Sparse differential resultant formulas: between the linear and the nonlinear case

Sonia L. Rueda

E.T.S. Arquitectura. Universidad Politécnica de Madrid
From algebraic to differential resultants
Differential resultant formulas
...for linear differential polynomials
...for nonlinear Laurent differential polynomials
Order and degree bounds for sparse differential resultants
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Differential resultant formulas

...for linear differential polynomials

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Order and degree bounds for sparse differential resultants
Macaulay Resultant (1916)

Let $P_1, \ldots, P_n \in \mathbb{C}[x_0, x_1, \ldots, x_{n-1}]$ be homogeneous polynomials.

$$d_i = \deg(P_i) \quad \text{and} \quad D = 1 + \sum_i (d_i - 1)$$

Given $\alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}^n$ denote by $x^\alpha = x_0^{\alpha_0} \cdots x_{n-1}^{\alpha_{n-1}}$. 
Macaulay Resultant (1916)

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Given $\alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}^n$ denote by $x^\alpha = x_0^{\alpha_0} \cdots x_{n-1}^{\alpha_{n-1}}$.

$$M(F_1, \ldots, F_n) = \begin{bmatrix}
\text{coefficients of } x^\alpha P_i \\
\text{with } \deg(x^\alpha P_i) = D \\
\leftrightarrow \text{L columns}
\end{bmatrix}, \quad L = \binom{n-1+D}{n-1}$$
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Let \( P_1, \ldots, P_n \in \mathbb{C}[x_0, x_1, \ldots, x_{n-1}] \) be homogeneous polynomials.

\[ d_i = \deg(P_i) \quad \text{and} \quad D = 1 + \sum_i (d_i - 1) \]

Given \( \alpha = (\alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{N}^n \) denote by \( x^\alpha = x_0^{\alpha_0} \cdots x_{n-1}^{\alpha_{n-1}} \).

\[
M(F_1, \ldots, F_n) = \begin{bmatrix}
\text{coefficients of } x^\alpha P_i \\
\text{with } \deg(x^\alpha P_i) = D
\end{bmatrix}, \quad L = \binom{n - 1 + D}{n - 1}
\]

\[
\text{Mac}(P_1, \ldots, P_n) = \gcd\{ \det(M) \mid M \text{ is an } L \times L \text{ submatrix of } M(P_1, \ldots, P_n) \}.
\]
Condition on the coefficients of $P_1, \ldots, P_n$ for $\{P_1 = 0, \ldots, P_n = 0\}$ to have a nontrivial solution.
Condition on the coefficients of $P_1, \ldots, P_n$ for $\{P_1 = 0, \ldots, P_n = 0\}$ to have a nontrivial solution

\begin{align*}
f_1 &= a_1 + a_{110} x_1 + a_{101} x_2 + a_{120} x_1^2 + a_{111} x_1 x_2 + a_{102} x_2^2 \\
f_2 &= a_2 + a_{210} x_1 + a_{201} x_2 \\
f_3 &= a_3 + a_{310} x_1 + a_{301} x_2
\end{align*}
From algebraic to differential resultants

Formula à la Macaulay.

\[
M(L) = \begin{bmatrix}
    x_1^2 & x_1 x_2 & x_2^2 & x_1 & x_2 & 1 \\
    a_{120} & a_{111} & a_{102} & a_{110} & a_{101} & a_1 \\
    a_{210} & a_{201} & 0 & a_2 & 0 & 0 \\
    0 & 0 & 0 & a_{210} & a_{201} & a_2 \\
    a_{310} & a_{301} & 0 & a_3 & 0 & 0 \\
    0 & a_{310} & a_{301} & 0 & a_3 & 0 \\
    0 & 0 & 0 & a_{310} & a_{301} & a_3
\end{bmatrix} \begin{bmatrix}
f_3 \\
x_1 f_2 \\
f_2 \\
x_1 f_1 \\
x_2 f_1 \\
f_1
\end{bmatrix}
\]

\[
A = [a_{201}]
\]
From algebraic to differential resultants

Formula à la Macaulay.

\[ M(L) = \begin{pmatrix} x_1^2 & x_1x_2 & x_2^2 & x_1 & x_2 & 1 \\ a_{120} & a_{111} & a_{102} & a_{110} & a_{101} & a_1 \\ a_{210} & a_{201} & 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & a_{210} & a_{201} & a_2 \\ a_{310} & a_{301} & 0 & a_3 & 0 & 0 \\ 0 & a_{310} & a_{301} & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_{310} & a_{301} & a_3 \end{pmatrix} \begin{pmatrix} f_3 \\ x_1f_2 \\ f_2 \\ x_1f_1 \\ x_2f_1 \\ f_1 \end{pmatrix} \]

\[ A = [a_{201}] \]

\[ \text{Mac}(f_1, f_2, f_2) = \frac{\det(M(L))}{\det(A)} \]
From algebraic to differential resultants

\[ f_1 = a_1 + a_{110} x_1 + a_{101} x_2 + a_{120} x_1^2 + a_{111} x_1 x_2 + 0 x_2^2 \]
\[ f_2 = a_2 + a_{210} x_1 + 0 x_2 \]
\[ f_3 = a_3 + a_{310} x_1 + 0 x_2 \]
From algebraic to differential resultants

\[ f_1 = a_1 + a_{110} x_1 + a_{101} x_2 + a_{120} x_1^2 + a_{111} x_1 x_2 + 0 x_2^2 \]

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\[ M(L) = \begin{bmatrix}
  x_1^2 & x_1 x_2 & x_2^2 & x_1 & x_2 & 1 \\
  a_{120} & a_{111} & 0 & a_{110} & a_{101} & a_1 \\
  a_{210} & 0 & 0 & a_2 & 0 & 0 \\
  0 & a_{210} & 0 & 0 & a_2 & 0 \\
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  0 & 0 & 0 & a_{310} & 0 & a_3 \\
\end{bmatrix}
\begin{bmatrix}
  f_3 \\
  x_1 f_2 \\
  x_2 f_2 \\
  f_2 \\
  x_1 f_1 \\
  x_2 f_1 \\
  f_1 \\
\end{bmatrix} \]
From algebraic to differential resultants

\[ f_1 = a_1 + a_{110} x_1 + a_{101} x_2 + a_{120} x_1^2 + a_{111} x_1 x_2 + 0 x_2^2 \]
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\[
M(L) = \begin{bmatrix}
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    a_{120} & a_{111} & 0 & a_{110} & a_{101} & a_1 \\
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\end{bmatrix}
\begin{bmatrix}
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    f_3 \\
    x_1 f_2 \\
    x_2 f_2 \\
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    x_2 f_1 \\
    f_1
\end{bmatrix}
\]

Differential resultants for **ordinary differential operators** defined by Berkovich and Tsirulik (1986) and studied by Chardin (1991).

Differential resultants for **partial differential operators** defined by Carrà-Ferro (1994) through formulas à la Macaulay. No existence was proved.
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Differential resultants of differential polynomials were introduced and studied by G. Carrà-Ferro (1997)

\[ \mathbb{D} \text{ differential integral domain, derivation } \partial, \]
\[ f_i \in \mathbb{D}\{U\} \text{ ordinary differential polynomial of order } o_i, i = 1, \ldots, n. \]

The **differential resultant of Carrà-Ferro** is the Macaulay’s algebraic resultant of the differential polynomial set

\[ \{ \partial^{N-o_i} f_i, \ldots, \partial f_i, f_i \mid i = 1, \ldots, n, \text{ where } N = \sum_{i=1}^{n} o_i \}. \]
From algebraic to differential resultants


Generic differential polynomial of order $o_i$ and degree $d_i$, $i = 1, \ldots, n$

$$F_i = c_i + \sum_{\omega \in \Omega_i} c^i_\omega \omega$$

$U = \{u_1, \ldots, u_{n-1}\}$

$\Omega_i = \{\text{monomials in } u_{j,k}, k = 1, \ldots, o_i\text{ of degree } \leq d_i\}$,

$C = \{c^i_\omega \mid i = 1, \ldots, n, \omega \in \Omega_i\}$ differential indeterminates over $\mathbb{Q}$. 
From algebraic to differential resultants


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$C = \{c^i_\omega \mid i = 1, \ldots, n, \omega \in \Omega_i\}$ differential indeterminates over $\mathbb{Q}$.

