LMS South East Mathematical Physics lecture series

Poisson structures and discrete integrability

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Nonlinear dynamical Systems

- Continuous (differential Equations)
- Discrete (difference equations)

Integrability

- Exact solutions, organised behaviour, conserved quantities, Lax representations, symmetries, vanishing algebraic entropy...
- Various definitions of integrability

Liouville integrable maps

- Suitable Poisson structures
- Liouville's theorem

Outline

1. Introduction

Let M be a smooth manifold. A *Poisson bracket on M* is an alternating bilinear map $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, such that:

(i) $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$ (Jacobi identity)

(ii) $\{f, gh\} = \{f,g\}h + \{f,h\}g$ (Leibniz rule)

for any $f, g, h \in C^\infty(M)$.

The pair $(M, \{\ ,\ \})$ is called a *Poisson manifold.*

For any $H \in C^\infty(M)$ the unique vector field X_H on M, with

$$X_H[f] = \{f, H\} \text{ for any } f \in C^\infty(M),$$

is called the *Hamiltonian vector field* of the Hamiltonian function H.

The map $H \mapsto X_H$ is a Lie algebra antihomomorphism, i.e.,

$$X_{\{H_1,H_2\}} = -[X_{H_1}, X_{H_2}] = [X_{H_2}, X_{H_1}]\ .$$

Example 1. On any manifold M the bracket $\{f, g\} = 0$ is a Poisson bracket.

Example 2. On $M = \mathbb{R}^{2n}$, with coordinates $(q_1, ..., q_n, p_1, ..., p_n)$, the bracket

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

is a Poisson bracket and it is called *canonical Poisson bracket.* The Hamiltonian vector field of a Hamiltonian function H is

$$X_H = \sum_{i=1}^{n} (\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i}).$$

Canonical Hamiltonian equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad \text{for } i = 1, ..., n.$$

On a local coordinate system $\{x_1, ..., x_n\}$

$$\{f_1,\ f_2\}(x) = \sum_{i,j=1}^{n} \{x_i, x_j\}(x) \frac{\partial f_1(x)}{\partial x_i} \frac{\partial f_2(x)}{\partial x_j}.$$

By setting $J_{ij}(x) = \{x_i, x_j\}(x),\ i, j = 1, ..., n,$ we have

$$\{f_1,\ f_2\}(x) = \sum_{i,j=1}^{n} J_{ij}(x) \frac{\partial f_1(x)}{\partial x_i} \frac{\partial f_2(x)}{\partial x_j} = (\sum_{i,j=1}^{n} J_{ij}(x) \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j})(d_x f_1, d_x f_2)$$

So, for any $x \in M$, there is a 2-tensor

$$\pi_x = \sum_{i,j=1}^{n} J_{ij}(x) \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} \in T_x M \otimes T_x M,$$

called *Poisson bivector* or *Poisson tensor* such that

$$\{f_1,\ f_2\}(x) = \pi_x(d_x f_1, d_x f_2).$$

Example The Poisson bivector of the canonical Poisson bracket on $\mathbb{R}^{2n}$ with coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ is $\pi = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \ldots \frac{\partial}{\partial q_n} \wedge \frac{\partial}{\partial p_n}$.

Proposition

Any

$$\pi = \sum_{i<j} J_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \in \wedge^2 TM.$$

defines a Poisson bracket on M *by* $\{f_1, f_2\} = \pi(df_1, df_2)$, *iff*

$$\sum_{l=1}^{n} (J_{il} \frac{\partial J_{jk}}{\partial x_l} + J_{jl} \frac{\partial J_{ki}}{\partial x_l} + J_{kl} \frac{\partial J_{ij}}{\partial x_l}) = 0, \qquad i,\ j,\ k = 1, ..., n.$$

The Hamiltonian vector field of a smooth function H is

$$X_H = \pi^{\#}(dH), \text{ where}$$

$\pi_x^{\#} : T_x^* M \to T_x M$, $< \xi_x, \pi_x^{\#}(\eta_x) >= \pi_x(\xi_x, \eta_x), \quad \forall\ \xi_x, \eta_x \in T_x^* M.$

- Rank of the Poisson structure at x: $rank(\pi_x^{\#}) = rank[J_{ij}(x)]$

Let(M, π) be a Poisson manifold. If for any $x \in M$, the linear map $\pi_x^{\#} : T_x^* M \to T_x M$ is an isomorphism, then M is called symplectic manifold.

The isomorphism $\omega_x^{\#} = (\pi_x^{\#})^{-1} : T_x M \to T_x^* M$, defines a closed 2-form ω (symplectic form) on M by

$$\omega_x(X, Y) =< \omega_x^{\#}(X), Y >= \pi_x(\omega_x^{\#}(X), \omega_x^{\#}(Y)), \quad \forall\ X, Y \in T_x M$$

- Any symplectic manifold is a Poisson manifold with Poisson bracket

$$\{f, g\} = \omega(X_f, X_g).$$

- A function F which is in involution with the Hamiltonian H of a Hamiltonian vector field X_H, i.e. $\{F, H\} = 0$, is called (*first*) *integral* of X_H. If ϕ_t is the flow of the vector field X_H then $F \circ \phi_t = F$.

  Poisson Theorem If $\{f, h\} = 0$ and $\{g, h\} = 0$ then $\{\{f, g\}, h\} = 0$.

- The functions which are in involution with any functions on M are called *Casimir functions*.
- We consider two Poisson manifolds $(M, \{\ ,\ \}_M))$, $(N, \{\ ,\ \}_N)$. A map $\varphi\colon M \to N$ is called a Poisson map if

$$\{f \circ \varphi, g \circ \varphi\}_M = \{f, g\}_N \circ \varphi, \text{ for all } f, g \in C^\infty(N).$$

  The flow $\phi_t : M \to M$ of any Hamiltonian vector field is a Poisson map.

- A submanifold N of a Poisson manifold M is called *Poisson submanifold* if there is a Poisson bracket on N such as the inclusion map $i : N \hookrightarrow M$ is a Poisson map.
  $N \subset M$ is a Poisson submanifold, iff every Hamiltonian vector field on M is tangent to N at the points of N.

  Minimal Poisson submanifolds: *symplectic leaves*

Symplectic leaves

We consider on (M, π) the relation: $x \sim y$ if there is a piecewise-smooth curve that joints x and y, such that each piece is an integral curve of a Hamiltonian vector field.

- The relation $\sim$ on M is an equivalence relation. The equivalence classes are called symplectic leaves
- The symplectic leaves of a Poisson manifold are symplectic manifolds, with $\dim \mathcal{O}_x = \operatorname{rank} \pi_x$
- The symplectic leaves are included on the level sets of the Casimir functions.
- Let $f_1, ..., f_k$ be Casimir functions, $S_i = \{x \in M \ / \ f_i(x) = c_i\}$ and $S = S_1 \cap ... \cap S_k$. If $dimS = rank(\pi(x))$ for every $x \in S$, then the connected components of S are symplectic leaves.

i) Linear Poisson structures (Lie-Poisson bracket)

$$\{f,g\}_{L-P}(x) =< x, [d_x f, d_x g] >, \ x \in \mathfrak{g}^*, \ f,g \in C^\infty(\mathfrak{g}^*),$$

where $\mathfrak{g}$ is a Lie algebra. For $x = (x_1, \dots x_n) \in \mathfrak{g}^*$ and c_{ij}^k the structure constants of $\mathfrak{g}$ with respect to the basis $\{v_1, \dots, v_n\}$
($[v_i, v_j] = \sum_{k=1}^n c_{ij}^k v_k, \ i,j = 1, \dots, n$)

$$J_{ij}(x) = \sum_{k=1}^n c_{ij}^k x_k, \ i,j = 1, \dots, n.$$

ii) Log-canonical Poisson structures

For any skew symmetric matrix $B = [b_{ij}]$, we consider

$$\{x_i, x_j\} = b_{ij} x_i x_j, \ i,j = 1, \dots, n.$$

iii) Cartesian product of Poisson manifolds

Let (M_1, π_1) and (M_2, π_2) be two Poisson manifolds, then $M_1 \times M_2$ is a Poisson manifold with Poisson tensor $\pi = \pi_1 \oplus \pi_2$.

Definition

A Hamiltonian system on a $2n$-dimensional symplectic manifold M is called *(Liouville) integrable* if it admits n functionally independent integrals $f_1, \ldots, f_n$, $(df_1 \wedge df_2 \wedge \cdots \wedge df_n \neq 0)$ in involution, i.e.

$$\{f_i, f_j\} = 0, \ i, j = 1, ..., n.$$

Theorem *(Liouville–Arnold)*

If a Hamiltonian system is Liouville integrable then the Hamilton's equations can be solved by quadratures and each connected component of a compact level set of $f_1, \ldots, f_n$:

$$\mathcal{L} = \{x \in M : f_i(x) = c_i, \ 1 \leq i \leq n\}$$

is diffemorphic to an n-dimensional torus T^n.

Definition

Let M be a $2n$-dimensional symplectic manifold. A symplectic map $\Phi : M \to M$ is called Liouville integrable if there are n functionally independent integrals in involution, i.e. there are n functionally independent functions $f_1, \ldots, f_n$ on M such that

$$f_i \circ \Phi = f_i, \ \textit{and} \ \{f_i, f_j\} = 0, \ i, j = 1, \ldots, n.$$

*Liouville's theorem for integrable maps**

$x_{n+1} = \Phi(x_n), \ x_n \in M$ (solutions by quadratures, Liouville tori)

Poisson case

Let (P, π) be an $n = (2r + s)$-dimensional Poisson manifold with $rank\pi = 2r$ and s funct. independent Casimir functions. A Poisson map $\Phi : P \to P$ is called Liouville integrable if there are $r + s$ functionally independent integrals in involution.

*Maeda S (1987), Veselov A P (1991), Bruschi M, Ragnisco O, Santini P M, Gui-Zhang T (1991)

2. A discretisation preserving the Poisson structure

A generalized Lotka-Volterra system

We consider on $\mathbb{R}^n$ the generalized Lotka-Volterra system

$$\dot{x}_i = x_i \sum_{j=1}^{n} A_{ij} x_j \ , \ \text{for } i = 1, \ldots, n, \ \text{for}$$

$$A = \begin{pmatrix} 0 & a_2 & a_3 & \ldots & a_n \\ -a_1 & 0 & a_3 & \ldots & a_n \\ -a_1 & -a_2 & 0 & \ldots & a_n \\ \vdots & \vdots & \vdots & \ddots & \\ -a_1 & -a_2 & -a_3 & \ldots & 0 \end{pmatrix}$$

and $(a_1, \ldots, a_n) \in \mathbb{R}^n$, i.e.,

$$\dot{x}_i = x_i (\sum_{j>i} a_j x_j - \sum_{j<i} a_j x_j).$$

This is a Hamiltonian system with respect to

$$\{x_i, x_j\} = x_i x_j, \ \text{for } 1 \leq i < j \leq n,$$

and $H = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$.

