Periodic and Mean-Periodic Solutions of LODEs with Constant Coefficients

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A review of our approach to obtaining periodic and mean-periodic solutions of LODE with constant coefficients is presented.

Let’s consider a non-zero polynomial with constant coefficients of degree $n$:

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

and the following ordinary linear differential equation with constant coefficients:

$$P\left(\frac{d}{dt}\right)y = f(t), \quad -\infty < t < \infty$$  \hspace{1cm} (1)

We are looking for a periodic solution $y(t)$ with period $T$ of this equation, i.e. a solution satisfying the identity:

$$y(t + T) = y(t), \quad -\infty < t < \infty$$  \hspace{1cm} (2)
An obvious necessary condition for existing of a periodic solution of (1) with period $T$ is: the function $f(t)$ to be periodic with period $T$, i.e. for each $t \in \mathbb{R}$ to be satisfied:

$$f(t + T) = f(t)$$ (3)

The following theorem could be proven: A solution of (1) with periodic right-hand side $f(t)$ with period $T$ is $T$-periodic if and only if the following “boundary” conditions are satisfied:

$$y(T) - y(0) = 0, \quad y'(T) - y'(0) = 0, \quad \ldots \quad y^{(n-1)}(T) - y^{(n-1)}(0) = 0$$ (4)

This theorem allows the problem of obtaining periodic solutions of (1) to be reduced to the problem of finding a solution of this equation in the interval $(-\infty, \infty)$, satisfying the “boundary” conditions (4).
Further we reduce this problem to the following intermediate (auxiliary) boundary-value problem:

\[ P \left( \frac{d}{dt} \right) y = f(t), \quad -\infty < t < \infty \]

\[ \int_0^T y(\tau) d\tau = \alpha_0, \quad y^{(k)}(T) - y^{(k)}(0) = \alpha_{k+1}, \quad k = 0, 1, \ldots n - 2. \] (5)

An operational method for solving the auxiliary problem.

Convolution of Dimovski.

The Heaviside algorithm is developed for solving initial value problems for LODE with constant coefficients and it can not be used directly for finding periodic solutions of such equations.
Use of Fourier transform and Laplace transform for obtaining periodic solutions can be found in some works of Kaplan, Rosenvasser, Lurie and some others. We use an alternative direct approach, similar to those of Mikusiński, but using another convolution, based on the operational calculus of Dimovski (see Dimovski, I.H., *Convolutional Calculus*, Kluwer Acad. Publishers, Dordrecht, 1990) and related to the nonlocal boundary value problem in $C(\mathbb{R})$:

$$y' = f(x), \int_0^T y(\tau)d\tau = 0,$$

where $T$ is a constant.

The solution

$$L f(t) = \int_0^t f(\tau)d\tau - \frac{1}{T} \int_0^T \left( \int_0^\tau f(\sigma)d\sigma \right) d\tau$$

is an analogue of the integration operator $l f(t) = \int_0^t f(\tau)d\tau$ of Mikusiński’s operational calculus.
The operational calculus of Dimovski for the operator $L$ is an analogue of the operational calculus of Mikusinški, but the following convolution of Dimovski is used:

$$(f^t g)(t) = \Phi_\tau\{\int_\tau^t f(t + \tau - \sigma)g(\sigma)d\sigma\},$$

with an arbitrary linear functional $\Phi$ in $C(\mathbb{R})$. In our case the functional $\Phi\{f\} = \frac{1}{T}\int_0^T f(\tau)d\tau$ is used. The convolution

$$(f^t g)(t) = \frac{1}{T}\int_0^T \left(\int_\tau^t f(t + \tau - \sigma)g(\sigma)d\sigma\right)d\tau$$

has the property $Lf(t) = \{1\}^t f$. 

Dimovski and Grozdev proposed a simpler convolution (without using of repeated integrals):

\[(f * g)(t) = \frac{f(t)}{T} \int_0^T g(\tau) \, d\tau + \frac{g(t)}{T} \int_0^T f(\tau) \, d\tau - \frac{1}{T} \int_0^t f(t - \tau) \, g(\tau) \, d\tau - \frac{1}{T} \int_t^T f(t + T - \tau) \, g(\tau) \, d\tau,\]

(6)

for which \(\{1\}_t t f = f\).