$I = [F_1, \ldots, F_n]$ differential ideal in $\mathbb{Q}\{C\}\{U\}$,

$$I \cap \mathbb{Q}\{C\} = \text{sat}(\partial \text{Res}(F_1, \ldots, F_n))$$

$$= \{f \in \mathbb{Q}\{C\} \mid I^a S^b f \in [\partial \text{Res}(F_1, \ldots, F_n)]\}$$
Generic differential polynomial of order \( o_i \) and degree \( d_i, i = 1, \ldots, n \)

\[
F_i = c_i + \sum_{\omega \in \Omega_i} c_i^\omega \omega
\]

\( U = \{ u_1, \ldots, u_{n-1} \} \)

\( \Omega_i = \{ \text{monomials in } u_{j,k}, k = 1, \ldots, o_i \text{ of degree } \leq d_i \} \),

\( C = \{ c_i^\omega \mid i = 1, \ldots, n, \omega \in \Omega_i \} \) differential indeterminates over \( \mathbb{Q} \).

\( I = [F_1, \ldots, F_n] \) differential ideal in \( \mathbb{Q}\{C\}\{U\} \),

\[
I \cap \mathbb{Q}\{C\} = \text{sat}(\partial \text{Res}(F_1, \ldots, F_n))
\]

\[
= \{ f \in \mathbb{Q}\{C\} \mid I^a S^b f \in [\partial \text{Res}(F_1, \ldots, F_n)] \}
\]

A sparse differential resultant has been also defined by Gao, Li, Yuan (2011) for Laurent differential polynomials.
From algebraic to differential resultants

Differential resultant formulas

...for linear differential polynomials

...for nonlinear Laurent differential polynomials

Order and degree bounds for sparse differential resultants
$\mathbb{D}$ ordinary differential domain, with derivation $\partial$ (e.g. $Q(t)$, $\partial = \frac{d}{dt}$)

$U = \{u_1, \ldots, u_{n-1}\}$ set of differential indeterminates over $\mathbb{D}$.

$k \in \mathbb{N}$, $u_{j,k} = \partial^k u_j$ and $u_j = u_{j,0}$. 
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Ring of Laurent differential polynomials in the differential indeterminates $U$,

$$
\mathbb{D}\{U^\pm\} := \mathbb{D}[u_{j,k}, u_{j,k}^{-1} \mid j = 1, \ldots, n - 1, k \in \mathbb{N}] \}
$$
Differential resultant formulas

\( \mathbb{D} \) ordinary differential domain, with derivation \( \partial \) (e.g. \( \mathbb{Q}(t), \partial = \frac{d}{dt} \))

\( U = \{ u_1, \ldots, u_{n-1} \} \) set of differential indeterminates over \( \mathbb{D} \).

\[ k \in \mathbb{N}, \ u_{j,k} = \partial^k u_j \text{ and } u_j = u_{j,0}. \]

Ring of Laurent differential polynomials in the differential indeterminates \( U \),

\[ \mathbb{D}\{U^{\pm}\} := \mathbb{D}[u_{j,k}, u_{j,k}^{-1} \mid j = 1, \ldots, n-1, k \in \mathbb{N}] \]

\( f \in \mathbb{D}\{U^{\pm}\}, \ f = \sum_{\ell=1}^{m} \theta_{\ell} \omega_{\ell}, \) were \( \theta_{\ell} \in \mathbb{D} \) and \( \omega_{\ell} \) is a Laurent differential monomial in \( \mathbb{D}\{U^{\pm}\} \). Differential support in \( u_j \) of \( f \)

\[ \mathcal{S}_j(f) = \{ k \in \mathbb{N} \mid u_{j,k}^{\pm1}/\omega_{\ell} \text{ for some } \ell \in \{1, \ldots, m\} \}. \]

\[ \text{ord}(f, u_j) := \max \mathcal{S}_j(f) \text{ and } \text{lord}(f, u_j) := \min \mathcal{S}_j(f) \text{ if } \mathcal{S}_j(f) \neq \emptyset, \]

otherwise \( \text{ord}(f, u_j) = \text{lord}(f, u_j) = -\infty. \)
The order of $f$ equals

$$\max\{\text{ord}(f, u_j)\}.$$ 

System of differential polynomials in $\mathbb{D}\{U^\pm\}$.

$$\mathcal{P} := \{f_1, \ldots, f_n\}$$

1. The order of $f_i$ is $o_i \geq 0$, $i = 1, \ldots, n$. So that no $f_i$ belongs to $\mathbb{D}$.
2. $\mathcal{P}$ contains $n$ distinct polynomials.
The order of $f$ equals

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System of differential polynomials in $\mathbb{D}\{U^\pm\}$.

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$[\mathcal{P}]_{\mathbb{D}\{U^\pm\}}$ differential ideal generated by $\mathcal{P}$ in $\mathbb{D}\{U^\pm\}$. 
The order of $f$ equals

$$\max\{ \text{ord}(f, u_j) \}.$$ 

System of differential polynomials in $\mathbb{D}\{U^\pm\}$.

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$[\mathcal{P}]_{\mathbb{D}\{U^\pm\}}$ differential ideal generated by $\mathcal{P}$ in $\mathbb{D}\{U^\pm\}$.

**Goal:** Define differential resultant formulas to compute elements of the elimination ideal

$$[\mathcal{P}]_{\mathbb{D}\{U^\pm\}} \cap \mathbb{D}.$$
Lotka-Volterra equations, with $\alpha$, $\beta$, $\gamma$ and $\rho$ algebraic constants,

$$\begin{cases} 
x' = \alpha x - \beta xy, \\
y' = \gamma y - \rho xy, 
\end{cases}$$
Lotka-Volterra equations, with $\alpha$, $\beta$, $\gamma$ and $\rho$ algebraic constants,

$$\begin{aligned}
    x' &= \alpha x - \beta xy, \\
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\end{aligned}$$

system of two linear differential polynomials in $\mathbb{D}\{x\}$,
Lotka-Volterra equations, with $\alpha$, $\beta$, $\gamma$ and $\rho$ algebraic constants,

\[
\begin{cases}
x' = \alpha x - \beta xy, \\
y' = \gamma y - \rho xy,
\end{cases}
\]

system of two linear differential polynomials in $\mathbb{D}\{x\}$, with coefficients $a_1, a_2, b_0, b_1$ in $\mathbb{D} = \mathbb{Q}[\alpha, \beta, \gamma, \rho]\{y\}$.

\[f_1(x) = (\beta y - \alpha)x + x' = a_1x + a_2x',\]

\[f_2(x) = y' - \gamma y + \rho yx = b_0 + b_1x,\]
Lotka-Volterra equations, with $\alpha$, $\beta$, $\gamma$ and $\rho$ algebraic constants,

$$\begin{cases} x' = \alpha x - \beta xy, \\ y' = \gamma y - \rho xy, \end{cases}$$

system of two linear differential polynomials in $\mathbb{D}\{x\}$, with coefficients $a_1, a_2, b_0, b_1$ in $\mathbb{D} = \mathbb{Q}[\alpha, \beta, \gamma, \rho]\{y\}$.

$$f_1(x) = (\beta y - \alpha)x + x' = a_1x + a_2x',$$

$$f_2(x) = y' - \gamma y + \rho yx = b_0 + b_1x,$$

Determinant of the coefficient matrix of $f_1(x)$, $f_2(x)$ and $f'_2(x)$,

$$\rho((y')^2 - yy'' + \alpha yy' - \alpha \gamma y^2 - \beta y^2 y' + \beta \gamma y^3)$$

in $[f_1(x), f_2(x)] \cap \mathbb{D}$. 
Differential resultant formulas

\[ \text{PS} \subset \partial \mathcal{P} := \{ \partial^k f_i \mid i = 1, \ldots, n, k \in \mathbb{N} \}, \mathcal{U} \subset \{ U \} \] and sets of Laurent differential monomials \( \Omega_f, \Omega, f \in \text{PS} \) in \( \mathbb{D}[\mathcal{U}^{\pm}] \), verifying:

1. (ps1) \( \text{PS} = \{ \partial^k f_i \mid k \in [0, L_i] \cap \mathbb{N}, L_i \in \mathbb{N}, i = 1, \ldots, n \} \),

2. (ps2) \( \text{PS} \subset \mathbb{D}[\mathcal{U}^{\pm}] \) and \( |\mathcal{U}| = |\text{PS}| - 1 \),

3. (ps3) \( \sum_{f \in \text{PS}} |\Omega_f| = |\Omega| \) and \( \cup_{f \in \text{PS}} \Omega_f f \in \bigoplus_{\omega \in \Omega} \mathbb{D} \omega \).
Differential resultant formulas

\( \mathcal{P} \subset \partial \mathcal{P} := \{ \partial^k f_i \mid i = 1, \ldots, n, k \in \mathbb{N} \} \), \( \mathcal{U} \subset \{ \mathcal{U} \} \) and sets of Laurent differential monomials \( \Omega_f, \Omega, f \in \mathcal{P} \) in \( \mathbb{D}[\mathcal{U}^\pm] \), verifying:

\begin{enumerate}
\item[(ps1)] \( \mathcal{P} = \{ \partial^k f_i \mid k \in [0, L_i] \cap \mathbb{N}, \ L_i \in \mathbb{N}, \ i = 1, \ldots, n \} \),
\item[(ps2)] \( \mathcal{P} \subset \mathbb{D}[\mathcal{U}^\pm] \) and \( |\mathcal{U}| = |\mathcal{P}| - 1 \),
\item[(ps3)] \( \sum_{f \in \mathcal{P}} \left| \Omega_f \right| = |\Omega| \) and \( \cup_{f \in \mathcal{P}} \left( \Omega_f \right) \in \bigoplus_{\omega \in \Omega} \mathbb{D} \omega \).
\end{enumerate}

Total set of differential polynomials \( \mathcal{T} \mathcal{P} := \cup_{f \in \mathcal{P}} \Omega_f f \) whose elements are

\[ p = \sum_{\omega \in \Omega} \theta_{p,\omega} \omega, \text{ with } \theta_{p,\omega} \in \mathbb{D}. \]

\( \mathcal{M}(\mathcal{T} \mathcal{P}, \Omega) = (\theta_{p,\omega}) \), is an \( |\Omega| \times |\Omega| \) matrix. We call

\[ \det(\mathcal{M}(\mathcal{T} \mathcal{P}, \Omega)) \]  

(1)
a differential resultant formula for \( \mathcal{P} \).
- From algebraic to differential resultants
- Differential resultant formulas
  - ...for linear differential polynomials
  - ...for nonlinear Laurent differential polynomials
- Order and degree bounds for sparse differential resultants


System of \textbf{LINEAR} differential polynomials $\mathcal{P} := \{f_1, \ldots, f_n\}$ in $\mathbb{D}\{U\}$.

1. The order of $f_i$ is $o_i \geq 0$, $i = 1, \ldots, n$. So that no $f_i$ belongs to $\mathbb{D}$.

2. $\mathcal{P}$ contains $n$ distinct polynomials.

3. $\mathcal{P}$ is a nonhomogeneous system.
System of **LINEAR** differential polynomials $\mathcal{P} := \{f_1, \ldots, f_n\}$ in $\mathbb{D}\{U\}$.

1. The order of $f_i$ is $o_i \geq 0$, $i = 1, \ldots, n$. So that no $f_i$ belongs to $\mathbb{D}$.

2. $\mathcal{P}$ contains $n$ distinct polynomials.

3. $\mathcal{P}$ is a nonhomogeneous system.

There exist differential operators $\mathcal{L}_{i,j} \in \mathbb{D}[\partial]$ such that

$$f_i = a_i + \sum_{j=1}^{n-1} \mathcal{L}_{i,j}(u_j), \quad a_i \in \mathbb{D}.$$ 

4. $\nu(\mathcal{P}) = n - 1$

$$\nu(\mathcal{P}) = |\{j \in \{1, \ldots, n - 1\} \mid \mathcal{L}_{i,j} \neq 0, \text{ for some } i \in \{1, \ldots, n\}\}|.$$
\[ \text{PS} \subset \partial \mathcal{P} := \{ \partial^k f_i \mid i = 1, \ldots, n, k \in \mathbb{N}_0 \} \text{ and } \mathcal{U} \subset \{ \mathcal{U} \} \text{ verifying: } \\
(\text{ps1}) \text{ PS } = \{ \partial^k f_i \mid k \in [0, L_i] \cap \mathbb{Z}, L_i \in \mathbb{N}_0, i = 1, \ldots, n \}, \\
(\text{ps2}) \text{ PS } \subset \mathbb{D}[\mathcal{U}] \text{ and } |\mathcal{U}| = |\text{PS}| - 1. \]
$\mathbf{PS} \subset \partial \mathcal{P} := \{ \partial^k f_i \mid i = 1, \ldots, n, k \in \mathbb{N}_0 \}$ and $\mathcal{U} \subset \{ U \}$ verifying:

(ps1) $\mathbf{PS} = \{ \partial^k f_i \mid k \in [0, L_i] \cap \mathbb{Z}, L_i \in \mathbb{N}_0, i = 1, \ldots, n \}$,

(ps2) $\mathbf{PS} \subset \mathbb{D}[\mathcal{U}]$ and $|\mathcal{U}| = |\mathbf{PS}| - 1$.

The coefficient matrix $\mathcal{M}(\mathbf{PS}, \mathcal{U})$ of the differential polynomials in $\mathbf{PS}$ as polynomials in $\mathbb{D}[\mathcal{U}]$ is a $|\mathbf{PS}| \times |\mathbf{PS}|$ matrix. We call

$$\det(\mathcal{M}(\mathbf{PS}, \mathcal{U}))$$

a differential resultant formula for $\mathcal{P}$. Given $N := \sum_{i=1}^n o_i$,
\( \text{PS} \subset \partial \mathcal{P} := \{ \partial^k f_i \mid i = 1, \ldots, n, k \in \mathbb{N}_0 \} \) and \( \mathcal{U} \subset \{ \mathcal{U} \} \) verifying:

(\text{ps1}) \( \text{PS} = \{ \partial^k f_i \mid k \in [0, L_i] \cap \mathbb{Z}, L_i \in \mathbb{N}_0, i = 1, \ldots, n \} \),

(\text{ps2}) \( \text{PS} \subset \mathbb{D}[\mathcal{U}] \) and \( |\mathcal{U}| = |\text{PS}| - 1 \).

The coefficient matrix \( \mathcal{M}(\text{PS}, \mathcal{U}) \) of the differential polynomials in \( \text{PS} \) as polynomials in \( \mathbb{D}[\mathcal{U}] \) is a \( |\text{PS}| \times |\text{PS}| \) matrix. We call

\[
\det(\mathcal{M}(\text{PS}, \mathcal{U}))
\]

a differential resultant formula for \( \mathcal{P} \). Given \( N := \sum_{i=1}^{n} o_i \),

\[L_i = N - o_i\] and
\[\mathcal{U} = \{ u_{j,k} \mid k \in [0, N] \cap \mathbb{Z}, j = 1, \ldots, n - 1 \} \].