O. I. Bogoyavlenskij (1987)
van der Kamp P., K.T., Quispel G. R. W., Tran D.T, Vanhaecke P. (2014)

For $k = 1, \ldots, \left[\frac{n+1}{2}\right]$ we consider the functions

$$F_k = \begin{cases} v_{2k-1} \frac{x_{2k+1} x_{2k+3} \cdots x_n}{x_{2k} x_{2k+2} \cdots x_{n-1}} & \text{if } n \text{ is odd} \\ v_{2k} \frac{x_{2k+2} x_{2k+4} \cdots x_n}{x_{2k+1} x_{2k+3} \cdots x_{n-1}} & \text{if } n \text{ is even} \end{cases}, C := \frac{x_1 x_3 \ldots x_n}{x_2 x_4 \ldots x_{n-1}} .$$

for $v_0 := 0$, $v_i := a_1 x_1 + \cdots + a_i x_i$, $i = 1, \ldots, n$ $(F_{[(n+1)/2]} = v_n = H)$.

Proposition

$\{F_k, F_l\} = \{F_k, H\} = 0$, for $k, l \in \{1, \ldots, \left[\frac{n+1}{2}\right]\}$. Moreover, when n is odd $C = F_1/a_1$ is a Casimir function of the Poisson bracket.

- For any n, the Hamiltonian system

$$\dot{x}_i = x_i \Big(\sum_{j>i} a_j x_j - \sum_{j<i} a_j x_j \Big)$$

is Liouville integrable.

We consider systems with quadratic vector fields

$$\frac{dx_i}{dt} = \sum_{j,k} a_{ijk} x_j x_k + \sum_j b_{ij} x_j + c_i.$$

The Kahan discretisation is given by

$$\frac{\tilde{x}_i - x_i}{h} = \sum_{j,k} a_{ijk} \frac{\tilde{x}_j x_k + x_j \tilde{x}_k}{2} + \sum_j b_{ij} \frac{x_j + \tilde{x}_j}{2} + c_i.$$

- $\dot{x} = f(x) \xrightarrow{Kahan} \frac{\tilde{x} - x}{h} = 2\, f\left(\frac{x + \tilde{x}}{2}\right) - \frac{1}{2} f(\tilde{x}) - \frac{1}{2} f(x)$

Kahan '93, Kahan, Li '97, Hirota, Kimura '00

The Kahan map of the system

The Kahan discretisation of the system

$$\dot{x}_i = x_i \left(\sum_{j=i+1}^{n} a_j x_j - \sum_{j=1}^{i-1} a_j x_j \right), \quad i = 1, \ldots, n,$$

is the rational map $\mathcal{K} : (x_1, \ldots, x_n) \mapsto (\tilde{x}_1, \ldots, \tilde{x}_n)$, given by

$$\tilde{x}_i = x_i \frac{(1 - hH_0)(1 + hH_0)}{(1 - hH_0 + 2hv_{i-1})(1 - hH_0 + 2hv_i)},$$

where $v_j = a_1 x_1 + \cdots + a_j x_j$ and $H_0 = v_n$.

The Kahan map of the system

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where $v_j = a_1 x_1 + \cdots + a_j x_j$ and $H_0 = v_n$.

Theorem

The Kahan map $\mathcal{K}$ is a Poisson map with respect to the Poisson bracket $\{x_i, x_j\} = x_i x_j$, for $1 \leq i < j \leq n$. Furthermore, $\mathcal{K}$ preserves all the integrals of the continues system.

- *$\mathcal{K}$ is Liouville integrable.*

K.T., Quispel G. R. W., Vanhaecke P. (2016)

Remarks - Perspectives for future work

- Determination of Kahan discretisations preserving the Poisson structure of the continuous system
- Investigation of Lotka–Volterra systems with different community matrices, or integrable deformations of them, and their Kahan discretisation

$$\dot{x_i} = \sum_{j=1}^{n} A_{ij} x_i x_j + r_i x_i + c_i$$

P. A. Damianou, C. A. Evripidou, P. Kassotakis, P. Vanhaecke '17,

H. Christodoulidi, A.N.W. Hone, T.K. '19

- Integrability aspects of Kahan discretisation

E. Celledoni, R.I. McLachlan, D.I. McLaren, B. Owren, G.R.W. Quispel '13, '14, '17

A.N.W. Hone, M. Petrera '09, A. N. W. Hone, G. R. W. Quispel '19

M. Petrera, A. Pfadler, Y.B. Suris '09, '11, M. Petrera, J. Smirin, Y.B. Suris '19

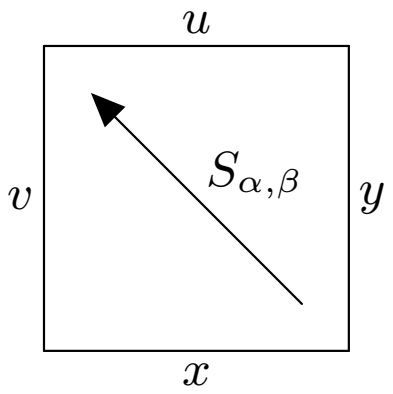
3. Refactorization problems and r-matrix Poisson structures

A *parametric refactorization system* $(S_{\alpha,\beta}, L_1, L_2)$ consists of a map $S_{\alpha,\beta} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$,

$$S_{\alpha,\beta}(x, y) \mapsto (u, v) := (u_{\alpha,\beta}(x, y), v_{\alpha,\beta}(x, y))$$

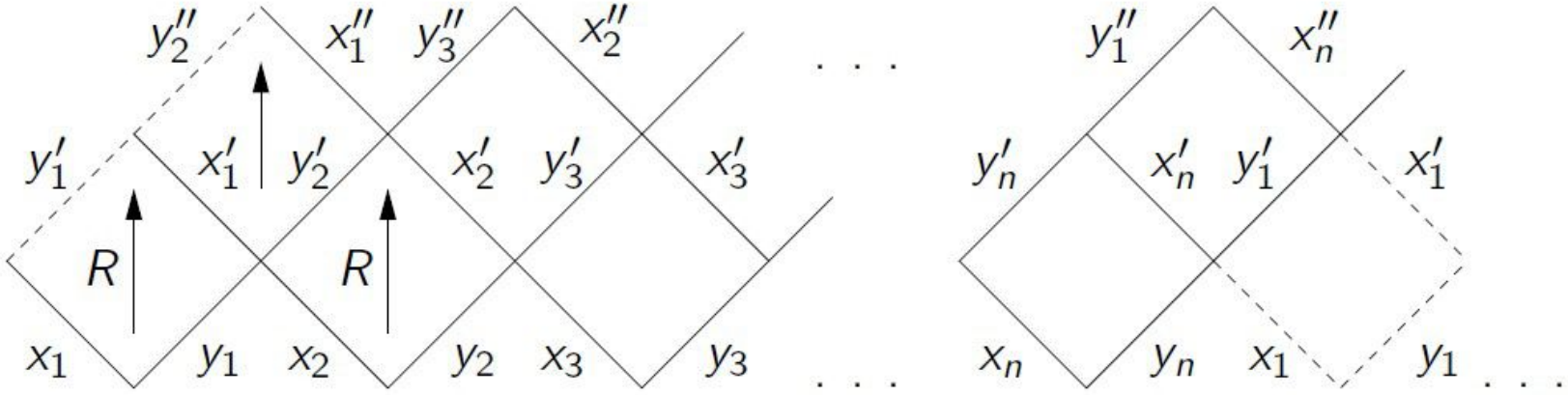
and two matrices $L_1(x, a, \zeta), L_2(y, \beta, \zeta)$ (*Lax pair*), such that

$$L_1(u, \alpha, \zeta)L_2(v, \beta, \zeta) = L_2(y, \beta, \zeta)L_1(x, \alpha, \zeta), \text{ for every } x, y \in \mathcal{X}.$$



Reshetikhin, Semenov-Tian-Shansky '88, Moser, Veselov '91, Adler 93, Adler, Yamilov '93, Veselov '02, Suris, Veselov '03, KT, Papageorgiou '09, '11

$(S_{\alpha,\beta}, L_1, L_2)$, $S_{\alpha,\beta} : (x, y) \mapsto (x', y')$ a refactorization system.



Transfer map $T_n : (x_1, ..., x_n, y_1, ..., y_n) \mapsto (x_1', ..., x_n', y_2', y_3' ..., y_n', y_1')$

The k-transfer map: $T_n^k = \underbrace{T_n \circ ... \circ T_n}_{k}$.

$(T_n^n(x_1, ..., x_n, y_1, ..., y_n) = (x_1^{(n)}, ..., x_n^{(n)}, y_1^{(n)}, ..., y_n^{(n)}))$

Papageorgiou V G, Nijhoff F W, Capel H W 1990

For any n-periodic initial value problem we define the **monodromy matrix** :

$$M_n(x_1, ..., x_n, y_1, ..., y_n) = \overset{\curvearrowleft}{\prod_{i=1}^{n}} L_2(y_i; \beta) L_1(x_i; \alpha)$$

Proposition

The transfer map preserves the spectrum of the monodromy matrix.

$$M_n(T_n(x_1, ..., x_n, y_1, ..., y_n)) L_2(y_1'; \beta) = L_2(y_1'; \beta) M_n(x_1, ..., x_n, y_1, ..., y_n).$$

Example

The equation $L(u_1, u_2; \bar{\alpha})L(v_1, v_2; \bar{\beta}) = L(y_1, y_2; \bar{\beta})L(x_1, x_2; \bar{\alpha})$, for

$$L(x_1, x_2, \bar{\alpha}) = \begin{pmatrix} \alpha_1\zeta + \frac{\alpha x_2}{\alpha_1 x_1 - \alpha_2 x_2} & -\frac{\alpha x_1 x_2}{\alpha_1 x_1 - \alpha_2 x_2} \\ \frac{\alpha}{\alpha_1 x_1 - \alpha_2 x_2} & \alpha_2\zeta - \frac{\alpha x_1}{\alpha_1 x_1 - \alpha_2 x_2} \end{pmatrix}, \bar{\alpha} = (\alpha, \alpha_1, \alpha_2),$$

admits the unique solution

$$u_1 = F(y_1, x_1, y_2, \bar{\alpha}, \bar{\beta}), u_2 = y_2, v_1 = x_1, v_2 = F(x_2, x_1, y_2, \bar{\alpha}, \bar{\beta}),$$

where $F(x, x_1, x_2, \bar{\alpha}, \bar{\beta}) = \frac{\beta\alpha_2 x_2(\alpha_1 x_1 - \alpha_2 x)x_2 + \alpha\beta_1 x_1(\beta_1 x - \beta_2 x_2)}{\beta\alpha_1(\alpha_1 x_1 - \alpha_2 x) + \alpha\beta_2(\beta_1 x - \beta_2 x_2)}$.