The constant function \(\{1\}\) plays the role of a unity in the convolution algebra \((C(\mathbb{R}), *)\). The operator \(L\) has the following representation:

\[L\{1\} = t - \frac{T}{2}, \text{ i.e. } Lf = \left\{t - \frac{T}{2}\right\}_t \star f.\]
Further, convolution fractions of the form $f/g$ are considered (with $f, g \in C[0, T]$, $g$ being a nondivisor of 0 of the operation (6)). The ring of the continuous functions on $(-\infty, \infty)$ is extended to the smallest ring $\mathcal{M}$, containing the convolution fractions $\frac{f}{g}$ with denominators which are nondivisors of 0. The most important convolution fraction

$$S = \frac{1}{L}$$

is considered as an algebraic analogue of $d/dt$.

The basic formula of the Operational Calculus of Dimovski is:

$$\left\{ f'(t) \right\} = S \{ f(t) \} - \frac{1}{T} \int_{0}^{T} f(\tau) d\tau. \quad (7)$$

Here $\frac{1}{T} \int_{0}^{T} f(\tau) d\tau$ is considered as a constant function.
For \( f^{(n)} \) the following formula can be derived from (7):

\[
f^{(n)} = S^n f - \frac{S^n}{T} \int_0^T f(\tau) d\tau - \sum_{k=1}^{n-1} \frac{S^k}{T} \left( f^{(n-1-k)}(T) - f^{(n-1-k)}(0) \right)
\] (8)

For the case \( T = 1 \), the integral operator \( L \) is called by Dimovski and Grozdev **Bernoullian integration operator** due to the following relation with the polynomials of Bernoulli:

\[
L^n \{1\} = \frac{T^n}{n!} B_n \left( \frac{t}{T} \right), \quad n = 0, 1, 2, \ldots,
\]

where \( B_n(t) \) is the polynomial of Bernoulli of degree \( n \).

**Further we can follow the scheme of Mikusiński**, using the convolution (6) and taking into account the following differences:

1) The operation (6) has a unit element.
2) This operation has divisors of 0.
The eigenfunctions of $L$ are divisors of 0 of (6). These functions have the form $\varphi_n(t) = Ce^{\frac{2\pi in}{T}}$, $n \in \mathbb{Z} \setminus \{0\}$.

For the application of the new operational calculus it is important we to have formulae for convolution fractions of the type $\frac{1}{(S - \lambda)^k}$, $k \in \mathbb{N}$. They exist iff $S - \lambda$ is a nondivisor of 0 and this is not truth iff $\lambda = \frac{2\pi in}{T}$ and $n \in \mathbb{Z} \setminus \{0\}$.

Thus for each $\lambda \neq \frac{2\pi in}{T}$, $n \in \mathbb{Z} \setminus \{0\}$ the following formulae hold:

$$\frac{1}{S - \lambda} = -\frac{1}{\lambda} + \frac{T e^{\lambda T}}{e^{\lambda T} - 1}$$ \hspace{1cm} (9)

$$\frac{S}{S - \lambda} = \frac{T \lambda e^{\lambda T}}{e^{\lambda T} - 1}$$ \hspace{1cm} (10)
Corollary. If \( \lambda \neq \frac{2\pi in}{T} \), \( n \in \mathbb{Z} \setminus \{0\} \), more general formulae hold (for each integer \( k \geq 1 \)):

\[
\frac{1}{(S - \lambda)^k} = \frac{1}{(k - 1)!} \frac{\partial^{k-1}}{\partial \lambda^{k-1}} \left( \frac{-1}{\lambda} + \frac{T e^{t \lambda}}{e^{\lambda T} - 1} \right)
\] (11)

\[
\frac{S}{(S - \lambda)^k} = \frac{1}{(k - 1)!} \frac{\partial^{k-1}}{\partial \lambda^{k-1}} \left( \frac{T \lambda e^{t \lambda}}{e^{\lambda T} - 1} \right)
\] (12)

The formulae (9)–(12) are intended to be used for interpretation of rational expressions in the extended Heaviside algorithm. For the purposes of the program implementation of this algorithm additional formulae were derived—for the case when the denominator is an integer power of a second degree polynomial.
Non-resonance case. Let’s apply the Operational Calculus of Dimovski for solving the auxiliary problem, formulated above:

\[ P \left( \frac{d}{dt} \right) y = f(t), \quad -\infty < t < \infty \]
\[ \int_0^T y(\tau) d\tau = \alpha_0, \quad y^{(k)}(T) - y^{(k)}(0) = \alpha_{k+1}, \quad k = 0, 1, \ldots n - 2. \]  

(13)

Using the formulae (7)–(8), we can make an “algebraization” of the problem, thus reducing it to one algebraic equation of 1\textsuperscript{st} degree:

\[ P(S)y = f + SQ(S), \]  

(14)

where \( P(S) \) and \( Q(S) \) are polynomials of \( S \) and the degree of \( Q(S) \) is less than the degree of \( P(S) \).