Carrà-Ferro:
\[ \mathcal{P}_1 = \begin{cases} 
F_1 = c_1 + u_1 + u_2 + u_{2,1} \\
F_2 = c_2 + tu_{1,1} + u_{2,2} \\
F_3 = c_3 + u_1 + u_{2,1} 
\end{cases} \text{ in } \mathbb{D}\{u_1, u_2\} \text{ with } K = \mathbb{Q}(t), \partial = \partial/\partial t, \mathbb{D} = K\{c_1, c_2, c_3\} 
\]

\[
M(11) = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & c_{1,3} \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & c_{1,2} \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & c_{1,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & c_1 \\
1 & 0 & 0 & t & 0 & 2 & 0 & 0 & 0 & 0 & c_{2,2} \\
0 & 0 & 1 & 0 & 0 & t & 0 & 1 & 0 & 0 & c_{2,1} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & t & 0 & 0 & c_2 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & c_{3,3} \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & c_{3,2} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & c_{3,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & c_3 
\end{bmatrix} \begin{bmatrix}
\partial^3 F_1 \\
\partial^2 F_1 \\
\partial F_1 \\
F_1 \\
\partial^2 F_2 \\
\partial F_2 \\
F_2 \\
\partial^3 F_3 \\
\partial^2 F_3 \\
\partial F_3 \\
F_3 
\end{bmatrix} 
\]
\( O(\mathcal{P}) = (o_{i,j}) \) the order matrix of \( \mathcal{P} \) with

\[
o_{i,j} := \text{ord}(f_i, u_j) = \max \mathcal{S}_j(f_i) = \deg(\mathcal{L}_{i,j}),
\]

\[
\text{lord}(f_i u_j) = \min \mathcal{S}_j(f_i).
\]
\( \mathcal{O}(\mathcal{P}) = (o_{i,j}) \) the order matrix of \( \mathcal{P} \) with

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\text{lord}(f_i u_j) = \min \mathcal{S}_j(f_i).
\]

For \( j = 1, \ldots, n - 1 \),

\[
\overline{\gamma}_j(\mathcal{P}) := \min \{ o_i - o_{i,j} \mid L_{i,j} \neq 0, \ i = 1, \ldots, n \}, \\
\underline{\gamma}_j(\mathcal{P}) := \min \{ \text{lord}(f_i u_j) \mid L_{i,j} \neq 0, \ i = 1, \ldots, n \},
\]

(2)
\( \mathcal{O}(\mathcal{P}) = (o_{i,j}) \) the order matrix of \( \mathcal{P} \) with

\[
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\]
\[
\text{lord}(f_i u_j) = \min \mathfrak{S}_j(f_i).
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For \( j = 1, \ldots, n - 1 \),

\[
\overline{\gamma}_j(\mathcal{P}) := \min \{ o_i - o_{i,j} \mid \mathcal{L}_{i,j} \neq 0, i = 1, \ldots, n \},
\]
\[
\underline{\gamma}_j(\mathcal{P}) := \min \{ \text{lord}(f_i, u_j) \mid \mathcal{L}_{i,j} \neq 0, i = 1, \ldots, n \},
\]

For all \( i \) such that \( \mathcal{L}_{i,j} \neq 0 \) we have

\[
\mathfrak{S}_j(f_i) \subseteq [\underline{\gamma}_j(\mathcal{P}), o_i - \overline{\gamma}_j(\mathcal{P})] \cap \mathbb{Z}.
\]
\( \mathcal{O}(\mathcal{P}) = (o_{i,j}) \) the order matrix of \( \mathcal{P} \) with

\[
\begin{align*}
o_{i,j} := \text{ord}(f_i, u_j) &= \max \mathcal{S}_j(f_i) = \deg(L_{i,j}), \\
\text{lord}(f_i u_j) &= \min \mathcal{S}_j(f_i).
\end{align*}
\]

For \( j = 1, \ldots, n - 1 \),

\[
\begin{align*}
\overline{\gamma}_j(\mathcal{P}) &= \min \{o_i - o_{i,j} \mid L_{i,j} \neq 0, i = 1, \ldots, n\}, \\
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\end{align*}
\] (2)

For all \( i \) such that \( L_{i,j} \neq 0 \) we have

\[
\mathcal{S}_j(f_i) \subseteq [\underline{\gamma}_j(\mathcal{P}), o_i - \overline{\gamma}_j(\mathcal{P})] \cap \mathbb{Z}.
\]

\[
\gamma_j(\mathcal{P}) := \underline{\gamma}_j(\mathcal{P}) + \overline{\gamma}_j(\mathcal{P}), \quad \gamma(\mathcal{P}) := \sum_{j=1}^{n-1} \gamma_j(\mathcal{P}).
\] (3)
If $N - o_i - \gamma(\mathcal{P}) \geq 0$, $i = 1, \ldots, n$,

$$
\text{ps}(\mathcal{P}) := \{ \partial^k f_i \mid k \in [0, N - o_i - \gamma(\mathcal{P})] \cap \mathbb{Z}, \ i = 1, \ldots, n \},
$$

containing $L := \sum_{i=1}^{n} (N - o_i - \gamma(\mathcal{P}) + 1)$ differential polynomials, in the set $\mathcal{V}(\mathcal{P})$ of $L - 1$ differential indeterminates

$$
\mathcal{V}(\mathcal{P}) := \{ u_{j,k} \mid k \in [\gamma_j(\mathcal{P}), N - \overline{\gamma}_j(\mathcal{P}) - \gamma(\mathcal{P})] \cap \mathbb{Z}, \ j = 1, \ldots, n - 1 \}.
$$
If \( N - o_i - \gamma(\mathcal{P}) \geq 0, \ i = 1, \ldots, n, \)

\[
\text{ps}(\mathcal{P}) := \{ \partial^k f_i \mid k \in [0, N - o_i - \gamma(\mathcal{P})] \cap \mathbb{Z}, \ i = 1, \ldots, n \},
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containing \( L := \sum_{i=1}^{n} (N - o_i - \gamma(\mathcal{P}) + 1) \) differential polynomials, in the set \( \mathcal{V}(\mathcal{P}) \) of \( L - 1 \) differential indeterminates

\[
\mathcal{V}(\mathcal{P}) := \{ u_{j,k} \mid k \in [\underline{\gamma}_j(\mathcal{P}), N - \overline{\gamma}_j(\mathcal{P}) - \gamma(\mathcal{P})] \cap \mathbb{Z}, \ j = 1, \ldots, n - 1 \}.
\]

The matrix \( \mathcal{M}(\mathcal{P}) := \mathcal{M}(\text{ps}(\mathcal{P}), \mathcal{V}(\mathcal{P})) \) is an \( L \times L \) matrix. We can define the differential resultant formula for \( \mathcal{P} \):

\[
\partial \text{FRes}(\mathcal{P}) := \det(\mathcal{M}(\mathcal{P})).
\]
$\mathcal{P}$ is differentially essential if, there exist $i \in \{1, \ldots, n\}$ and a bijection

$$
\mu_i : \{1, \ldots, n\} \setminus \{i\} \longrightarrow \{1, \ldots, n - 1\}
$$

such that

$$
\begin{cases}
\mathcal{L}_{j, \mu_i(j)} \neq 0, & j = 1, \ldots, i - 1, \\
\mathcal{L}_{j, \mu_i(j)} \neq 0, & j = i + 1, \ldots, n.
\end{cases}
$$

...for linear differential polynomials
$\mathcal{P}$ is differentially essential if, there exist $i \in \{1, \ldots, n\}$ and a bijection

$$\mu_i : \{1, \ldots, n\} \setminus \{i\} \rightarrow \{1, \ldots, n - 1\}$$

(5)

such that

$$\left\{ \begin{array}{l}
\mathcal{L}_{j, \mu_i(j)} \neq 0, \quad j = 1, \ldots, i - 1, \\
\mathcal{L}_{j, \mu_i(j)} \neq 0, \quad j = i + 1, \ldots, n.
\end{array} \right.$$  

(6)

A linear differential system $\mathcal{P}$ is called super essential if, for every $i \in \{1, \ldots, n\}$, there exists $\mu_i$ verifying (6).
\[ \mathcal{P} \text{ is differentially essential if, there exist } i \in \{1, \ldots, n\} \text{ and a bijection } \mu_i : \{1, \ldots, n\} \setminus \{i\} \longrightarrow \{1, \ldots, n - 1\} \] (5)

such that

\[
\begin{align*}
\mathcal{L}_{j, \mu_i(j)} &\neq 0, \ j = 1, \ldots, i - 1, \\
\mathcal{L}_{j, \mu_i(j)} &\neq 0, \ j = i + 1, \ldots, n.
\end{align*}
\] (6)