So, $(S_{\bar{\alpha},\bar{\beta}}, L, L)$, with $S_{\bar{\alpha},\bar{\beta}}(x_1, x_2, y_1, y_2) \mapsto (u_1, u_2, v_1, v_2)$ is a (strong) refactorization system.

$$T_1 = S_{\bar{\alpha},\bar{\beta}},\ M_1(x_1, x_2, y_1, y_2) = L(y_1, y_2; \bar{\beta})L(x_1, x_2, \bar{\alpha}).$$

Two integrals from $trM_1 : I_1 = \frac{(x_2 - y_1)(x_1 - y_2)}{(\alpha_1 x_1 - \alpha_2 x_2)(\beta_1 y_1 - \beta_2 y_2)}, I_2 =$ ...too big!

We denote by $\mathcal{L}_m^n$ the set of $m \times m$, $n-$degree polynomial matrices

$$L(\zeta) = X_0 + \zeta X_1 + ... + \zeta^n X_n, \quad X_i \in Mat_{m\times m}(\mathbb{K}), \ \zeta \in \mathbb{K}$$

The Sklyanin bracket:

$$\{L(\zeta) \overset{\otimes}{,} L(\eta)\} = [\frac{r}{\zeta - \eta}, L(\zeta) \otimes L(\eta)], \quad r(x \otimes y) = y \otimes x.$$

For $L(\zeta) = [a_{ij}(\zeta)]$

$$\{L(\zeta) \overset{\otimes}{,} L(\eta)\} = \begin{pmatrix} A_{11} & ... & A_{1m} \\ \vdots & \vdots & \vdots \\ A_{m1} & ... & A_{mm} \end{pmatrix}, A_{ij} = \begin{pmatrix} \{a_{ij}(\zeta), a_{11}(\eta)\} ... \{a_{ij}(\zeta), a_{1m}(\eta)\} \\ \vdots \qquad\qquad \vdots \\ \{a_{ij}(\zeta), a_{2m}(\eta)\} ... \{a_{ij}(\zeta), a_{mm}(\eta)\} \end{pmatrix}$$

$$L(\zeta) \otimes L(\eta) = \begin{pmatrix} a_{11}(\zeta)L(\eta) & ... & a_{1m}(\zeta)L(\eta) \\ \vdots & & \vdots \\ a_{m1}(\zeta)L(\eta) & ... & a_{mm}(\zeta)L(\eta) \end{pmatrix}, \qquad dim\mathcal{L}_m^n = m^2(n+1)$$

E.K. Sklyanin 1983

For $L(\zeta) = X - \zeta A \in \mathcal{L}_2^1$, (here $r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$) with

$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ the Poisson structure matrix is

$$J_A(X) = \begin{pmatrix} 0 & -x_2a_1 + x_1a_2 & x_3a_1 - x_1a_3 & x_3a_2 - x_2a_3 \\ * & 0 & x_4a_1 - x_1a_4 & x_4a_2 - x_2a_4 \\ * & * & 0 & -x_4a_3 + x_3a_4 \\ * & * & * & 0 \end{pmatrix}$$

where $J_A(X)_{ij} = \{x_i - \zeta a_i, x_j - \eta a_j\}$ for $i, j = 1, ..., 4$.

Six Casimir functions on $\mathcal{L}_2^1$: $\quad a_i$ for $i = 1, ..., 4$ and

$$f_0(X; A) = \det X, \;\; f_1(X; A) = a_4x_1 - a_3x_2 - a_2x_3 + a_1x_4,$$

i.e. the coefficients of the polynomial

$$\det L(\zeta) = f_2(X; A)\zeta^2 - f_1(X; A)\zeta + f_0(X; A)$$

$(f_2(X; A) = \det A)$

Let $(S_{\alpha,\beta}, L_1, L_2)$, $S_{\alpha,\beta} : (\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{u}, \mathbf{v})$ a ref. system, such that

$$\begin{aligned}
\{L_1(\zeta) \overset{\otimes}{,} L_1(\eta)\} &= [\frac{r}{\zeta - \eta}, L_1(\zeta) \otimes L_1(\eta)], \\
\{L_2(\zeta) \overset{\otimes}{,} L_2(\eta)\} &= [\frac{r}{\zeta - \eta}, L_2(\zeta) \otimes L_2(\eta)] \\
\{L_1(\zeta) \overset{\otimes}{,} L_2(\eta)\} &= 0. \text{ Then for } i = 1, 2
\end{aligned}$$

$$\{L_i(\zeta)L_j(\zeta) \overset{\otimes}{,} L_i(\eta)L_j(\eta)\} = [\frac{r}{\zeta - \eta}, L_i(\zeta)L_j(\zeta) \otimes L_i(\eta)L_j(\eta)].$$

- Poisson transfer maps with respect to the Sklyanin bracket
- The integrals are in involution $\{TrM_n(\bar{x}, \bar{y}, \zeta), TrM_n(\bar{x}, \bar{y}, \eta)\} = 0$ (*Babelon, Viallet '90*)

Example

$$L(\mathbf{x},\zeta) := L(x_1,x_2,\bar{\alpha}) = \begin{pmatrix} \alpha_1\zeta + \frac{\alpha x_2}{\alpha_1 x_1 - \alpha_2 x_2} & -\frac{\alpha x_1 x_2}{\alpha_1 x_1 - \alpha_2 x_2} \\ \frac{\alpha}{\alpha_1 x_1 - \alpha_2 x_2} & \alpha_2\zeta - \frac{\alpha x_1}{\alpha_1 x_1 - \alpha_2 x_2} \end{pmatrix},$$

satisfies $\{L(\mathbf{x},\zeta) \overset{\otimes}{,} L(\mathbf{x},h)\} = [\frac{r}{\zeta - h}, L(\mathbf{x},\zeta) \otimes L(\mathbf{x},h)]$, iff

$$\{x_1,x_2\} = -\frac{(\alpha_1 x_1 - \alpha_2 x_2)^2}{\alpha}.$$

$T_1 = S_{\bar{\alpha},\bar{\beta}}$ is Poisson map with respect to

$$\{x_1,x_2\} = -\frac{(\alpha_1 x_1 - \alpha_2 x_2)^2}{\alpha}, \{y_1,y_2\} = -\frac{(\beta_1 y_1 - \beta_2 y_2)^2}{\beta}, \{x_i,y_j\} = 0.$$

Furthermore, $\{I_1,I_2\} = 0$. So, $S_{\bar{\alpha},\bar{\beta}}$ is Liouville integrable map.

- Similarly, for any n, T_n is Poisson and $\{I_i,I_j\} = 0$.

The Adler–Yamilov map

We consider $L(\zeta) = X - \zeta A$, with $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

- Poisson structure matrix: $J_A(X) = \begin{pmatrix} 0 & -x_2 & x_3 & 0 \\ * & 0 & x_4 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{pmatrix}$
- Casimir functions: $f_0(X) = x_1x_4 - x_2x_3,\ f_1(X) = x_4$.

  We set $f_0(X) = \alpha$, $f_1(X) = 1$ and solve with respect to x_4 and x_1.
- Symplectic leaves: $\{ \begin{pmatrix} x_2x_3 + \alpha - \zeta & x_2 \\ x_3 & 1 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \}$

  (reduced Poisson bracket: $\{x_2, x_3\} = 1$).

The equation $L(u_1, u_2, \alpha)L(v_1, v_2, \beta) = L(y_1, y_2, \beta)L(x_1, x_2, \alpha)$, for

$$L(x_1, x_2, \alpha) = \begin{pmatrix} x_1x_2 + \alpha - \zeta & x_1 \\ x_2 & 1 \end{pmatrix}$$

admits the unique solution

$$u_1 = y_1 - \frac{(\alpha - \beta)x_1}{1 + x_1y_2}, \ u_2 = y_2, \ v_1 = x_1, \ v_2 = x_2 + \frac{(\alpha - \beta)y_2}{1 + x_1y_2} \ .$$

- The map $S_{\alpha,\beta} : (x_1, x_2, y_1, y_2) \mapsto (u_1, u_2, v_1, v_2)$ is symplectic with respect to
$$\{x_1, x_2\} = 1, \ \{y_1, y_2\} = 1, \ \{x_i, y_j\} = 0.$$

Two integrals from $\mathrm{tr}L(y_1, y_2; \beta)L(x_1, x_2, \alpha)$:

$$I_1 = x_1x_2 + y_1y_2, \ I_2 = \alpha y_1y_2 + \beta x_1x_2 + x_1y_2 + x_2y_1 + x_1x_2y_1y_2 \ ,$$

with $\{I_1, I_2\} = 0$.

- The map $S_{\alpha,\beta}$ is Liouville integrable.

Remarks

- Refactorization systems with $L_1 = L_2 \longleftrightarrow$ *Yang-Baxter maps*

  ($S_{23} \circ S_{13} \circ S_{12} = S_{12} \circ S_{13} \circ S_{23}$, with $S_{12}(x, y, z) = (u(x, y), v(x, y), z)$,
  $S_{13}(x, y, z) = (u(x, z), y, v(x, z))$, $S_{23}(x, y, z) = (x, u(y, z), v(y, z))$)

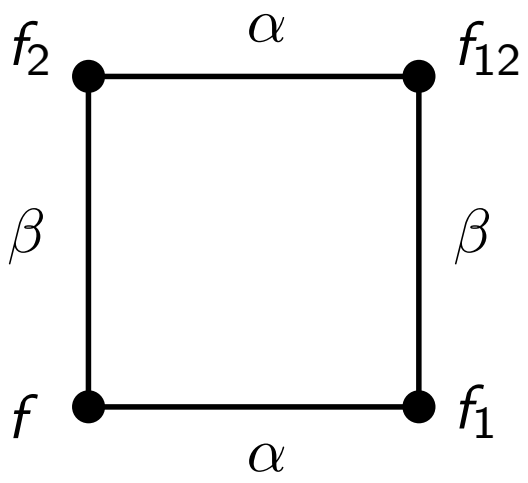
- Lax matrices of YB maps as symplectic leaves of the Sklyanin bracket and integrability

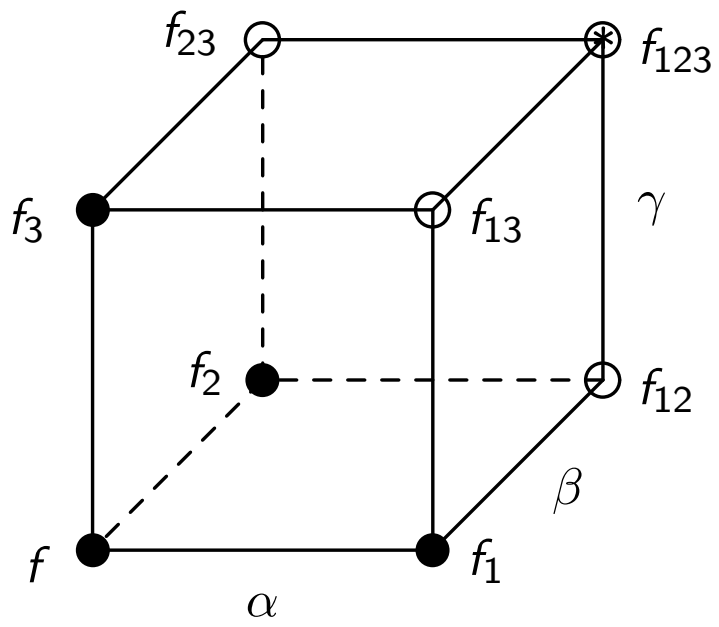
- Yang-Baxter maps $\longleftrightarrow$ Integrable equations on quad-graphs (3D consistency)

Yang '67, Baxter '72, Sklyanin '83,'88, Drinfeld '92, Weinstein, Xu, '92, Bukhshtaber '98, Etingof, Schedler, Soloviev '99, Adler, Bobenko, Suris '03, Papageorgiou, Tongas '07,'08, Shibukawa '07, Kouloukas, Papageorgiou '09,'11,'12, Papageorgiou, Suris, Tongas, Veselov '10, Konstantinou-Rizos, Mikhailov '13, Kassotakis '19, Dimakis, Müller-Hoissen '19, '20 Konstantinou-Rizos, Papamikos '19, ...

