline{\nu} < 1 \). Therefore, with \( \omega \geq \alpha \), the degenerate distribution \( \nu = 0 \) delivers a higher value of expected social welfare than any other probability distribution. The welfare-maximizing monetary policy in this case is to make nominal house-price inflation perfectly predictable.

With democratic equal weights, \( \omega_t \) is given by (3.8). Using the expression for \( \alpha \) from (2.35), the condition \( \omega_t \geq \alpha \) is equivalent to:

\[
\frac{1}{1 + \gamma + \gamma^2} \geq \frac{1}{1 + \delta \kappa + (\delta \kappa)^2},
\]

and hence to \( \gamma \leq \delta \kappa \). Step 4 establishes that \( \alpha > 1/3 \), which means that \( \delta \kappa < 1 \). Therefore \( \gamma \leq \delta \kappa \) requires a declining population.

(iii) Now suppose that \( \gamma > \delta \kappa \). The analysis above in part (ii) still implies that financial stability is chosen if and only if \( \omega_t \geq \alpha \). With no commitment, the expression for \( \omega_t \) in (3.6b) has \( t = t_0 \) and hence:

\[
\omega_t = \frac{N_{o,t} \Omega_{t-2}}{N_{y,t} \Omega_t + N_{m,t} \Omega_{t-1} + N_{o,t} \Omega_{t-2}}.
\]

By using this formula and rearranging the inequality \( \omega_t \geq \alpha \):

\[
\Omega_{t-2} \geq \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{N_{m,t} \Omega_{t-1} + N_{y,t} \Omega_t}{N_{o,t}} \right) = \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{N_{m,t} + N_{y,t}}{N_{o,t}} \right) \left( \frac{N_{m,t} \Omega_{t-1} + N_{y,t} \Omega_t}{N_{m,t} + N_{y,t}} \right).
\]

Equation (2.1) implies \( (N_{m,t} + N_{y,t})/N_{o,t} = \gamma + \gamma^2 = \gamma(1 + \gamma) \). The formula for \( \alpha \) in (2.35) implies \( \alpha/(1 - \alpha) = 1/((\delta \kappa(1 + \delta \kappa)) \). Substituting these expressions into the inequality above demonstrates that it is equivalent to the given threshold for \( \Omega_{t-2} \). Moreover, the coefficient \( \gamma(1 + \gamma)/(\delta \kappa(1 + \delta \kappa)) \) is greater than 1 in this case, so \( \Omega_{t-2} \) must be greater than the population-weighted average of \( \Omega_{t-1} \) and \( \Omega_t \) to achieve financial stability. The proof is complete.

A.14 Proof of Step 8

With complete financial markets, the budget identities (2.6) are replaced by the following:

\[
C_{y,t} + \frac{V_I H_{m,t+1}}{P_t} + \mathbb{E}_t[K_{t+1}A_{m,t+1}] = 0; \quad [A.14.1a]
\]

\[
C_{m,t} + \mathbb{E}_t[K_{t+1}A_{o,t+1}] = y_{m,t} + \frac{V_I H_{m,t}}{P_t} + A_{m,t}; \quad [A.14.1b]
\]

and \( C_{o,t} = A_{o,t} \). \quad [A.14.1c]

Since nominal bonds can be spanned by a portfolio of contingent securities, it is no longer necessary to account for these separately. But housing has a non-pecuniary return (the flow value of housing services), so it must be treated separately \( (A_{t,t} \text{ includes all financial wealth, but not housing wealth}) \). Maximizing expected utility (2.2) with respect to portfolios of contingent securities \( A_{m,t+1} \) and \( A_{o,t+1} \) subject to the budget identities in (A.14.1) implies the Euler equations:

\[
\frac{K_{t+1}}{C_{y,t}} = \frac{\delta}{C_{m,t+1}}, \quad \text{and} \quad \frac{K_{t+1}}{C_{m,t}} = \frac{\delta}{C_{o,t+1}}.
\]