A linear differential system \( \mathcal{P} \) is called super essential if, for every \( i \in \{1, \ldots, n\} \), there exists \( \mu_i \) verifying (6).

\[
N - o_i - \gamma(\mathcal{P}) = \sum_{j \in \{1, \ldots, n\} \setminus \{i\}} (o_j - \gamma_{\mu_i(j)}(\mathcal{P})) \geq 0
\]
Theorem Given a super essential system $\mathcal{P}$, then

$$N - o_i - \gamma(\mathcal{P}) \geq 0$$

and $\partial \text{FRes}(\mathcal{P}) := \det(\mathcal{M}(\mathcal{P}))$ can be defined. Furthermore, $\mathcal{M}(\mathcal{P})$ has no zero columns.
The system

\[ f_1 = a_1 + a_{1,1,0}u_1 + a_{1,1,1}u_{1,1} + a_{1,2,1}u_{2,1} + a_{1,2,2}u_{2,2}, \]
\[ f_2 = a_2 + a_{2,2,2}u_{2,2} + a_{2,2,3}u_{2,3}, \]
\[ f_3 = a_3 + a_{3,1,1}u_{1,1} + a_{3,2,1}u_{2,1} + a_{3,2,2}u_{2,2}. \]
The system

\[
\begin{align*}
  f_1 &= a_1 + a_{1,1,0} u_1 + a_{1,1,1} u_{1,1} + a_{1,2,1} u_{2,1} + a_{1,2,2} u_{2,2}, \\
  f_2 &= a_2 + a_{2,2,2} u_{2,2} + a_{2,2,3} u_{2,3}, \\
  f_3 &= a_3 + a_{3,1,1} u_{1,1} + a_{3,2,1} u_{2,1} + a_{3,2,2} u_{2,2}.
\end{align*}
\]

\[\mathcal{O}(\mathcal{P}) = \begin{pmatrix} 1 & 2 \\ -\infty & 3 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} \mathcal{L}_{1,1} & \mathcal{L}_{1,2} \\ 0 & \mathcal{L}_{2,2} \\ \mathcal{L}_{3,1} & \mathcal{L}_{3,2} \end{pmatrix}\]

is super essential, let us construct \(\mathcal{M}(\mathcal{P})\).
The system

\[ f_1 = a_1 + a_{1,1,0}u_1 + a_{1,1,1}u_{1,1} + a_{1,2,1}u_{2,1} + a_{1,2,2}u_{2,2}, \]
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\[ \mathcal{O}(\mathcal{P}) = \begin{pmatrix} 1 & 2 \\ -\infty & 3 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} \mathcal{L}_{1,1} & \mathcal{L}_{1,2} \\ 0 & \mathcal{L}_{2,2} \\ \mathcal{L}_{3,1} & \mathcal{L}_{3,2} \end{pmatrix} \]

is super essential, let us construct \( \mathcal{M}(\mathcal{P}) \).

Denote \( \partial^l a_{i,j,k} \) by \( a_{i,j,k}^{(l)} \) and \( \partial^l a_i \) by \( a_i^{(l)} \), \( l \in \mathbb{N} \).
...for linear differential polynomials

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...for linear differential polynomials

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</tbody>
</table>
Nonzero differential operator $\mathcal{L} = \sum_{k \in \mathbb{N}_0} a_k(\mathcal{L}) \partial^k \in \mathbb{D}[\partial]$.
The $\gamma$-symbol matrix $\sigma_\gamma(\mathcal{P})$ of $\mathcal{P}$ is the $n \times (n - 1)$ matrix whose $i$th row contains
\[
(a_{o_i - \gamma_{n-1}}(\mathcal{P})(\mathcal{L}_{i,n-1}), \ldots, a_{o_i - \gamma_1}(\mathcal{P})(\mathcal{L}_{i,1})).
\]
Nonzero differential operator $\mathcal{L} = \sum_{k \in \mathbb{N}_0} a_k(\mathcal{L}) \partial^k \in \mathbb{D}[\partial]$.

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$$(a_{o_i - \gamma_{n-1}}(\mathfrak{P})(\mathcal{L}_{i,n-1}), \ldots, a_{o_i - \gamma_1}(\mathfrak{P})(\mathcal{L}_{i,1})).$$

(7)

$$f_1 = a_1 + a_{1,1,2}u_{1,2} + a_{1,2,0}u_2 + a_{1,3,0}u_3,$$

$$f_2 = a_2 + a_{2,1,2}u_{1,2} + a_{2,2,0}u_2 + a_{2,3,0}u_3,$$

$$f_3 = a_3 + a_{3,1,2}u_{1,2} + a_{3,2,0}u_2 + a_{3,3,0}u_3,$$

$$f_3 = a_4 + a_{4,1,0}u_1 + a_{4,2,1}u_{2,1} + a_{4,3,2}u_{3,2}.$$
From algebraic to differential resultants

Differential resultant formulas

...for linear differential polynomials

...for nonlinear Laurent differential polynomials

Order and degree bounds for sparse differential resultants
\[ \mathcal{P} = \{f_1, \ldots, f_n\} \in \mathbb{D}\{U^\pm\} \text{ and } \mathcal{P}_i := \mathcal{P} \setminus f_i \]

\[ \mathcal{O}(\mathcal{P}) = (o_{i,j}) \text{ order matrix of } \mathcal{P}. \]
\( \mathcal{P} = \{ f_1, \ldots, f_n \} \in \mathbb{D}\{U^\pm\} \) and \( \mathcal{P}_i := \mathcal{P} \backslash f_i \)

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The diagonals of the matrix \( \mathcal{O}(\mathcal{P}_i) \) are indexed by the set \( \Gamma_i \) of all possible bijections

\[ \{1, \ldots, n\} \backslash \{i\} \rightarrow \{1, \ldots, n - 1\}. \]
\[ \mathcal{P} = \{f_1, \ldots, f_n\} \in \mathbb{D}\{U^\pm\} \text{ and } \mathcal{P}_i := \mathcal{P} \setminus f_i \]

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\[ \{1, \ldots, n\}\setminus\{i\} \rightarrow \{1, \ldots, n - 1\}. \]

The Jacobi number \( J_i(\mathcal{P}) \) of the matrix \( \mathcal{O}(\mathcal{P}_i) \), Li, Yuan, Gao 2012

\[ J_i(\mathcal{P}) := \text{Jac}(\mathcal{O}(\mathcal{P}_i)) := \max \left\{ \sum_{j \in \{1, \ldots, n\}\setminus\{i\}} o_{j,\mu(j)} \mid \mu \in \Gamma_i \right\}. \]
The situation where \( J_i(\mathcal{P}) \geq 0, \ i = 1, \ldots, n \) is of special interest.
The situation where $J_i(\mathcal{P}) \geq 0$, $i = 1, \ldots, n$ is of special interest.

$x_{i,j}, i = 1, \ldots, n, j = 1, \ldots, n - 1$ be algebraic indeterminates over $\mathbb{Q}$ $X(\mathcal{P}) = (X_{i,j})$ the $n \times (n - 1)$ matrix, such that

$$X_{i,j} := \begin{cases} x_{i,j}, & \mathcal{S}_j(f_i) \neq \emptyset, \\ 0, & \mathcal{S}_j(f_i) = \emptyset. \end{cases}$$

(8)

entries in the field $\mathbb{K} := \mathbb{Q}(X_{i,j} \mid X_{i,j} \neq 0)$. 

The situation where $J_i(\mathcal{P}) \geq 0$, $i = 1, \ldots, n$ is of special interest.