4. Poisson structures for integrable quadrilateral equations

We consider equations of the type $Q(f, f_1, f_2, f_{12}; \alpha, \beta) = 0$, that can be uniquely solved for any one of $f, f_1, f_2, f_{12} \in \mathbb{C}$.





The value f_{123} can be determined by three different ways. If all the three values coincide then we call the equation Q *3D consistent*.

Nijhoff, Walker '01, Nijhoff '02, Bobenko, Suris '02, Adler, Bobenko, Suris '03

The 3D consistency of a quadrilateral equation gives rise to Lax representations, i.e. an equation

$$L(f_2, f_{12}, \alpha)L(f, f_2, \beta) = L(f_1, f_{12}, \beta)L(f, f_1, \alpha)$$

for some matrix L, equivalent to $Q(f, f_1, f_2, f_{12}; \alpha, \beta) = 0$. The matrix L is called a *Lax matrix* of the equation.

- *Example* The KdV quad-graph equation[1],

$$(f_{12} - f)(f_1 - f_2) = \alpha - \beta,$$

is a 3D consistent equation with Lax matrix

$$L(x_1, x_2, \alpha) = \begin{pmatrix} x_1 & \alpha + x_1 x_2 - \zeta \\ -1 & -x_2 \end{pmatrix}.$$

[1] *Nijhoff, Capel '95*

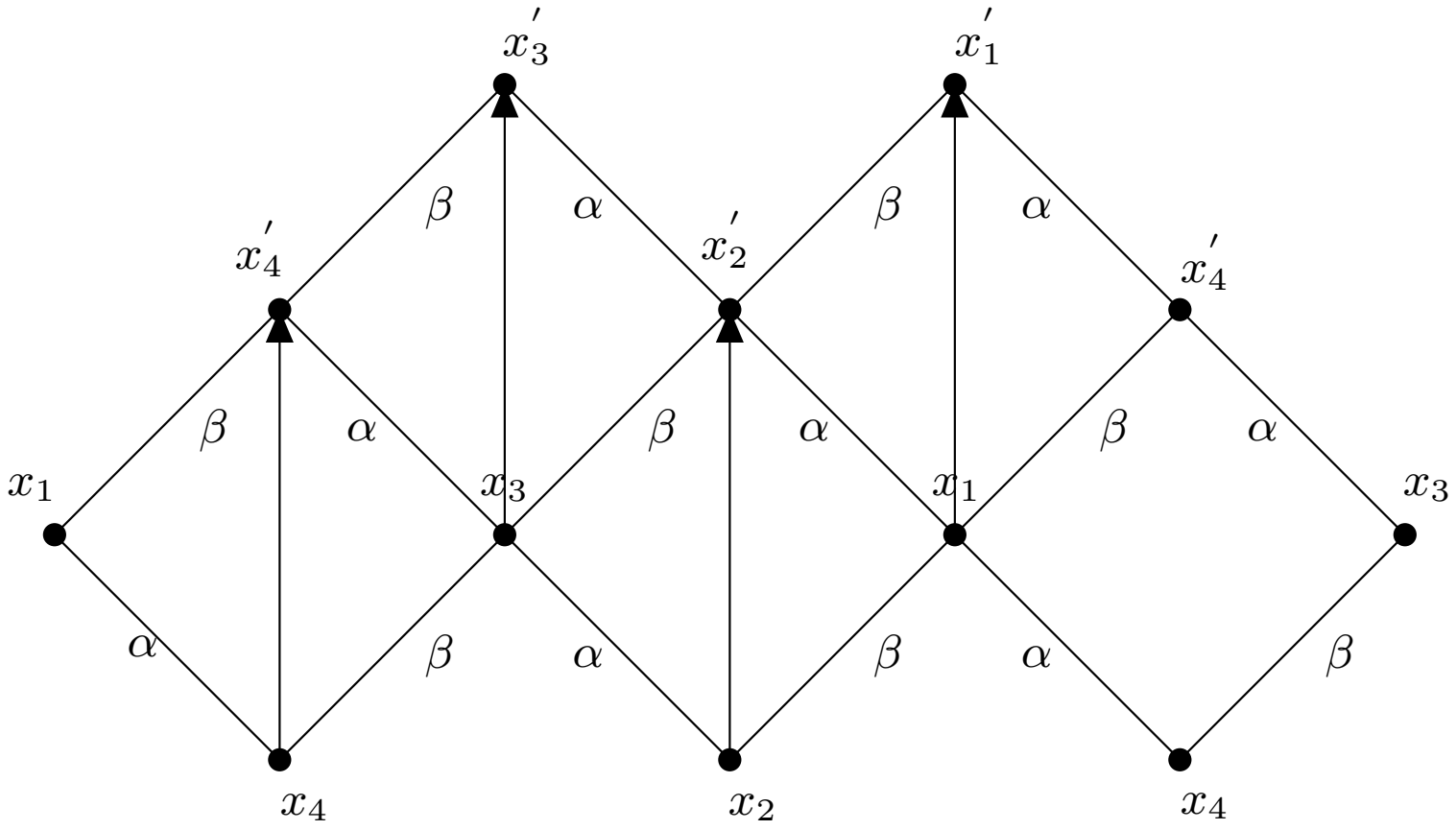


Figure: $\mathcal{S}_{\alpha,\beta} : (x_1, x_2, x_3, x_4) \mapsto (x_1', x_2', x_3', x_4')$

- Monodromy matrix: $M = L(x_2, x_1, \beta)L(x_3, x_2, \alpha)L(x_4, x_3, \beta)L(x_1, x_4, \alpha)$

Papageorgiou, Nihoff, Capel '90, Quispel, Capel, Papageorgiou, Nijhoff '91

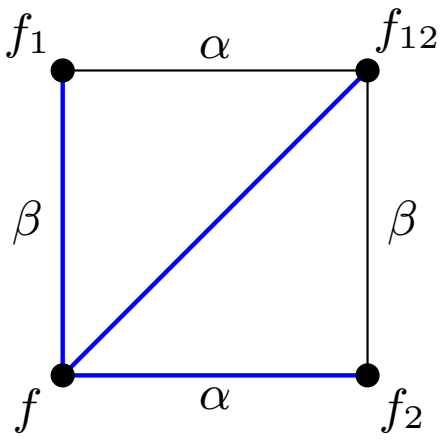
A three-leg form of a quad-graph equation

$$Q(f, f_1, f_2, f_{12}; \alpha, \beta) = 0,$$

centered at f, is an equivalent equation

$$\psi(f, f_1, \alpha) - \psi(f, f_2, \beta) = \phi(f, f_{12}, \alpha, \beta),$$

for some functions ψ and ϕ. All equations in the ABS list (after some transformations) have a three leg-form.



Adler, Bobenko, Suris '03

We consider a quad-graph equation $Q(f, f_1, f_2, f_{12}; \alpha, \beta) = 0$ with three-leg form $\psi(f, f_1, \alpha) - \psi(f, f_2, \beta) = \phi(f, f_{12}, \alpha, \beta)$.