The formal solution of the above equation has the form

\[ y = \frac{1}{P(s)}f + S \frac{Q(s)}{P(s)}. \]  

(15)
The above representation contains division by $P(S)$ and this is possible if $P(S)$ is not a divisor of 0 in $\mathcal{M}$, i.e. iff $P\left(\frac{2\pi im}{T}\right) \neq 0$ for each $m \in \mathbb{Z} \setminus \{0\}$. This is the so–called non-resonance case.

Main steps of the extended Heaviside algorithm for solving the intermediate problem in the non-resonance case:

1) Finding the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation $P(\lambda) = 0$.
2) Finding out that none of the roots have the form $\frac{2\pi im}{T}$ with $m \in \mathbb{Z} \setminus \{0\}$.
3) Finding the polynomial $Q(S)$.
4) Expanding $\frac{1}{P(S)}$ and $\frac{Q(S)}{P(S)}$ into a sum of partial fractions.
5) Interpretation of the fractions $w = \frac{1}{P(S)}$ and $v = S\frac{Q(S)}{P(S)}$ as functions.
6) Representation of the solution in the form $u = w \ast f + v$. 

A comparison with the classical Heaviside algorithm:

– for algebraization of the problem the formulae (7)-(8) are used now.

– we have here an additional step (step 2);

– new interpretation formulae (such as (9)-(12)) are used on step 5);

– the operation $\ast$ on step 6) is not the Duhamel convolution; it is the convolution (6).
**Resonance case.**
If the above condition $\lambda \neq \frac{2\pi in}{T}, \ n \in \mathbb{Z} \setminus \{0\}$ fails for one or more roots of $P$, we have the so-called **resonance case** and the corresponding roots are called resonance roots.

Let’s denote with $n_1, n_2, \ldots, n_p$ all integer numbers, for which $P\left(\frac{2\pi in_k}{T}\right) = 0, \ k = 1, 2, \ldots, p$, and let $C_{n_1, n_2, \ldots, n_p}$ be the sub-algebra of $(C(\mathbb{R}), \ast)$, such that the convolution (6) plays the role of multiplication in it. It was mentioned above that the eigenfunctions of the operator $L$ have the form $\varphi_n(t) = e^{\frac{2\pi int}{T}}, \ n \in \mathbb{Z} \setminus \{0\}$. It is shown in Grozdev S., I. Dimovski. Bernoullian operational calculus, *Mathematics and Math. education, Pcoc. of the 9th Spring Conf. of the UBS, April 3–6, 1980, Sofia, BAS, 1980*, 30–36 (in Bulgarian), that if $f \in C[0, T]$, then

$$f \ast \{e^{\frac{2\pi int}{T}}\} = \chi_n(f) e^{\frac{2\pi int}{T}}, \ n = \pm 1, \pm 2, \ldots,$$

where

$$\chi_n(f) = \frac{1}{T} \int_0^1 (e^{\frac{2\pi int}{T}} - 1)f(t)dt, \ n = \pm 1, \pm 2, \ldots,$$

(16)
is a complete system of multiplicative functionals. We call them Fourier coefficients of \( f \) with respect to \( \left\{ e^{\frac{2\pi int}{T}} \right\}, \ n \in \mathbb{Z} \setminus \{0\} \).

Due to a theorem proven in the above cited paper, at least one of the Fourier coefficients of the function \( f \) has to be equal to zero in order this function to be a divisor of 0 in the algebra \( (C(\mathbb{R}), \ast) \). One can prove that this condition is necessary as well.

Let’s denote by \( \tilde{L} \) the restriction of the operator \( L \) to \( C_{n_1, n_2, \ldots, n_p} \). Then instead of \( Lf = r \ast f \), for \( r(t) = t - \frac{T}{2} \) in \([0, T]\), the following presentation in \( C_{n_1, n_2, \ldots, n_p} \) will hold: \( \tilde{L}f = \tilde{r} \ast f \), where

\[
\tilde{r}(t) = r(t) - \sum_{k=1}^{p} \chi_{n_k}(r) e^{\frac{2\pi in_k t}{T}} = t - \frac{T}{2} - \sum_{k=1}^{p} \frac{T}{2\pi in_k t} e^{\frac{2\pi in_k t}{T}}.
\]

We denote by \( \mathcal{M}_{n_1, n_2, \ldots, n_p} \) the ring of the convolution fractions of \( C_{n_1, n_2, \ldots, n_p} \), whose denominators are nondivisors of 0 of the convolution (6). Denote the algebraic inverse element of \( \tilde{L} \) by \( \tilde{S} \), i.e. \( \tilde{S} = \frac{1}{\tilde{L}} \).
Two important theorems, proven by Dimovski and Grozdev, are denoted here by T1 and T2, respectively:

**T1.** The elements $\tilde{S} - \frac{2\pi in_k}{T}, k = 1, 2, \ldots, p$ of the ring $\mathcal{M}_{n_1, n_2, \ldots, n_p}$ are reversible and

\[
\frac{1}{(\tilde{S} - \frac{2\pi in_k}{T})^m} = \left\{ \frac{(-1)^{m-1}}{(\frac{2\pi in_k}{T})^m} + \frac{e^{\frac{2\pi in_k t}{T}}}{m!} B_m\left(\frac{t}{T}\right) \right\}^* \tag{17}
\]

for $m = 1, 2, \ldots$, where $B_m$ is the polynomial of Bernoulli of degree $m$ (the sign $*$ means a convolution operator).

**T2.** If $P\left(\frac{2\pi in_k}{T}\right) = 0$ for $k = 1, 2, \ldots, p$ and $P\left(\frac{2\pi in}{T}\right) \neq 0$ for all other integer numbers $n \neq 0$, an necessary and sufficient condition for solvability of (13) is:

\[
\frac{1}{T} \int_0^1 f(t)\left(e^{\frac{2\pi in_k t}{T}} - 1\right)dt = 0, \ k = 1, 2, \ldots, p, \tag{18}
\]

i.e. the Fourier coefficients of $f(t)$ with numbers $n_1, n_2, \ldots, n_p$ to be equal to 0.
We can formulate now the algorithm for solving (13) in the resonance case:

1) As in the non-resonance case, we can make an algebraization of the problem, i.e. we can reduce it to a single equation but in $C_{n_1, n_2, \ldots, n_p}$:

$$P(\tilde{S}) \tilde{y} = f + Q(\tilde{S}).$$  \hspace{1cm} (19)

2) We consider the homogenous BVP:

$$P \left( \frac{d}{dt} \right) y = 0, \int_0^T y(\tau) d\tau = 0, y^{(j)}(T) - y^{(j)}(0) = 0, j = 0, 1, \ldots n - 2.$$  

It is equivalent to the equation $P(\tilde{S}) y = 0$ and its solutions have the form:

$$y = \left\{ C_1 e^{\frac{2\pi i k_1 t}{T}} + \ldots + C_m e^{\frac{2\pi i k_m t}{T}} \right\},$$

where $C_1, C_2, \ldots, C_m$ are constants.
3) The solution of (13) has the form:

$$y = \tilde{y} + \left\{ C_1 e^{\frac{2\pi i k_1 t}{T}} + \ldots + C_m e^{\frac{2\pi i k_m t}{T}} \right\},$$

(20)

where $\tilde{y}$ is the solution of (19).

The reducing of the problem for obtaining periodic solutions of LODE with constant coefficients to the auxiliary problem deserves special attention. This consideration is omitted here.
**General algorithm** for obtaining a periodic solution.

1) Algebraization of the given problem and finding roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the equation $P(\lambda) = 0$.

2) a) Finding out roots of the form $\frac{2\pi m}{T}$ ($m \in \mathbb{Z} \setminus \{0\}$).
    b) Verifying whether the roots selected in 2 a) satisfy the conditions (18). If for some of the selected roots these conditions are not satisfied, periodic solutions do not exist.

3) Forming the polynomial $Q(S)$.

4) Partial fraction decomposition of $\frac{1}{P(S)}$ and $\frac{Q(S)}{P(S)}$ and separation of the resonance and non-resonance parts.

5) Interpretation of the fractions $w = \frac{1}{P(S)}$ and $v = \frac{Q(S)}{P(S)}$ as functions. As was mentioned above, different groups of formulae are used for interpretation of the fractions from the resonance and the non-resonance parts.

6) Presentation of the solution in the form:

$$u_{nr} = w_1 \ast f + v_1, \quad u_r = w_2 \ast f + v_2$$

$$u = u_{nr} + u_r,$$

(21)
where $w_1$ and $w_2$ are functions, obtained at step 5) after interpretation of the non-resonance and resonance parts respectively of the partial fraction decomposition of $w$; $v_1$ and $v_2$ are functions obtained at step 5) after interpretation of the non-resonance and resonance parts respectively of the partial fraction decomposition of $v$.

The general solution $u$ is the sum of both parts of the solution—the non-resonance part $u_{nr}$ and the resonance part $u_r$. It is possible, of course, for each of these parts to be equal to zero.