$x_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, n - 1$ be algebraic indeterminates over $\mathbb{Q}$

$X(\mathcal{P}) = (X_{i,j})$ the $n \times (n - 1)$ matrix, such that

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entries in the field $\mathbb{K} := \mathbb{Q}(X_{i,j} \mid X_{i,j} \neq 0)$.

$$J_i(\mathcal{P}) \geq 0 \iff \det(X(\mathcal{P}_i)) \neq 0.$$
The situation where \( J_i(\mathcal{P}) \geq 0, i = 1, \ldots, n \) is of special interest.

\( x_{i,j}, i = 1, \ldots, n, j = 1, \ldots, n - 1 \) be algebraic indeterminates over \( \mathbb{Q} \).

\( X(\mathcal{P}) = (X_{i,j}) \) the \( n \times (n - 1) \) matrix, such that

\[
X_{i,j} := \begin{cases} x_{i,j}, & \mathcal{S}_j(f_i) \neq \emptyset, \\ 0, & \mathcal{S}_j(f_i) = \emptyset. \end{cases}
\]

(8)

entries in the field \( \mathbb{K} := \mathbb{Q}(X_{i,j} \mid X_{i,j} \neq 0) \).

\[
J_i(\mathcal{P}) \geq 0 \iff \det(X(\mathcal{P}_i)) \neq 0.
\]

\( \mathcal{P} \) is super essential if \( \det(X(\mathcal{P}_i)) \neq 0, i = 1, \ldots, n \).
For $j = 1, \ldots, n - 1$ let us define integers in $\mathbb{N}$

$$\gamma_j(P) := \min\{\text{lord}(f_i, u_j) \mid S_j(f_i) \neq \emptyset, i = 1, \ldots, n\},$$

$$\gamma(P) := \sum_{j=1}^{n-1} \gamma_j(P).$$
For $j = 1, \ldots, n - 1$ let us define integers in $\mathbb{N}$

$$
\gamma_j(\mathcal{P}) := \min\{\text{lord}(f_i, u_j) \mid \mathcal{G}_j(f_i) \neq \emptyset, i = 1, \ldots, n\},
$$

$$
\gamma(\mathcal{P}) := \sum_{j=1}^{n-1} \gamma_j(\mathcal{P}).
$$

If $J_i(\mathcal{P}) \geq 0, i = 1, \ldots, n$

$$
\text{ps}(f_i) := \{\partial^k f_i \mid k \in [0, J_i(\mathcal{P}) - \gamma(\mathcal{P})] \cap \mathbb{N}\} \quad \text{and} \quad \text{ps}(\mathcal{P}) := \bigcup_{i=1}^{n} \text{ps}(f_i),
$$

containing $L := \sum_{i=1}^{n} (J_i(\mathcal{P}) - \gamma(\mathcal{P}) + 1)$ differential polynomials, whose variables belong to

$$
\mathcal{V}(\mathcal{P}) := \{u_{j,k} \mid k \in [\gamma(\mathcal{P})_j, M_j] \cap \mathbb{N}, j = 1, \ldots, n - 1\},
$$

with $M_j := m_j - \gamma(\mathcal{P})$ and $m_j := \max\{o_{i,j} + J_i(\mathcal{P}) \mid i = 1, \ldots, n\}$. 

...for nonlinear Laurent differential polynomials
\[ J_i(\mathcal{P}) \geq 0, \ i = 1, \ldots, n \Rightarrow \sum_{i=1}^{n} J_i(\mathcal{P}) = \sum_{j=1}^{n-1} m_j \]

Thus the number of elements of \( \mathcal{V}(\mathcal{P}) \) equals

\[
\sum_{j=1}^{n-1} (M_j - \gamma_j + 1) = \sum_{j=1}^{n-1} (m_j - \gamma_j - \gamma + 1) = \sum_{i=1}^{n} J_i(\mathcal{P}) - n\gamma + n - 1 = L - 1.
\]
\[ J_i(\mathcal{P}) \geq 0, \ i = 1, \ldots, n \Rightarrow \sum_{i=1}^{n} J_i(\mathcal{P}) = \sum_{j=1}^{n-1} m_j \]

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\]

Given \( j \in \{1, \ldots, n-1\} \) we have

\[
\bigcup_{f \in \text{ps}(\mathcal{P})} \mathcal{G}_j(f) \subseteq [\gamma(\mathcal{P})_j, M_j] \cap \mathbb{N}, \tag{9}
\]

and we cannot guarantee that the equality holds.
for nonlinear Laurent differential polynomials

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Given \( j \in \{1, \ldots, n - 1\} \) we have

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\bigcup_{f \in \text{ps}(\mathcal{P})} \mathcal{S}_j(f) \subseteq [\gamma(\mathcal{P})_j, M_j] \cap \mathbb{N}, \quad (9)
\]

and we cannot guarantee that the equality holds.

If there exists \( j \) such that (9) is not an equality, \( \mathcal{P} \) is sparse in the order.
...for nonlinear Laurent differential polynomials

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Given \( j \in \{1, \ldots, n - 1\} \) we have

\[ \bigcup_{f \in \text{ps}(\mathcal{P})} \mathcal{S}_j(f) \subseteq [\gamma(\mathcal{P})_j, M_j] \cap \mathbb{N}, \quad (9) \]

and we cannot guarantee that the equality holds.

If there exists \( j \) such that (9) is not an equality, \( \mathcal{P} \) is sparse in the order.

- Every system \( \mathcal{P} \) contains a super essential subsystem \( \mathcal{P}^* \).

- If \( \text{rank}(X(\mathcal{P})) = n - 1 \) then \( \mathcal{P}^* \) is unique.
...for nonlinear Laurent differential polynomials

Systems $\mathcal{P} = \{f_1, f_2, f_3, f_4\}$ and $\mathcal{P}' = \{f_1, f_2, f_3, f_5\}$

\[ f_1 = 2 + u_1u_{1,1} + u_{1,2}, \quad f_2 = u_1u_{1,2}, \quad f_3 = u_2u_{3,1}, \quad f_4 = u_{1,1}u_2, \quad f_5 = u_{1,2}, \]

\[ X(\mathcal{P}) = \begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & 0 & 0 \\ 0 & x_{3,2} & x_{3,3} \\ x_{4,1} & x_{4,2} & 0 \end{pmatrix} \quad \text{and} \quad X(\mathcal{P}') = \begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & 0 & 0 \\ 0 & x_{3,2} & x_{3,3} \\ x_{4,1} & 0 & 0 \end{pmatrix}. \]
...for nonlinear Laurent differential polynomials

Systems $\mathcal{P} = \{f_1, f_2, f_3, f_4\}$ and $\mathcal{P}' = \{f_1, f_2, f_3, f_5\}$

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$\mathcal{P}$ is not super essential but since $\text{rank}(X(\mathcal{P})) = 3$, it has a unique super essential subsystem, which is $\{f_1, f_2\}$. 
Systems $\mathcal{P} = \{f_1, f_2, f_3, f_4\}$ and $\mathcal{P}' = \{f_1, f_2, f_3, f_5\}$

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 0 & x_{3,2} & x_{3,3} \\
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and

$$X(\mathcal{P}') = \begin{pmatrix} x_{1,1} & 0 & 0 \\
 x_{2,1} & 0 & 0 \\
 0 & x_{3,2} & x_{3,3} \\
 x_{4,1} & 0 & 0 \end{pmatrix}.$$

$\mathcal{P}$ is not super essential but since $\text{rank}(X(\mathcal{P})) = 3$, it has a unique super essential subsystem, which is $\{f_1, f_2\}$.

$\mathcal{P}'$ is not super essential and $\text{rank}(X(\mathcal{P}')) < 3$, super essential subsystems are $\{f_1, f_2\}, \{f_1, f_3\}$ and $\{f_2, f_3\}$. 
Theorem If $\mathcal{P}$ is super essential then
\[ \bigcup_{f \in \text{ps}(\mathcal{P})} \mathcal{S}_j(f) = [0, M_j] \cap \mathbb{N}, \quad j = 1, \ldots, n - 1. \]

$\mathcal{P}$ is a system of $L$ polynomials in $L - 1$ algebraic indeterminates.
Differential resultant formulas, S.L. Rueda

- From algebraic to differential resultants
- Differential resultant formulas
- ...for linear differential polynomials
- ...for nonlinear Laurent differential polynomials
- Order and degree bounds for sparse differential resultants
Ordering on $\mathcal{V}(\mathcal{P})$ through a bijection $\beta : \mathcal{V} \to \{1, \ldots, L - 1\}$.