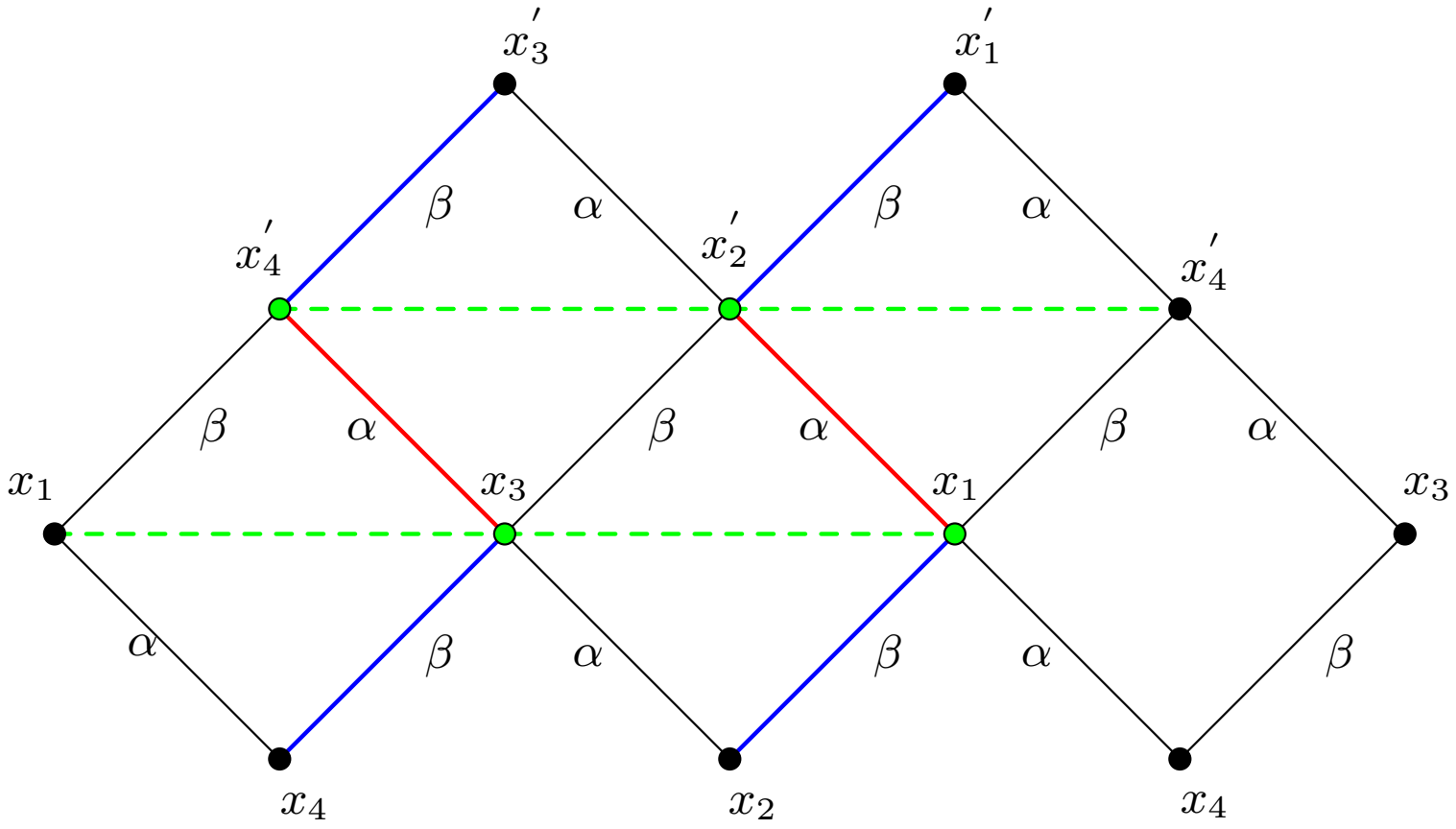
Proposition

The $(2,2)$ per. reduction $\mathcal{S}_{\alpha,\beta} : (x_1, x_2, x_3, x_4) \mapsto (x_1', x_2', x_3', x_4')$ is a Poisson map with respect to

$$\pi = \frac{1}{s(x_1, x_2, \beta)} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{1}{s(x_3, x_4, \beta)} \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4},$$

where

$$s(f, f_1, \alpha) = \frac{\partial \psi(f, f_1, \alpha)}{\partial f_1} = \frac{\partial \psi(f_1, f, \alpha)}{\partial f}.$$



$$\frac{\partial\phi(x_i, x_j, \alpha, \beta)}{\partial x_j} = \frac{\partial\phi(x_j, x_i, \alpha, \beta)}{\partial x_i}$$

$$s(x_1, x_2, \beta)dx_1 \wedge dx_2 + s(x_3, x_4, \beta)dx_3 \wedge dx_4 = s(x_1, x_2', \alpha)dx_1 \wedge dx_2' + s(x_3, x_4', \alpha)dx_3 \wedge dx_4$$
$$s(x_1, x_2', \alpha)dx_1 \wedge dx_2' + s(x_3, x_4', \alpha)dx_3 \wedge dx_4 = s(x_1', x_2', \beta)dx_1' \wedge dx_2' + s(x_3', x_4', \beta)dx_3' \wedge dx_4'$$

For equations in the ABS list, the function s is given as follows:

- H_1: $s(f, f_1, \alpha) = 1$
- H_2: $s(f, f_1, \alpha) = \frac{1}{f+f_1+\alpha}$
- $H_3^{\delta=0}$: $s(f, f_1, \alpha) = \frac{1}{ff_1}$, $\quad H_3^{\delta=1}$: $s(f, f_1, \alpha) = \frac{1}{ff_1+\alpha}$
- $Q_1^{\delta=0}$: $s(f, f_1, \alpha) = \frac{\alpha}{(f-f_1)^2}$, $Q_1^{\delta=1}$: $s(f, f_1, \alpha) = \frac{1}{f-f_1-\alpha} - \frac{1}{f-f_1+\alpha}$
- Q_2:$s(w, w_1\alpha) = \frac{\alpha}{(f-f_1)^2-2\alpha^2(f+f_1)+\alpha^4}$
- $Q_3^{\delta=0}$: $s(f, f_1, \alpha) = \frac{\alpha^2-1}{(f_1-\alpha f)(f-\alpha f_1)}$, $Q_3^{\delta=1}$: $s(f, f_1, \alpha) = N/D$,

  $N = 2\left(r(f)f + f^2 - 1\right)\left(r(f_1)f_1 + {f_1}^2 - 1\right)(f + r(f))(f_1 + r(f_1))\left(\alpha^3 - \alpha\right)$
  $D = \left(\alpha\, r(f_1)r(f) + \alpha\, r(f_1)f + \alpha\, f_1 r(f) + \alpha\, f_1 f - 1\right)(\alpha\, r(f) - r(f_1) + \alpha\, f - f_1)$
  $(r(f_1)r(f) + r(f_1)f + f_1 r(f) + f_1 f - \alpha)(r(f) - \alpha r(f_1) - \alpha\, f_1 + f)\, r(f)r(f_1)$ and
  $r(f) = \sqrt{f^2 - 1}$.
- Q_4: $s(f, f_1, \alpha) = \frac{1}{\alpha^2 f^2 f_1^2 + 2af f_1 + \alpha^2 - f^2 - f_1^2}$, $a = \sqrt{\alpha^4 + \delta\alpha^2 + 1}$.

The cross-ratio equation (Q_1, $\delta = 0$)

We consider the 3D consistent quad-graph equation

$$Q_1 : \qquad \alpha(x - x_2)(x_1 - x_{12}) - \beta(x - x_1)(x_2 - x_{12}) = 0$$

with Lax matrix

$$L(x_1, x_2, \alpha) = \begin{pmatrix} \zeta + \frac{\alpha x_2}{x_1 - x_2} & -\frac{\alpha x_1 x_2}{x_1 - x_2} \\ \frac{\alpha}{x_1 - x_2} & \zeta - \frac{\alpha x_1}{x_1 - x_2} \end{pmatrix}.$$

Q_1 is equivalent to $x_2 = F(x_1, x, x_{12}, \alpha, \beta)$, where

$$F(x, x_1, x_2, \alpha, \beta) = \frac{\alpha x_1(x - x_2) + \beta x_2(x_1 - x)}{\alpha(x - x_2) + \beta(x_1 - x)} \ .$$

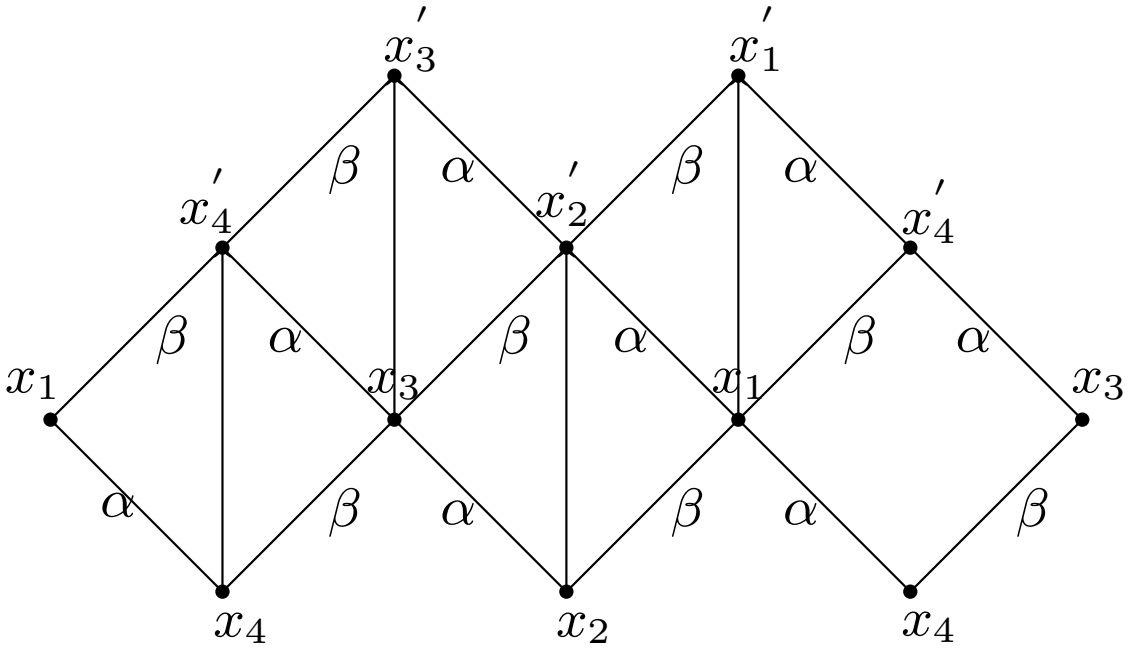
In this terms, the map obtained by the $(2,2)$ periodic reduction is

$$\mathcal{S}_{\alpha,\beta} : (x_1, x_2, x_3, x_4) \mapsto (x_1^{'}, x_2^{'}, x_3^{'}, x_4^{'}),$$

where

$$x_2^{'} = F(x_2, x_3, x_1, \alpha, \beta), \quad x_4^{'} = F(x_4, x_1, x_3, \alpha, \beta),$$
$$x_3^{'} = F(x_3, x_4^{'}, x_2^{'}, \alpha, \beta), \quad x_1^{'} = F(x_1, x_2^{'}, x_4^{'}, \alpha, \beta),$$

for $F(x, x_1, x_2, \alpha, \beta) = \frac{\alpha x_1(x-x_2)+\beta x_2(x_1-x)}{\alpha(x-x_2)+\beta(x_1-x)}$.



- Three-leg form:

$$\psi(x, x_1, \alpha) - \psi(x, x_2, \beta) = \phi(x, x_{12}, \alpha, \beta),$$

for

$$\psi(x, x_1, \alpha) = \frac{\alpha}{x - x_1} \text{ and } \phi(x, x_{12}, \alpha, \beta) = \frac{\alpha - \beta}{x - x_{12}} .$$

- $$s(x, x_1, \alpha) = \frac{\partial \psi(x, x_1, \alpha)}{\partial x_1} = \frac{\partial \psi(x_1, x, \alpha)}{\partial x} = \frac{\alpha}{(x - x_1)^2}$$

Proposition

The $(2, 2)$ per. reduction $\mathcal{S}_{\alpha,\beta} : (x_1, x_2, x_3, x_4) \mapsto (x_1', x_2', x_3', x_4')$ is a Poisson map with respect to

$$\pi = \frac{1}{s(x_1, x_2, \beta)} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{1}{s(x_3, x_4, \beta)} \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}$$
$$= \frac{(x_1 - x_2)^2}{\beta} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{(x_3 - x_4)^2}{\beta} \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} .$$

Integrals are derived from the trace of the monodromy matrix:

$$M(x_1, x_2, x_3, x_4) = L(x_2, x_1, \beta)L(x_3, x_2, \alpha)L(x_4, x_3, \beta)L(x_1, x_4, \alpha).$$

Here we get

$$I_1 = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_3)(x_1 - x_4)}, \quad I_2 = \frac{\alpha x_1}{x_4 - x_1} + \frac{\beta x_2}{x_1 - x_2} + \frac{\alpha x_3}{x_2 - x_3} + \frac{\beta x_4}{x_3 - x_4}$$

which are in involution with respect to π.