**Program implementation** of the algorithm

The program implementation of the general algorithm follows the successive steps considered above.

A program package for the computer algebra system *Mathematica* was developed.
Two illustrative examples are presented – for a resonance case and for a “mixed” case. An option for visualization of the solution, together with the right-hand side function is used in the first example.

**Example 1:**

\[
\text{DSolveOCP}[\{y''(t) + 4y(t) = \cos(3t), \alpha[1] = 0\}, \ y[t], \ t, \ 2\pi, \ \text{Graph} \rightarrow \text{True}]
\]

\[y(t) \rightarrow -\frac{1}{8} \cos(3t)\]

**Visualization of the solution:**

[Graph of the solution]

**Example 2:**

\[
de = y'''[t] + y''[t] + 4y'[t] + 4y[t] = \cos(5t);
\]

\[
\text{DSolveOCP}\left[\{\text{de}, \alpha[1] = 0, \alpha[2] = 0\}, y[t], t, 2\pi\right]
\]

\[y[t] \rightarrow \frac{1}{540} \left(-\cos(5t) - 5\sin(5t)\right)\]
Main part of the interpretation formulae used by our program implementation

For the non-resonance case:

\[
\frac{1}{(S-a)^2} = \frac{(-1)^m}{a^m} + \frac{T \delta_{(a,-1,m)} \frac{a^m}{(-1+m)!}}{(-1+m)!}
\]

\[
\frac{1}{S^2 + a^2} = \frac{1}{a^2} - \frac{T \cos\left[a t - \frac{a T}{2}\right] \csc\left[\frac{a T}{2}\right]}{2 a}
\]

\[
\frac{S}{S^2 + a^2} = \frac{1}{2} T \csc\left[\frac{a T}{2}\right] \sin\left[a t - \frac{a T}{2}\right]
\]

\[
\frac{c S + d}{S^2 + a^2} = c \left(\frac{1}{a^2} - \frac{T \cos\left[a t - \frac{a T}{2}\right] \csc\left[\frac{a T}{2}\right]}{2 a}\right) + d \left(\frac{1}{2} T \csc\left[\frac{a T}{2}\right] \sin\left[a t - \frac{a T}{2}\right]\right)
\]

\[
\frac{1}{S^2 + p S + q} = \frac{1}{q} + \frac{T \left(\frac{e^{\frac{1}{\tau} t} (\pm \sqrt{p^2 - 4 q})}{\sqrt{p^2 - 4 q}} - \frac{e^{\frac{1}{\tau} t} (\pm \sqrt{p^2 - 4 q})}{\sqrt{p^2 - 4 q}}\right)}{\delta}
\]

For \(\sqrt{p^2 - 4 q} = \delta\), \(p + \delta = \alpha\), \(-p + \delta = \beta\):

\[
\frac{1}{S^2 + p S + q} = \frac{1}{q} + \frac{T \left(\frac{e^{\frac{1}{\tau} t} \alpha}{\sqrt{p^2 - 4 q}} + \frac{e^{\frac{1}{\tau} t} \beta}{\sqrt{p^2 - 4 q}}\right)}{\delta}
\]

\[
\frac{S}{S^2 + p S + q} = \frac{T}{\delta} \left(\frac{e^{\frac{1}{\tau} t} \alpha}{2 (1 + e^{\frac{1}{\tau} t})} + \frac{e^{\frac{1}{\tau} t} \beta}{2 (1 + e^{\frac{1}{\tau} t})}\right)
\]
For the resonance case:

\[
\frac{1}{s^2 + a^2} = \frac{1}{a^2} + \frac{t \sin(at)}{a} - \frac{T \sin[at]}{2a}
\]

\[
\frac{S}{s^2 + a^2} = t \cos[at] - \frac{1}{2} T \cos[at] + \frac{\sin[at]}{a}
\]

\[
\frac{cS + d}{s^2 + a^2} = c \left( \frac{1}{a^2} + \frac{t \sin[at]}{a} - \frac{T \sin[at]}{2a} \right) + d \left( t \cos[at] - \frac{1}{2} T \cos[at] + \frac{\sin[at]}{a} \right)
\]

\[
\frac{1}{s^2 + pS + q} = \frac{e^{-\frac{t}{T} (p+\delta)}}{2q \delta} \left( e^{\delta q (2t-T)} + q (-2t+T) + 2 e^{\frac{t}{T} (p+\delta)} \right), \quad p^2 - 4q \neq 0
\]

where \( B[m, \frac{t}{T}] \) is the polynomial of Bernoulli of degree \( m \).
Mean-periodic solutions of LODE with constant coefficients