$\mathcal{V} = \{y_1, \ldots, y_{L-1}\}$ be a set of $L - 1$ algebraic indeterminates over $\mathbb{Q}$. A bijection $\nu : \mathcal{V} \to \mathcal{V}$, by $\nu(y_l) = \beta^{-1}(l)$ extends to a ring isomorphism

$$\nu : \mathbb{D}[\mathcal{V}^\pm] \to \mathbb{D}[\mathcal{V}^\pm].$$
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$$
\nu : \mathbb{D}[\mathcal{V}^\pm] \rightarrow \mathbb{D}[\mathcal{V}^\pm].
$$

Monomials in $\mathbb{D}[\mathcal{V}^\pm]$

$$
y^\alpha = y_1^{\alpha_1} \cdots y_{L-1}^{\alpha_{L-1}}, \ \alpha = (\alpha_1, \ldots, \alpha_{L-1}) \in \mathbb{Z}^{L-1}.
$$

Algebraic support of $f = \sum_{\alpha \in \mathbb{N}^{L-1}} a_\alpha \nu(y^\alpha)$ in $\mathbb{D}[\mathcal{V}^\pm]$

$$
\mathcal{A}(f) := \{\alpha \in \mathbb{Z}^{L-1} \mid a_\alpha \neq 0\}.
$$
Ordering on $\text{ps}(\mathcal{P})$ through a bijection $\lambda : \text{ps}(\mathcal{P}) \rightarrow \{1, \ldots, L\}$. 
Ordering on $\text{ps}(\mathcal{P})$ through a bijection $\lambda : \text{ps}(\mathcal{P}) \rightarrow \{1, \ldots, L\}$.

We define the \textit{algebraic generic system} associated to $\mathcal{P}$ as

\[ \text{ags}(\mathcal{P}) := \left\{ \sum_{\alpha \in A(f)} c^\lambda(f) y^\alpha \mid f \in \text{ps}(\mathcal{P}) \right\}, \]

where $c^\lambda(f)$ are algebraic indeterminates over $\mathbb{Q}$.
Ordering on $\text{ps}(\mathcal{P})$ through a bijection $\lambda : \text{ps}(\mathcal{P}) \rightarrow \{1, \ldots, L\}$.

We define the algebraic generic system associated to $\mathcal{P}$ as

$$\text{ags}(\mathcal{P}) := \left\{ \sum_{\alpha \in A(f)} c^{\lambda(f)}_{\alpha} y^{\alpha} \mid f \in \text{ps}(\mathcal{P}) \right\},$$

where $c^{\lambda(f)}_{\alpha}$ are algebraic indeterminates over $\mathbb{Q}$.

$\rho := \lambda^{-1}$ we have

$$\text{ags}(\mathcal{P}) = \left\{ P_l := \sum_{\alpha \in A(\rho(l))} c^{l}_{\alpha} y^{\alpha} \mid l = 1, \ldots, L \right\}.$$
EXAMPLE System $\mathcal{P} = \{ f_1, f_2 \}$ in $\mathbb{D}\{u\}$, $\partial = \frac{\partial}{\partial t}$.

\[
\begin{align*}
  f_1 &= a_{(0,0)} + a_{(1,0)}u + a_{(0,1)}u' + a_{(2,0)}u^2 + a_{(3,0)}u^3, \\
  f_2 &= b_{(0,0)} + b_{(1,0)}u + b_{(2,0)}u^2 + b_{(3,0)}u^3,
\end{align*}
\]
EXAMPLE System $\mathcal{P} = \{f_1, f_2\}$ in $\mathbb{D}\{u\}$, $\partial = \frac{\partial}{\partial t}$.

$f_1 = a_{(0,0)} + a_{(1,0)}u + a_{(0,1)}u' + a_{(2,0)}u^2 + a_{(3,0)}u^3,$
$f_2 = b_{(0,0)} + b_{(1,0)}u + b_{(2,0)}u^2 + b_{(3,0)}u^3,$

$\text{ps}(\mathcal{P}) = \{f_1, f_2, \partial f_2\}$, with $\partial f_2$

$\partial b_{(0,0)} + \partial b_{(1,0)}u + b_{(1,0)}u' + \partial b_{(2,0)}u^2 + 2b_{(2,0)}uu' + \partial b_{(3,0)}u^3 + 3b_{(3,0)}u^2u'$

and $\mathcal{V} = \{u, u'\}$. 
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\end{align*}
\]

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\end{align*}
\]

and $\mathcal{V} = \{u, u'\}$.

System $\text{ags}(\mathcal{P}) = \{ P_1, P_2, P_3 \}$ of algebraic generic polynomials in $y_1, y_2$

\[
\begin{align*}
P_1 &= c_1 + c_{(1,0)}^1y_1 + c_{(0,1)}^1y_2 + c_{(2,0)}^1y_1^2 + c_{(3,0)}^1y_1^3, \\
P_2 &= c_2 + c_{(1,0)}^2y_1 + c_{(2,0)}^2y_1^2 + c_{(3,0)}^2y_1^3, \\
P_3 &= c_3 + c_{(1,0)}^3y_1 + c_{(0,1)}^3y_2 + c_{(2,0)}^3y_1^2 + c_{(1,1)}^3y_1y_2 + c_{(3,0)}^3y_1^3 + c_{(2,1)}^3y_1^2y_2.
\end{align*}
\]
Order and degree bounds for sparse differential resultants

\( \text{ags}(\mathcal{P}) \) is included in \( \mathbb{K}[\mathcal{Y}^{\pm}] \), with \( \mathbb{K} := \mathbb{Q}(\mathcal{C}) \)

\[ C_l := \{ c^l_{\alpha} \mid \alpha \in \mathcal{A}(\rho(l)) \} \quad \text{and} \quad \mathcal{C} := \bigcup_{l=1}^{L} C_l. \]
ags(\mathcal{P}) is included in \mathbb{K}[\mathcal{Y}^{\pm}], with \mathbb{K} := \mathbb{Q}(\mathcal{C})
\begin{align*}
\mathcal{C}_l := \{ c^l_\alpha | \alpha \in \mathcal{A}(\rho(l)) \} \text{ and } \mathcal{C} := \bigcup_{l=1}^L \mathcal{C}_l.
\end{align*}

\mathcal{P} \text{ super essential } \Rightarrow \text{ags}(\mathcal{P}), \text{ } L \text{ polynomials in } L - 1 \text{ indeterminates } \mathcal{Y}.
ags(\mathcal{P}) \text{ is included in } \mathbb{K}[\mathcal{Y}^\pm], \text{ with } \mathbb{K} := \mathbb{Q}(\mathcal{C})

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\mathcal{C}_l := \{c^l_\alpha \mid \alpha \in \mathcal{A}(\rho(l))\} \text{ and } \mathcal{C} := \bigcup_{l=1}^{L} \mathcal{C}_l.

\mathcal{P} \text{ super essential } \Rightarrow \text{ags}(\mathcal{P}), \text{ } L \text{ polynomials in } L - 1 \text{ indeterminates } \mathcal{Y}.