Proposition

The (2,2)-periodic reduction $\mathcal{S}_{\alpha,\beta}$ of the cross-ratio equation is Liouville integrable.

- All the (2,2)-periodic reductions of the equations in the ABS list are Liouville integrable with respect to

$$\pi = \frac{1}{s(x_1, x_2, \beta)} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{1}{s(x_3, x_4, \beta)} \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} .$$

- There is a connection between Poisson structures for 3D consistent equations from three-leg forms and r-matrix Poisson structures for particular refactorisation problems associated with them (lift of 3D cons. equations as Yang-baxter maps).

T.K, D T Tran '15

(n.n) periodic reductions

Presymplectic structure that involves all the short legs [1]

$$s(x_1, x_2, \beta) dx_1 \wedge dx_2 + s(x_2, x_3, \alpha) dx_3 \wedge dx_4 \ldots s(x_{2n}, x_1, \alpha) d_{x_{2n}} \wedge dx_1.$$

This form is degenerate in some cases (e.g. H_1 and $Q_1^{\delta=0}$).

[1] *V.E. Adler, A.I. Bobenko, Yu.B. Suris '03*

5. Poisson structures for cluster maps

We consider a *seed* $(B, \mathbf{x})$ consisting of a skew-symmetric *exchange matrix* $B = (b_{ij}) \in \mathrm{Mat}_N(\mathbb{Z})$ and an N-tuple of *cluster variables* $\mathbf{x} = (x_1, \ldots, x_N)$.

For each integer $k \in [1, N]$ we define a *mutation* $\boldsymbol{\mu}_k$ which produces a new seed $(B', \mathbf{x}') = \boldsymbol{\mu}_k(B, \mathbf{x})$, where $B' = (b'_{ij})$ with

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \mathrm{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise,} \end{cases}$$

where $[a]_+ = \max(a, 0)$, and $\mathbf{x}' = (x'_j)$ with

$$x'_j = \begin{cases} \frac{1}{x_k}\left(\prod_{i=1}^N x_i^{[b_{ki}]_+} \prod_{i=1}^N x_i^{[-b_{ki}]_+}\right), & \text{for } j = k \\ x_j\ , & \text{for } j \neq k. \end{cases}$$

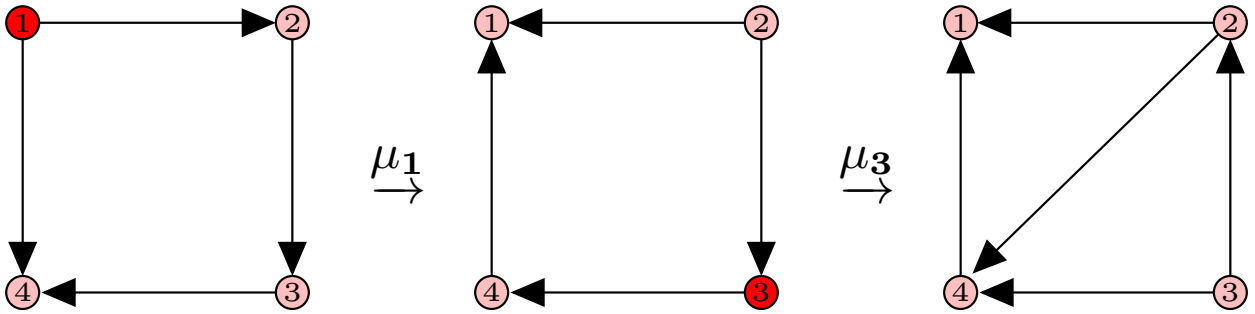
The matrix B can be associated with a quiver Q without 1- or 2-cycles with

$$b_{ij} = \begin{cases} \#i \to j, & \text{for } b_{ij} \geq 0, \\ -\#j \to i, & \text{otherwise.} \end{cases}$$

S. Fomin, A. Zelevinsky '02,'03,'07

Example

For $N = 4$ and $B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$.



$$\underset{(x_1,x_2,x_3,x_4)}{\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}} \xrightarrow{\mu_1} \underset{(x_1',x_2,x_3,x_4)}{\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}} \xrightarrow{\mu_3} \underset{(x_1',x_2,x_3',x_4)}{\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & -1 & -1 & 0 \end{pmatrix}}$$

where

$$x_1' = \frac{x_2x_4 + 1}{x_1} \text{ and } x_3' = \frac{x_4 + x_2}{x_3}\ .$$

Quiver mutation at vertex k:

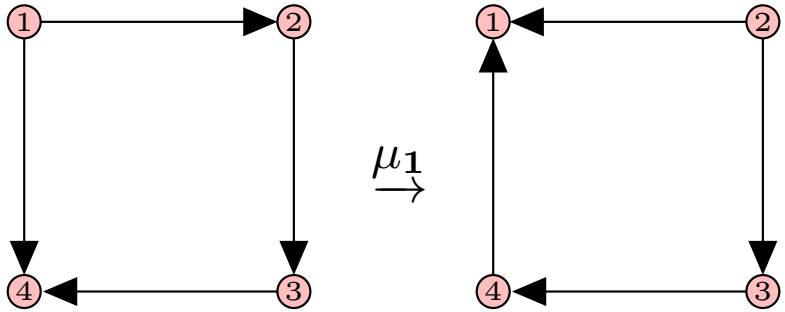
i If there are p arrows $i \to k$ and q arrows $k \to j$ add pq arrows $i \to j$

ii reverse all arrows in Q that go in/out of vertex k

iii delete any 2-cycles created in the first step.

Example

$$N = 4,\ B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$



$$\underset{(x_1,x_2,x_3,x_4)}{\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}} \xrightarrow{\mu_1} \underset{(x_1',x_2,x_3,x_4)}{\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}} \xrightarrow{\mu_2} \underset{(x_1',x_2',x_3,x_4)}{\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}}$$

$$\xrightarrow{\mu_3} \underset{(x_1',x_2',x_3',x_4)}{\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}} \xrightarrow{\mu_4} \underset{(x_1',x_2',x_3',x_4')}{\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}} \xrightarrow{\mu_1} \cdots$$

$$x_1' = \frac{x_2x_4+1}{x_1},\ x_2' = \frac{x_1'x_3+1}{x_2},\ x_3' = \frac{x_2'x_4+1}{x_3},\ x_4' = \frac{x_1'x_3'+1}{x_4}$$

or by setting $x_1' = x_5,\ x_2' = x_6,\ x_3' = x_7,\ x_4' = x_8, \ldots$

$$x_5 = \frac{x_2x_4+1}{x_1},\ x_6 = \frac{x_3x_5+1}{x_2},\ x_7 = \frac{x_4x_6+1}{x_3}, \ldots,\ i.e.\ x_{m+4} = \frac{x_{m+1}x_{m+3}+1}{x_m}$$

Definition

An exchange matrix B is said to be *cluster mutation-periodic with period m* if (for a suitable labelling of indices) $\mu_m \mu_{m-1} \dots \mu_1(B) = \rho^m(B)$, where ρ is the cyclic permutation $\rho : (1, 2, 3, ..., N) \mapsto (N, 1, 2, ..., N-1)$.

Proposition

The exchange matrix $B = (b_{ij}) \in \mathrm{Mat}_N(\mathbb{Z})$ defines a cluster-mutation periodic quiver with period 1, iff

$$b_{j,N} = b_{1,j+1} \text{ and } b_{j+1,k+1} = b_{j,k} + b_{1,j+1}[-b_{1,k+1}]_+ - b_{1,k+1}[-b_{1,j+1}]_+,$$

for $1 \leq j, k \leq N-1$.

- If B is a a cluster-mutation periodic quiver with period 1, then the sequence of mutations $\mu_1, \mu_2, \dots, \mu_N, \mu_1, \mu_2, \dots$ *etc.* applied on the initial data $(x_1, \dots, x_N)$ is equivalent to the iterations of the recurrence

$$x_{m+N} x_m = \prod_{j=1}^{N-1} x_{m+j}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{m+j}^{[-b_{1,j+1}]_+}.$$

A.P. Fordy, R.J. Marsh 2011

We consider recurrences of the form

$$x_{m+N}x_m = \prod_{j=1}^{N-1} x_{m+j}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{m+j}^{[-b_{1,j+1}]_+},$$

where $[b]_+ = \max(b, 0)$ and $B = [b_{i,j}]$ is an $N \times N$ skew-symmetric integer matrix with

$b_{j,N} = b_{1,j+1}$ and $b_{j+1,k+1} = b_{j,k} + b_{1,j+1}[-b_{1,k+1}]_+ - b_{1,k+1}[-b_{1,j+1}]_+,$

for $1 \leq j, k \leq N-1$.

B is a cluster mutation-periodic quiver with period 1.

Let $(a_1, \ldots, a_{N-1})$ an $(N-1)$-tuple of integers that is palindromic, i.e. $a_j = a_{N-j}$. Then the skew-symmetric matrix B with entries specified by

$$b_{1,j+1} = a_j \text{ and } b_{i+1,j+1} = b_{ij} + a_i[-a_j]_+ - a_j[-a_i]_+,$$

for all $i, j \in [1, N-1]$, is cluster mutation-periodic with period 1.

Theorem

The map $\phi : (x_1, x_2, \ldots, x_{N-1}, x_N) \mapsto (x_2, x_3, \ldots, x_N, x_{N+1})$, *with*

$$x_{N+1} = \frac{\prod_{j=1}^{N-1} x_{j+1}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{j+1}^{[-b_{1,j+1}]_+}}{x_1}$$

preserves the two form

$$\omega = \sum_{i<j} \frac{b_{ij}}{x_i x_j} dx_i \wedge dx_j, \quad i.e.\ \phi^* \omega = \omega.$$

- If B is non-degenerate ($rankB = N$) then ω is a symplectic form and ϕ is a symplectic map.

M. Gekhtman, M. Shapiro, A. Vainshtein '05, A.P. Fordy, A.N.W. Hone '14

Example

For $N = 4$, we consider $(a_1, a_2, a_3) = (1, 0, 1)$ $(a_j = a_{N-j})$ and the skew-symmetric matrix B with entries specified by

$$b_{1,j+1} = a_j \text{ and } b_{i+1,j+1} = b_{ij} + a_i[-a_j]_+ - a_j[-a_i]_+,$$

for all $i, j \in [1, 3]$,

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

The corresponding recurrence

$$x_{m+N} x_m = \prod_{j=1}^{N-1} x_{m+j}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{m+j}^{[-b_{1,j+1}]_+}$$

becomes

$$x_{m+4} x_m = x_{m+1} x_{m+3} + 1$$

and the cluster map

$$\phi : (x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4, \frac{x_2 x_4 + 1}{x_1}) \ .$$

Example

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}.$$

$\mathrm{Det} B \neq 0$, so the two-form

$$\omega = \sum_{i<j} \frac{b_{ij}}{x_i x_j} dx_i \wedge dx_j = dx_1 \wedge dx_2 + dx_1 \wedge dx_4 + dx_2 \wedge dx_3 + dx_3 \wedge dx_4$$

is a symplectic form.