Let \( P(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1} \lambda + a_n \) be a non-zero polynomial with constant coefficients of degree \( n \) and let us consider an ordinary linear differential equation of the form:

\[
P \left( \frac{d}{dt} \right) y = f(t), \quad -\infty < t < \infty
\]

(1)

Let \( \Phi \) be a linear functional on \( C(\mathbb{R}) \). We are looking for solutions of (1) which satisfy the relation

\[
\Phi \{ y(t + \tau) \} = 0
\]

(2)

for all \( t \in \mathbb{R} \), i.e. for mean-periodic solutions of (1) with respect to the functional \( \Phi \).
Definition 1  The boundary value problem

\[
P \left( \frac{d}{dt} \right) y = f(t), \quad \Phi\{y^{(k)}\} = \alpha_k, \quad k = 0, 1, \ldots, n - 1, \quad f \in C(\Delta) \quad (3)
\]

is said to be a non-local Cauchy problem, associated with the functional \( \Phi \).

For technical convenience it is preferable to consider homogeneous problems, i.e. problems of the form

\[
P(D) u = f, \quad \Phi\{u^{(k)}\} = 0, \quad k = 0, 1, \ldots, \deg P - 1.
\]
The simplest non-local Cauchy problem for a polynomial $P$ of first degree is: \( \frac{dy}{dt} - \lambda y = f(t) \), $\Phi\{y\} = 0$. Its solution $y = R_\lambda f$ is the resolvent operator

$$R_\lambda f = \frac{e^{\lambda t}}{\Phi\{e^{\lambda \tau}\}} \Phi_\tau \left\{ \int_{\tau}^{t} e^{\lambda(\tau - \sigma)} f(\sigma) d\sigma \right\}$$

of the differentiation operator $D = \frac{d}{dt}$, with the boundary value condition $\Phi\{y\} = 0$.

Further, we denote $E(\lambda) = \Phi_\tau\{e^{\lambda \tau}\}$. This is an entire function of exponential type, called the indicatrix of the functional $\Phi$ (or the exponential indicatrix).
**Definition 2** Let \( \Phi \in [C(\mathbb{R})]^* \) be a given linear functional on the space of the continuous functions on the real line. A function \( f \in C(\mathbb{R}) \) is said to be mean-periodic with respect to the functional \( \Phi \) if

\[
\Phi_{\tau}\{f(t + \tau)\} = 0 \text{ for } t \in \mathbb{R}.
\]

The periodic functions with a period \( T > 0 \) are mean-periodic with respect to the functional

\[
\Phi\{f\} = f(T) - f(0).
\]

The antiperiodic functions with an antiperiod \( T > 0 \), i.e. the functions, satisfying the functional equation \( f(T + t) = -f(t) \), are mean-periodic functions with respect to the linear functional

\[
\Phi\{f\} = \frac{1}{2}\{f(0) + f(T)\}. \]
Further we denote by $MP_{\Phi}$ the space of the mean-periodic functions, determined by the functional $\Phi$.

A necessary condition for a solution of the LODE $P \left( \frac{d}{dt} \right) u = f$ to be mean-periodic is $f$ to be mean-periodic.

**Lemma 1** Let $f \in MP_{\Phi}$. If $u$ is a solution of the nonlocal Cauchy problem

$$P(D)u = f, \Phi \{u^{(k)}\} = 0, k = 0, 1, ..., degP - 1,$$

then $u \in MP_{\Phi}$.

**Example 1.** A function $u \in C(\mathbb{R})$ is a $T$-periodic solution of LODE $P(D)u = f$ with $n = degP$ and $T$-periodic $f$, iff it is solution of the nonlocal Cauchy problem

$$P(D)u = f, \ u^{(k)}(T) - u^{(k)}(0) = 0, k = 0, 1, ..., n - 1.$$

**Example 2.** $u$ is $T$-antiperiodic solution of $P(D)u = f$, where $f$ is antiperiodic, iff $P(D)u = f, \ u^{(k)}(T) + u^{(k)}(0) = 0, \ k = 0, 1, ..., n - 1.$
Mikusinški Type Operational Calculi for Solving Nonlocal Cauchy Problems.