Finite sets of monomials \Lambda_1, \ldots, \Lambda_L, \Lambda \text{ in } \mathbb{K}[\mathcal{Y}^\pm] \text{ are determined.}

\downarrow

The matrix Syl(ags(\mathcal{P})) in the monomial bases of the linear map

\langle \Lambda_1 \rangle_\mathbb{K} \oplus \cdots \oplus \langle \Lambda_L \rangle_\mathbb{K} \to \langle \Lambda \rangle_\mathbb{K} : (g_1, \ldots, g_L) \mapsto \sum g_l P_l,

verifies \text{det}(Syl(ags(\mathcal{P}))) \neq 0.
\[
\det(Syl(\text{ags}(\mathcal{P}))) \in (\text{ags}(\mathcal{P})) \cap \mathbb{Q}[\mathcal{C}].
\]

\(S_1(\mathcal{P}) := Syl(\text{ags}(\mathcal{P}))\) assigns a special role to \(P_1\). The same construction can be done choosing \(P_l, l = 2, \ldots, L\) as a distinguished polynomial, obtaining a matrix denoted by \(S_l(\mathcal{P})\).
\[ \det(Syl(ags(\mathcal{P}))) \in (ags(\mathcal{P})) \cap \mathbb{Q}[C]. \]

\( S_1(\mathcal{P}) := Syl(ags(\mathcal{P})) \) assigns a special role to \( P_1 \). The same construction can be done choosing \( P_l, l = 2, \ldots, L \) as a distinguished polynomial, obtaining a matrix denoted by \( S_l(\mathcal{P}) \).

\( S_l(\mathcal{P}) \) has the minimum number of rows containing coefficients of \( P_l \).

If \( \text{Res}(\mathcal{P}) := \text{Res}(ags(\mathcal{P})) \) exists,

\[
\deg(D_l, C_l) = \deg(\text{Res}(\mathcal{P}), C_l) = MV_{-l}(\mathcal{P}) := \\
\mathcal{M}(\mathcal{Q}_h \mid h \in \{1, \ldots, L\} \setminus \{l\}) = \sum_{J \subset \{1, \ldots, L\} \setminus \{l\}} \left(-1\right)^{L-|J|} \text{vol}(\sum_{j \in J} \mathcal{Q}_j)
\]

where \( \sum_{j \in J} \mathcal{Q}_j \) is the Minkowski sum of \( \mathcal{Q}_j, j \in J, \mathcal{Q}_l \) be the convex hull of \( \mathcal{A}(\rho(l)) \) in \( \mathbb{R}^{L-1} \) and \( \text{vol}(\mathcal{Q}_l) \) its \( L - 1 \) dimensional volume.
SPECIALIZATION to the coefficient set of $f = \sum_{\alpha \in A(f)} a^f_{\alpha} \nu(y^\alpha)$ in $\text{ps}(\mathcal{P})$

$A(\mathcal{P}) := \bigcup_{f \in \text{ps}(\mathcal{P})} A_f$, with $A_f := \{ a^f_{\alpha} \mid \alpha \in A(f) \}$. 
SPECIALIZATION to the coefficient set of $f = \sum_{\alpha \in \mathcal{A}(f)} a^f_{\alpha} \nu(y^\alpha)$ in $\text{ps}(\mathcal{P})$

$A(\mathcal{P}) := \bigcup_{f \in \text{ps}(\mathcal{P})} A_f$, with $A_f := \{a^f_{\alpha} \mid \alpha \in \mathcal{A}(f)\}$.

Given $l \in \{1, \ldots, L\}$, such that $\rho(l) = f$, and $c^l_\alpha \in C_l$, $a^\rho(l)_\alpha \in A_f$.

Ring epimorphism

$$\Xi : \mathbb{Q}[C][Y^\pm] \to \mathbb{Q}[A(\mathcal{P})][\nu^\pm],$$

$$\Xi(c^l_\alpha) = a^\rho(l)_\alpha$$

$$\Xi(y_l) = \nu(y_l)$$
SPECIALIZATION to the coefficient set of $f = \sum_{\alpha \in A(f)} a_{\alpha}^f v(y^\alpha)$ in $ps(P)$

$$A(P) := \bigcup_{f \in ps(P)} A_f, \text{ with } A_f := \{a_{\alpha}^f \mid \alpha \in A(f)\}.$$ 

Given $l \in \{1, \ldots, L\}$, such that $\rho(l) = f$, and $c_{\alpha}^l \in C_l$, $a_{\alpha}^{\rho(l)} \in A_f$.

Ring epimorphism

$$\Xi : \mathbb{Q}[C][Y^\pm] \to \mathbb{Q}[A(P)][V^\pm],$$

$$\Xi(c_{\alpha}^l) = a_{\alpha}^{\rho(l)}$$

$$\Xi(y_l) = v(y_l)$$

$$\Xi(D_{\rho}(P)) \in [P] \cap D$$ is a differential resultant formula for $P$ with

$$L_i = J_i(P) - \gamma(P), \ U = V(P) \text{ and } \Omega_f = \Xi(\Lambda_{\lambda(f)}), \Omega = \Xi(\Lambda),$$

$f \in ps(P)$.  

Order and degree bounds for sparse differential resultants
ORDER AND DEGREE BOUNDS Let us consider sets of differential indeterminates over $\mathbb{Q}$,

$$A_i := \{ a^i_\alpha \mid \alpha \in A(f_i) \}, \quad i = 1, \ldots, n,$$
ORDER AND DEGREE BOUNDS Let us consider sets of differential indeterminates over \( \mathbb{Q} \),

\[
A_i := \{ a^i_\alpha | \alpha \in \mathcal{A}(f_i) \}, \quad i = 1, \ldots, n,
\]

The generic polynomial \( F_i \) in \( D\{U^\pm\} \) with algebraic support \( \mathcal{A}(f_i) \) is

\[
F_i := \sum_{\alpha \in A_i} a^i_\alpha v(y^\alpha).
\]

\( \mathcal{P} = \{F_1, \ldots, F_n\} \) of sparse generic Laurent differential polynomials in \( D\{U^\pm\}, D = \mathbb{Q}\{\bigcup^n_{i=1} A_i\} \).
ORDER AND DEGREE BOUNDS Let us consider sets of differential indeterminates over \( \mathbb{Q} \),

\[
A_i := \{ a^i_\alpha \mid \alpha \in \mathcal{A}(f_i) \}, \quad i = 1, \ldots, n,
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The generic polynomial \( F_i \) in \( D\{U^\pm\} \) with algebraic support \( \mathcal{A}(f_i) \) is

\[
F_i := \sum_{\alpha \in A_i} a^i_\alpha \nu(y^\alpha).
\]

\( \mathcal{P} = \{F_1, \ldots, F_n\} \) of sparse generic Laurent differential polynomials in \( D\{U^\pm\} \), \( D = \mathbb{Q}\{\bigcup_{i=1}^n A_i\} \).

If the differential elimination ideal \([\mathcal{P}] \cap D\) has dimension \( n - 1 \) then

\[
[\mathcal{P}] \cap D = \text{sat}(\partial\text{Res}(\mathcal{P})),
\]

\( \partial\text{Res}(\mathcal{P}) \) is the sparse differential resultant of \( \mathcal{P} \).

From $D_l(\mathfrak{F})$ a nonzero $H$ in $[\mathfrak{F}] \cap \mathcal{D}$ can be computed such that

$$\text{ord}(H, A_i) \leq J_i(\mathfrak{F}) - \gamma(\mathfrak{F}), \quad i = 1, \ldots, n$$
From $D_l(\mathfrak{P})$ a nonzero $H$ in $[\mathfrak{P}] \cap \mathcal{D}$ can be computed such that

$$\text{ord}(H, A_i) \leq J_i(\mathfrak{P}) - \gamma(\mathfrak{P}), \quad i = 1, \ldots, n$$

and

$$\deg(H, \partial^k A_i) \leq \deg(D_{\lambda(\partial^k f_i)}, C_{\lambda(\partial^k f_i)}), \quad k = 1, \ldots, J_i(\mathfrak{P}) - \gamma(\mathfrak{P}).$$

Furthermore, if $\text{Res}(\mathfrak{P})$ exists then

$$\deg(H, \partial^k A_i) \leq MV_{-\lambda(\partial^k f_i)}(\mathfrak{P}), \quad k = 1, \ldots, J_i(\mathfrak{P}) - \gamma(\mathfrak{P}).$$