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which supersede those in (2.9) for nominal bonds. The Euler equation (2.8) with respect to housing services is unchanged. The market-clearing condition for nominal bonds (2.13) is replaced by:

$$N_{y,t-1}A_{m,t} + N_{m,t-1}A_{o,t} = 0,$$  \[A.14.3\]

but the market clearing conditions (2.11) and (2.12) for goods and housing markets are unchanged. The following new definitions are introduced for the complete financial markets economy:

$$1 + r_t^* = \frac{A_{m,t}}{E_t[K_tA_{m,t}]}, \quad d_t^* = -\frac{N_{m,t}P_tA_{m,t}}{V_tH_t}, \quad \text{and} \quad l_t^* = -\frac{N_{y,t}E_t[K_{t+1}A_{m,t+1}]}{Y_t},$$  \[A.14.4\]

where the superscript * is used to refer to the case of complete financial markets, distinguishing these variables from the ones defined in (2.14) and (2.17) for the incomplete-markets economy. The variable $r_t^*$ is the ex-post real return on the complete-markets portfolio, $d_t^*$ is the value of the contingent securities that pay out relative to the value of all houses in the realized state of the world, and $l_t^*$ is the value of all contingent securities when issued relative to GDP.

Let $c_{y,t}^*, c_{m,t}^*, c_{o,t}^*$, and $h_t^*$ denote the complete financial markets equilibrium values of the variables defined in (2.14). The definitions in (A.14.4) and the equations (2.1), (2.4), and (2.5) immediately imply that the first accounting identity in (2.19a) holds for the complete financial markets economy in terms of the new variables $d_t^*$, $l_t^*$, and $r_t^*$ (the derivation of the second accounting identity is unchanged). Using the market-clearing conditions (2.12) and (A.14.3) (and equations 2.1, 2.3, and 2.4, and the definitions in 2.14 and A.14.4), the budget identities (2.19b) hold in terms of the new variables $d_t^*$ and $l_t^*$. The housing Euler equation (2.8) implies (2.19c) just as before. It follows immediately from the definitions in (A.14.4) that

$$E_t[(1 + r_{t+1}^*)K_{t+1}] = 1,$$

and by substituting the contingent-securities Euler equations from (A.14.2) into the above and using (2.5) and (2.14), the Euler equations in (2.19d) must continue to hold in terms of the new variable $r_t^*$. Equation (2.19e) is not required to hold for $r_t^*$ in the complete-markets economy. In place of (2.19e) is an additional restriction derived from (A.14.2) (and using the definitions in 2.14):

$$\frac{c_{m,t+1}^*}{c_{y,t}^*} = \frac{c_{o,t+1}^*}{c_{m,t}^*},$$  \[A.14.5\]

which equates consumption growth rates for generations alive at the same time. It implicitly defines a complete-markets portfolio with ex-post real return $r_t^*$ that implements a consumption allocation with the property (A.14.5).

Observe that the complete-markets equilibrium satisfies all the requirements for equilibrium in the incomplete-markets economy with the exception of (2.19e). By substituting the budget identity of the old from (2.19b) into the new equilibrium condition (A.14.5):

$$\frac{c_{m,t+1}^*}{c_{y,t}^*} = \frac{d_{t+1}^*h_{t+1}^*}{c_{m,t}^*}, \quad \text{and hence} \quad d_{t+1}^* = \frac{c_{m,t}^*c_{m,t+1}^*}{h_{t+1}^*}.$$

The results derived in Step 1 and Step 2 continue to hold, thus:

$$\frac{c_{m,t}^*}{c_{y,t}^*} = \frac{c_{m,t}^*}{\kappa l_t^*} = \frac{1}{\delta K c_{m,t}^*} = \frac{1}{\delta K}, \quad \text{and} \quad \frac{c_{m,t+1}^*}{h_{t+1}^*} = \frac{c_{m,t+1}^*}{(1 - \kappa)l_t^*} = \frac{1}{\delta (1 - \kappa)c_{m,t}^*} = \frac{1}{\delta (1 - \kappa)}.$$

Substituting these results into (A.14.6) shows that:

$$d_{t+1}^* = \frac{1}{\delta^2 \kappa (1 - \kappa)} = \frac{\lambda}{\beta},$$

using the definitions of $\lambda$ and $\beta$ from (2.24) and (2.25). The equilibrium of the complete-markets economy is equivalent to (2.31) with $\nu_t^* = 0$ irrespective of monetary policy, confirming the claim that $d_t^* = d^*$, with $d^*$ as in (4.4). This means that the equilibrium of the economy is given by equations (2.35) and (2.41) from Result 1 with $\nu_t^* = 0$ replacing the term $\nu_t$. It follows immediately that $c_{y,t}^* = c_{y,t}^*, c_{m,t}^* = c_{m}^*$, and $c_{o,t}^* = c_{o}^*$, where $c_{y,t}^*$, $c_{m}^*$, and $c_{o}^*$ are specified in (2.42). Furthermore, $h_t^* = h^*$ and $l_t^* = l^*$, with $h^*$ and $l^*$ as given in (4.4), confirming the claims. This equilibrium is independent of the conduct of monetary policy.