The Mikusinški operational calculus is based on the Duhamel convolution

$$(f * g)(t) = \int_{0}^{t} f(t - \tau) g(\tau) d\tau.$$  

A Mikusinški type operational calculus, based on the convolution:

$$(f * g)(t) = \Phi_{\tau} \left\{ \int_{\tau}^{t} f(t + \tau - \sigma) g(\sigma) d\sigma \right\}$$  

(4)

with an arbitrary linear functional $\Phi$ in $C = C'(\Delta)$ with $\Phi\{1\} = 1$
is developed by I. Dimovski.
Consider the right inverse operator \( L : C \rightarrow C^1 \) of \( \frac{d}{dt} \), defined by the requirement \( \Phi\{Lf\} = 0 \) for \( f \in C \). It has the following explicit representation:

\[
Lf(t) = \int_0^t f(\tau)d\tau - \Phi_\tau \left\{ \int_0^\tau f(\sigma)d\sigma \right\}.
\]

This is the convolution operator \( \{1\}* \), i.e.

\[
Lf = \{1\} * f.
\]

Next, we consider the space \( C = C(\Delta) \) as a ring with multiplication, defined by (4). In this ring the operator \( L \) may be considered as the constant function \( \{1\} \), i.e. \( L = \{1\} \). Denote the ring by \( (C,*) \).

For the possibility to develop an operational calculus, based on the convolution (4), it is important that the set \( N \) of its nonzero non-divisors of 0 is not empty.
The next step consists in introducing convolution fractions of the form \( \frac{f}{g} \) with \( f \in C, \ g \in N \). This could be done in a standard way: each fraction \( \frac{f}{g} \) is considered as an equivalence class in \( C \times N \) using the equivalence relation \((f, g) \sim (f_1, g_1) \iff f \ast g_1 = f_1 \ast g\). The ring \( M = (C \times N)/\sim \) of the convolution fractions is the basic algebraic system of our operational calculus.

A special role is played by the following element of \( M \):

\[
S = \frac{1}{L},
\]

which may be called operator of algebraic differentiation.
If $f \in C^1$, then not always $Sf = f'$. The connection between $Sf$ and $f'$ is given by the basic relation of the operational calculus:

**Theorem 1** If $f \in C^1$, then

$$f' = Sf - \Phi\{f\},$$

where $\Phi\{f\}$ is considered as a numerical operator.

**Corollary.** If $f \in C^n$, then

$$f^{(n)} = S^n f - S^{n-1} \Phi\{f\} - S^{n-2} \Phi\{f'\} - \ldots - \Phi\{f^{(n-1)}\}.$$ 

These identities allow each non-local Cauchy problem to be algebraized.
Further we consider the elements of $\mathcal{M}$ of the form $P(S)$, where $P$ is a polynomial.

**Theorem 2** $P(S)$ is a non-divisor of 0 in $\mathcal{M}$ iff $\{\lambda : P(\lambda) = 0\} \cap \{\lambda : E(\lambda) = 0\} = \emptyset$, i.e. when no one of the roots of $P(\lambda) = 0$ is an eigenvalue of the non-local eigenvalue problem $y' - \lambda y = 0$, $\Phi\{y\} = 0$.

**Lemma 2** Let $\lambda \in \mathbb{C}$ be such that $E(\lambda) \neq 0$. Then

$$\frac{1}{S - \lambda} = \left\{ \frac{e^{\lambda t}}{E(\lambda)} \right\}.$$

**Corollary.** For arbitrary $n \in \mathbb{N}$ it holds:

$$\frac{1}{(S - \lambda)^n} = \frac{1}{(n - 1)!} \frac{\partial^{n-1}}{\partial \lambda^{n-1}} \left( \frac{e^{\lambda t}}{E(\lambda)} \right).$$
Operational Calculus Approach to Nonlocal Cauchy Boundary Value Problems.

Let us consider the general non-local Cauchy problem (3), determined by a given linear functional \( \Phi \) in \( C = C(\Delta) \). We considered above a developed by I. Dimovski operational calculus for the differentiation operator \( \frac{d}{dt} \), associated with \( \Phi \). We assumed that \( \Phi\{1\} \neq 0 \), or, which is the same, that \( \lambda = 0 \) is not a zero of the exponential indicatrix \( E(\lambda) = \Phi_\tau\{e^{\lambda\tau}\} \) of the functional \( \Phi \).