- The map $\phi : (x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4, \frac{x_2 x_4 + 1}{x_1})$ is symplectic with respect to ω ($\phi^* \omega = \omega$).

Let $\mathbf{w} = (w_j) \in KerB$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2K}$ be an integer basis of imB

- Scaling symmetries: $\mathbf{x} \mapsto \tilde{\mathbf{x}} = \lambda^{\mathbf{w}} \cdot x$, (i.e. $\tilde{x}_j = \lambda^{w_j} x_j$),
- Invariant Laurent monomials: $u_i = \mathbf{x}^{\mathbf{v}_i}, \quad i = 1, \ldots, 2K$
  (where $\mathbf{x}^{\mathbf{a}} = \prod_i x_i^{a_i}$, for any integer vector $\mathbf{a} = (a_i)$).

Theorem

For $rankB = 2K < N$, there is a rational map $\pi : \mathbb{C}^N \to \mathbb{C}^{2K}$ ($\pi : \mathbf{x} \mapsto \mathbf{u} = (u_i)$) and a symplectic birational map $\hat{\phi} : \mathbb{C}^{2K} \to \mathbb{C}^{2K}$ with respect to a symplectic form $\hat{\omega}$, such that $\hat{\phi} \circ \pi = \pi \circ \phi$ and $\pi^* \hat{\omega} = \omega$.

$$\begin{tikzcd}
\mathbb{C}^N \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathbb{C}^N \arrow[d, "\pi"] \\
\mathbb{C}^{2K} \arrow[r, "\hat{\phi}"'] & \mathbb{C}^{2K}
\end{tikzcd}$$

The recurrence associated with $\hat{\phi}$ is called U-system.

A.P. Fordy, A.N.W.Hone'14

Discrete Hirota reductions

The *discrete Hirota equation* is the bilinear partial difference equation

$$T_1 T_{-1} = T_2 T_{-2} + T_3 T_{-3},$$

where $T = T(m_1, m_2, m_3)$ and $T_{\pm i} = T|_{m_i \to m_i \pm 1}$, for $i = 1, 2, 3$.
Plane wave reduction:

$$T(m_1, m_2, m_3) = a_1^{{m_1}^2} a_2^{{m_2}^2} a_3^{{m_3}^2} \tau_m, \quad m = m_0 + \delta_1 m_1 + \delta_2 m_2 + \delta_3 m_3 \ ,$$

then τ_m satisfies the ordinary difference equation

$$\tau_{m+\delta_1} \tau_{m-\delta_1} = \alpha \tau_{m+\delta_2} \tau_{m-\delta_2} + \beta \tau_{m+\delta_3} \tau_{m-\delta_3},$$

with $\alpha = \frac{a_2^2}{a_1^2}$ and $\beta = \frac{a_3^2}{a_1^2}$, or

$$\tau_{m+N} \tau_m = \alpha \tau_{m+K} \tau_{m+N-K} + \beta \tau_{m+L} \tau_{m+N-L},$$

for $N = 2\delta_1, \ K = \delta_1 + \delta_2$ and $L = \delta_1 + \delta_3$.

- The discrete Hirota reductions are cluster mutation-periodic quiver recurrences.

R. Hirota '81, T. Miwa '82, A.V. Zabrodin '97

A class of discrete Hirota reductions

We consider discrete Hirota reductions of the form,

$$\tau_{m+2N+M}\tau_m = \alpha\tau_{m+2N}\tau_{m+M} + \beta_m\tau_{m+N+M}\tau_{m+N}, \text{ for } N > M, M \neq 0. \quad (1)$$

Exchange matrix $B = [b_{i,j}] \in \mathrm{Mat}_{2N+M}(\mathbb{Z})$, with

$$b_{1,M+1} = b_{1,2N+1} = -1, \; b_{1,N+M+1} = b_{1,N+1} = 1$$

and $b_{1,j} = 0$, for $j \neq M+1, N+1, 2N+1, N+M+1$.

- Invariant presymplectic structure $\omega = \sum_{i<j} \frac{b_{ij}}{x_i x_j} dx_i \wedge dx_j$.
- $\mathrm{rank} B = \begin{cases} N+M-1, & for\ N+M\ odd, \\ N+M-2, & for\ N+M\ even\ . \end{cases}$

Proposition

For $N+M$ odd, τ_m satisfies (1) iff $u_m = \frac{\tau_m\tau_{m+N+1}}{\tau_{m+1}\tau_{m+N}}$ satisfies the U-system

$$u_m u_{m+1} \dots u_{m+N+M-1} = b + a u_{m+M} u_{m+M+1} \dots u_{m+N-1},$$

For $N+M$ even, τ_m satisfies (1) iff $u_m = \frac{\tau_m\tau_{m+N+2}}{\tau_{m+2}\tau_{m+N}}$ satisfies the U-system

$$u_m u_{m+2} \dots u_{m+N+M-2} = b + a u_{m+M} u_{m+M+2} \dots u_{m+N-2},$$

A. N.W. Hone, T.K, C. Ward '17

The discrete or lattice KdV equation (*Hirota, 1977*) :

$$V_{k+1,l} - V_{k,l+1} = \alpha \left(\frac{1}{V_{k,l}} - \frac{1}{V_{k+1,l+1}} \right).$$

Lax Representation

$$\mathbf{L}(V_{k,l+1}, V_{k+1,l+1})\mathbf{M}(V_{k,l}) = \mathbf{M}(V_{k+1,l})\mathbf{L}(V_{k,l}, V_{k+1,l}), \text{ where}$$

$$\mathbf{L}(V,W) = \begin{pmatrix} V - \frac{\alpha}{W} & \lambda \\ 1 & 0 \end{pmatrix}, \ \mathbf{M}(V) = \begin{pmatrix} V & \lambda \\ 1 & \frac{\alpha}{V} \end{pmatrix}$$

and λ is a spectral parameter.

Periodic reductions

The (N, M) periodic reduction is derived by considering

$$V_{k+N,l+M} = V_{k,l} \implies V_{k,l} = v_m, \qquad m = kM - lN.$$

Thus, we obtain the following ordinary difference equation:

$$v_{m+N+M} - v_m = \alpha \left(\frac{1}{v_{m+N}} - \frac{1}{v_{m+M}} \right).$$

The iteration of the recurrence

$$v_{m+N+M} - v_m = \alpha \left(\frac{1}{v_{m+N}} - \frac{1}{v_{m+M}} \right)$$

is equivalent to iteration of the birational map

$$\phi(v_0, v_1, \ldots, v_{N+M-1}) = (v_1, v_2, \ldots, v_0 + \alpha(\frac{1}{v_N} - \frac{1}{v_M}))$$

ϕ preserves the spectrum of the monodromy matrix:

$$\mathcal{M} = \prod_{i=0}^{M-1} \mathbf{M}(v_{r_i+N}) \mathbf{L}(v_{r_i+N-M}, v_{r_i+N}) \mathbf{L}(v_{r_i+N-2M}, v_{r_i+N-M}) \\ \ldots \mathbf{L}(v_{r_{i+1}}, v_{r_{i+1}+M}),$$

where $r_k = kN mod M$.

- Integrals are derived from the trace of the monodromy matrix

Proposition

$$v_m = \frac{\tau_m \tau_{m+N+M}}{\tau_{m+M}\tau_{m+N}}$$

is a solution of the dKdV reduction, $v_{m+N+M} - v_m = \alpha\left(\frac{1}{v_{m+N}} - \frac{1}{v_{m+M}}\right)$, *if and only if* τ_m *satisfies the bilinear equation (discrete Hirota reduction)*

$$\tau_{m+2N+M}\tau_m = \beta_m \tau_{m+N+M}\tau_{m+N} - \alpha\tau_{m+2N}\tau_{m+M},$$

with periodic coefficients $\beta_{m+M} = \beta_m$.

U-systems for $N+M$ *odd:* $u_m = \frac{\tau_m\tau_{m+N+1}}{\tau_{m+1}\tau_{m+N}}$,

$$u_m u_{m+1}\dots u_{m+N+M-1} = \beta_m - a u_{m+M}u_{m+M+1}\dots u_{m+N-1}.$$

U-system to dKdV : $v_m = u_m u_{m+1}\dots u_{m+M-1}$.

- Invariant presymplectic forms for discrete Hirota reductions

$\downarrow$

- Invariant symplectic forms for the U-systems

$\downarrow$

- Invariant Poisson structures for the KdV reductions

Example: (3,2) KdV periodic reductions

For $N = 3, M = 2$, $(b_{1,4} = b_{1,6} = 1, b_{1,3} = b_{1,7} = -1,\ rankB = 4)$

Discrete Hirota reduction

$$\tau_{m+8}\tau_m + \alpha\tau_{m+6}\tau_{m+2} = \beta_m\tau_{m+3}\tau_{m+5},\ \ \beta_{m+2} = \beta_m$$

Invariant two-form: $\omega = \sum_{i<j} \frac{b_{ij}}{\tau_i\tau_j} d\tau_i \wedge d\tau_j$.

$\mathcal{U}$-system

$$u_m u_{m+1} u_{m+2} u_{m+3} u_{m+4} = \beta_m - \alpha u_{m+2}$$

$\pi : \mathbb{C}^8 \to \mathbb{C}^4$, $\pi : (\tau_0, \tau_1, \ldots, \tau_7) \mapsto (u_0, u_1, u_2, u_3)$, where $u_j = \frac{\tau_j\tau_{j+4}}{\tau_{j+1}\tau_{j+3}}$.

$$\hat{\omega} = \frac{1}{u_0u_2} du_0 \wedge du_2 + \frac{1}{u_1u_2} du_1 \wedge du_2 + \frac{1}{u_1u_3} du_1 \wedge du_3, \quad \pi^*\hat{\omega} = \omega.$$

$\hat{\phi} : (u_0, u_1, u_2, u_3) \mapsto (u_1, u_2, u_3, \dfrac{\beta_0 - \alpha u_2}{u_0u_1u_2u_3})$ *is a symplectic map.*

Example: (3,2) KdV periodic reductions

dKdV reduction

$$v_{m+5} - v_m = \alpha \left(\frac{1}{v_{m+3}} - \frac{1}{v_{m+2}} \right), \text{ with}$$

$$v_m = \frac{\tau_m \tau_{m+5}}{\tau_{m+2}\tau_{m+3}} = u_m u_{m+1} \text{ and } u_m u_{m+1} u_{m+2} u_{m+3} u_{m+4} = b_m - a u_{m+2}.$$

$$v_0 = u_0 u_1, v_1 = u_1 u_2, v_2 = u_2 u_3, v_3 = \frac{\beta_0 - \alpha u_2}{u_0 u_1 u_2}, v_4 = \frac{\beta_1 - \alpha u_3}{u_1 u_2 u_3}$$

Poisson bracket associated with $\hat{\omega}$:

$$\{u_0, u_2\} = -u_0 u_2, \ \{u_0, u_3\} = u_0 u_3, \ \{u_1, u_3\} = -u_1 u_3,$$
$$\{u_0, u_1\} = \{u_1, u_2\} = \{u_2, u_3\} = 0.$$

Poisson bracket in v-variables: $\{v_i, v_j\} = c_{j-i} v_i v_j + \alpha d_{j-i}, \ 0 \le i < j \le 4,$

with $c_1 = -1, c_2 = -1, c_3 = 1, c_4 = 1, d_3 = 1$ and $d_k = 0$ for $k \neq 3$.