The equilibrium of the complete financial markets economy is the same as the equilibrium of the incomplete-markets economy when monetary policy stabilizes nominal house-price inflation. That equilib-
rium has already been shown to imply a Pareto-efficient allocation of resources by Result 5, so the complete
financial markets equilibrium is efficient.

In the complete-markets economy, a nominal bond is just a particular portfolio of contingent securities.
By definition, one nominal bond purchased at date \( t \) delivers a future state-contingent real payoff \( \frac{1}{P_{t+1}} + 1 \),
and it must therefore have real value \( E_t[K_{t+1}/P_{t+1}] \) in time period \( t \). The nominal bond price \( Q_t \) (and the
equivalent risk-free nominal interest rate \( i_t \) satisfying 2.10) is thus determined by:

\[
Q_t = P_t E_t \left[ \frac{K_{t+1}}{P_{t+1}} \right].
\]

Given \( Q_t \), the return on the nominal bond is calculated as before using the definitions in (2.17). Using
the equation above together with (2.15) and (2.17), the ex-post real return \( r_t \) on the nominal bond must
satisfy:

\[
E_t [(1 + r_{t+1}) K_{t+1}] = 1.
\]

Using (2.10) and the definition of inflation in (2.15), the ex-post real return on the nominal bond is given
by the ex-post Fisher equation (2.19e). By substituting that into the equation above:

\[
E_t \left[ \frac{(1 + i_t) K_{t+1}}{(1 + \pi_{t+1})} \right] = 1.
\]  

[A.14.7]

The equilibrium asset pricing kernel \( K_{t+1} \) can be obtained from either of the equations in (A.14.2). Taking
the first equation:

\[
K_{t+1} = \delta C_{y,t} C_{m,t+1} = \delta \frac{N_{m,t+1} Y_{t+1} C_{y,t}^*}{N_{y,t} Y_{t+1} C_{m,t+1}^*} = \delta \frac{c_{y,t}^*}{(1 + g_{t+1}) c_{m,t+1}^*},
\]

which uses the definitions of \( c_{y,t} \) in (2.14), the definition of real GDP growth \( g_{t+1} \) from (2.3), and the fact
that \( N_{m,t+1} = N_{y,t} \). Substituting the complete-markets equilibrium values of \( c_{y,t}^* \) and \( c_{m,t+1}^* \) derived earlier,
which are respectively \( c_{y}^* \) and \( c_{m}^* \) from (2.42):

\[
K_{t+1} = \delta \frac{\alpha \delta^2 \kappa^2}{(1 + g_{t+1}) \alpha \delta \kappa} = \delta^2 \frac{\kappa}{1 + g_{t+1}} = \frac{\beta}{1 + g_{t+1}},
\]

where the final step uses the definition of \( \beta \) from (2.25). Substituting this equilibrium asset pricing kernel
into (A.14.7) leads to:

\[
1 = E_t \left[ \frac{\beta (1 + i_t)}{(1 + \pi_{t+1})(1 + g_{t+1})} \right].
\]

Finally, substituting the second accounting identity from (2.19a) into the above:

\[
1 = \beta E_t \left[ \frac{(1 + i_t) h_t^*}{(1 + \pi_{t+1}) h_{t+1}^*} \right] = \beta E_t \left[ \frac{1 + i_t}{1 + \pi_{t+1}} \right],
\]

which uses the complete-markets equilibrium \( h_t^* = h^* \). Rearranging this equation implies that (2.25)
continues to hold in the complete-markets economy. As the optimality condition for housing is unchanged,
the expression for the housing risk premium \( \xi_t \) in (2.26) from Step 3 remains the same. Finally, given that
(2.25) continues to hold, the derivation of Step 5 remains valid. This completes the proof.

A.15 Proof of Result 7

[To be added.]

A.16 Proof of Result 8

[To be added.]