$\phi_v : (v_0, ..., v_4) \mapsto (v_1, ..., v_0 + \alpha(\frac{1}{v_3} - \frac{1}{v_2}))$ *is a Poisson map.*

Integrals are derived from the trace of the monodromy matrix:

$$\mathcal{M} = \mathbf{M}(v_3)\mathbf{L}(v_1, v_3)\mathbf{M}(v_4)\mathbf{L}(v_2, v_4)\mathbf{L}(v_0, v_2),$$

where $\mathbf{L}(v_1, v_2) = \begin{pmatrix} v_1 - \frac{\alpha}{v_2} & \zeta \\ 1 & 0 \end{pmatrix}$, $\mathbf{M}(v) = \begin{pmatrix} v & \zeta \\ 1 & \frac{\alpha}{v} \end{pmatrix}$.

$$\begin{aligned} I_1 &= \frac{1}{v_2}(\alpha - v_0 v_2)(\alpha - v_1 v_3)(\alpha - v_2 v_4), \\ I_2 &= v_0 + v_1 + v_2 + v_3 + v_4 - \frac{\alpha}{v_2}, \\ I_3 &= v_2(\alpha + v_0 v_3)(\alpha + v_1 v_4) \quad (Casimir) \end{aligned}$$

$$\{I_i, I_j\} = 0, \quad i, j = 1, 2, 3.$$

The birational map $\phi_v : \mathbb{C}^5 \to \mathbb{C}^5$ is Liouville integrable.

The birational map $\hat{\phi} : \mathbb{C}^4 \to \mathbb{C}^4$ is Liouville integrable ($I_3 \to \beta_0\beta_1$).

Plot of a solution of (3,2) KdV periodic reduction

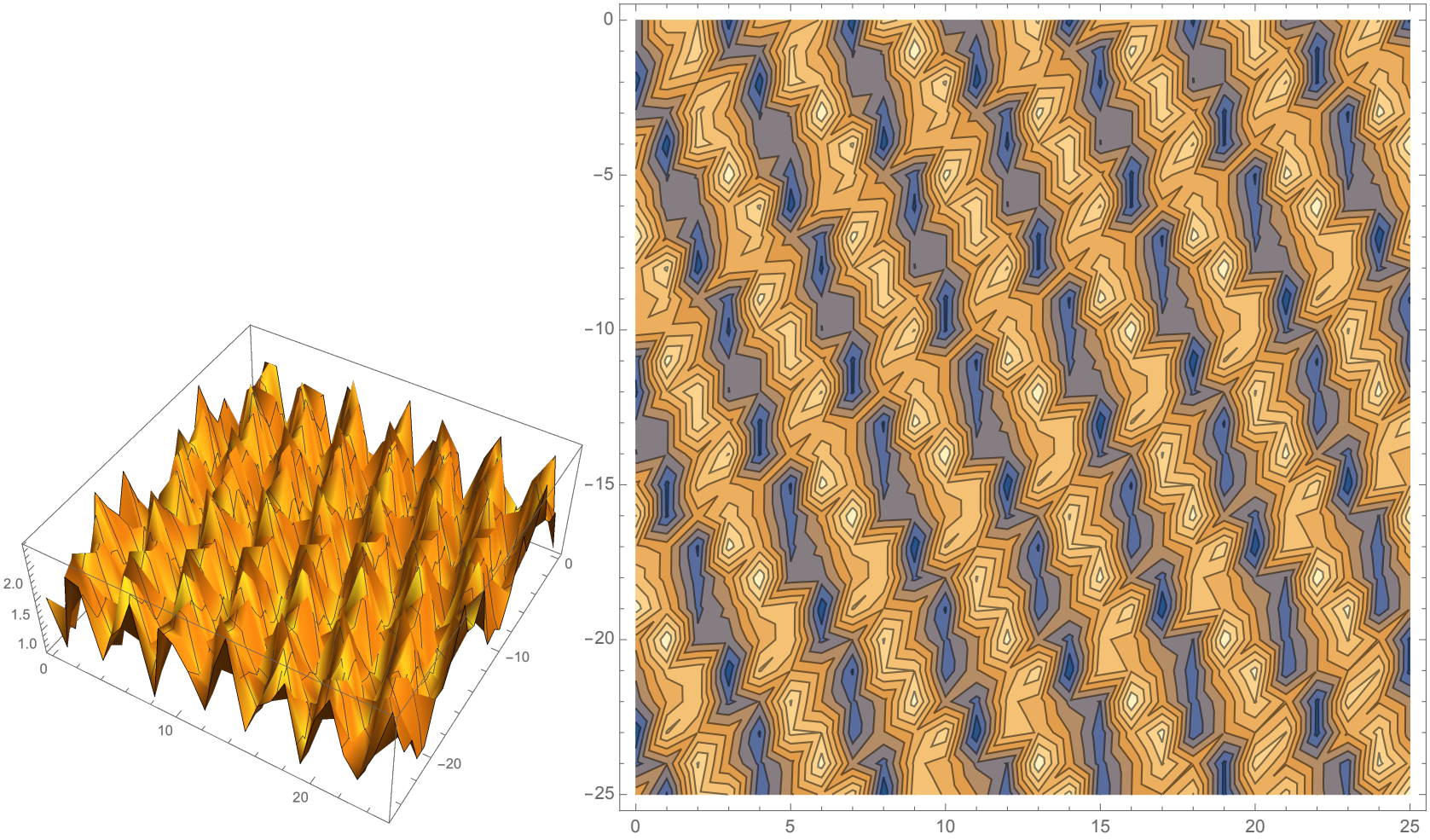


Figure: A solution of dKdV periodic reduction $V_{k,l} = v_{2k-3l}$, $\alpha = 1$, with $v_0 = 1, v_1 = 1.2, v_2 = 2, v_3 = 1, v_4 = 2$ and $v_{m+5} - v_m = \left(\frac{1}{v_{m+3}} - \frac{1}{v_{m+2}}\right)$.

A Liouville torus of (3,2) KdV periodic reduction

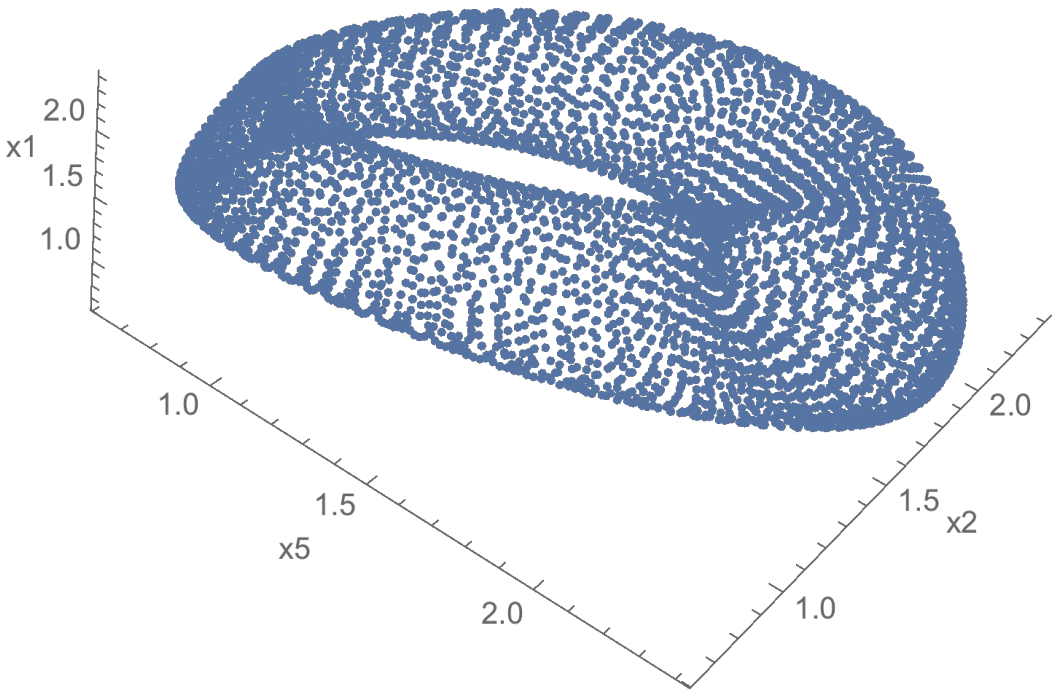


Figure: A projection on $\mathbb{R}^3$ of $\mathbf{v}_{n+1} = \phi_v(\mathbf{v}_n)$ with $\mathbf{v}_0 = (1, 1.2, 2, 1, 2)$, $\alpha = 1$.

Discrete Hirota reductions associated with integrable lattice equations

- Integrable aspects from the underlying cluster algebra structure
- Periodic reductions of the lattice KdV
  - Two discrete Hirota reductions associated with the dKdV

$$\tau_{m+2N+M}\tau_m = \beta_m \tau_{m+N+M}\tau_{m+N} - \alpha\tau_{m+2N}\tau_{m+M}\,,\ \ \beta_{m+M} = \beta_m$$
$$\tau_{m+2M+N}\tau_m = \beta'_m \tau_{m+N+M}\tau_{m+M} + \alpha\tau_{m+2M}\tau_{m+N}\,,\ \ \beta'_{m+N} = \beta'_m$$

  - Bi-hamiltonian formalism
  - Refactorization of the monodromy matrices
  - Liouville integrability of U-systems and the KdV periodic reductions

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- Similar approach for various integrable lattice equations (e.g. discrete Toda, Boussinesq lattice equation)
- Somos/Gale-Robinson recurrences $\left(x_{m+N}x_m = \sum_{j=1}^{\lfloor \frac{N}{2} \rfloor} a_j x_{m+N-j}x_{m+j}\right)$

Cluster algebras and discrete integrable systems

Laurent property, algebraic entropy, singularity confinement, tropical dynamics, Poisson-Lie groups, QRT maps, Y-systems, discrete Painlevé equations, pentagram maps, dimer models and many more!

Thank you for your attention!