Now we assume that \( \Phi\{1\} = E(0) = 1 \). The solution of the simplest non-local Cauchy problem for \( \frac{d}{dt} \):

\[
\frac{dy}{dt} - \lambda y = f(t), \ t \in \Delta, \ \Phi\{y\} = 0
\]

is

\[
y = \frac{1}{S - \lambda} f = \left\{ \frac{e^{\lambda t}}{E(\lambda)} \right\} * f, \ \text{when} E(\lambda) \neq 0.
\]
Consider the general problem (3). Using the Corollary of Theorem 2 we get:

\[ y^{(k)} = S^k y - \sum_{j=0}^{k-1} \Phi\{y^{(j)}\} S^{k-j-1} = S^k y - \sum_{j=0}^{k-1} \alpha_j S^{k-j-1} \quad (k = 1, 2, \ldots, n). \]

After substitution in the differential equation, we get

\[ P(S)y - \sum_{j=0}^{n} \sum_{k=0}^{k-j-1} a_{n-k} \alpha_k S^{n-k-1} = f \]

or

\[ P(S)y = f + Q(S), \]

where \( Q(S) \) is a polynomial of \( S \) with \( \deg Q < \deg P \). The formal solution is

\[ y = \frac{1}{P(S)} f + \frac{Q(S)}{P(S)}, \quad \text{if } P(S) \text{ is a non-divisor of zero in } \mathcal{M}. \]
Definition 3  The non-local Cauchy boundary value problem (3) is said to be non-resonance when $P(\lambda)$ and $E(\lambda)$ have no common zeroes. In the case when $P(\lambda)$ and $E(\lambda)$ have common zeroes, the problem is called resonance.

In the non-resonance case the solution of (3) may be obtained by the Heaviside algorithm.

Let us consider the general Cauchy problem

$$P \left( \frac{d}{dt} \right) y = f(t), \quad \Phi\{y^{(k)}\} = \alpha_k, \quad k = 0, 1, \ldots, \deg P - 1.$$  

We assume that $\Phi\{1\} = 1$. Using a set of Appell polynomials, the general problem can be reduced to a problem with homogeneous “initial” conditions.

Therefore, without any restriction of the generality we may assume that the general nonlocal Cauchy problem has homogeneous initial conditions. The algebraic equivalent of the general problem with homogeneous “initial” conditions reduces to the single algebraic equation

$$P(S)y = f \quad (5)$$

in the ring $\mathcal{M}$ of the convolution fractions.
**Definition 4** If $P(S)$ is a divisor of zero in $\mathcal{M}$, then each solution of the nonlocal Cauchy problem (3) is said to be a resonance solution.

We have characterized above the divisors of zero of $P(S)$ in $\mathcal{M}$: $P(S)$ is a divisor of zero iff $P(\lambda)$ and $E(\lambda)$ have common zeroes, i.e. iff \[ \{\lambda : P(\lambda) = 0\} \cap \{\lambda : E(\lambda) = 0\} \neq \emptyset. \]

Let $\mu_1, \mu_2, ..., \mu_l$ be the common zeroes with corresponding multiplicities $\kappa_1, \kappa_2, ..., \kappa_l$ and $\sigma_1, \sigma_2, ..., \sigma_l$ (for $P(\lambda)$ and $E(\lambda)$, respectively). For simplicity sake here we assume that $\sigma_1 = \sigma_2 = ... = \sigma_l = 1$, i.e. that the zeroes of $E(\lambda)$, common with zeros of $P(\lambda)$, are simple zeroes of $E(\lambda)$. Then the exponentials $e^{\mu_1 t}, e^{\mu_2 t}, ..., e^{\mu_l t}$ are solutions of (5) with $f \equiv 0$, i.e. $P(S)\{e^{\mu_j t}\} = 0$, $j = 1, 2, ..., l$, hence $f \ast \{e^{\mu_j t}\} = 0$, $j = 1, 2, ..., l$ are necessary conditions for the existence of solution of (5). These conditions are sufficient too.
Now we are looking for solution of (5) not in the whole space \( C \) but in its subspace:

\[
\tilde{C} = C_{\mu_1, \mu_2, \ldots, \mu_l} = \{ f \in C : f \ast e^{\mu_j t} = 0, \ j = 1, 2, \ldots, l \}.
\]

Similar considerations for the case of resonance periodic solutions were made in the first part of this presentation.

**Lemma 3** \( \tilde{C} = C_{\mu_1, \mu_2, \ldots, \mu_l} \) is an ideal in \( (C, \ast) \).

Denote by \( \tilde{L} \) the restriction of \( L \) to \( \tilde{C} = C_{\mu_1, \mu_2, \ldots, \mu_l} \). Let \( \tilde{M} \) be the ring of the convolution fractions of \( \tilde{C} \). We denote the unit of \( \tilde{M} \) by 1, as the unit of \( M \). The convolution operators \( f \ast \) with \( f \in C \) can be considered as inner operators in \( \tilde{C} = C_{\mu_1, \mu_2, \ldots, \mu_l} \). Further, we will use this notation freely, understanding \( f \ast \) as the operator \( (f \ast)g = f \ast g \).

**Lemma 4** \( P(\tilde{S}) \) is not a divisor of zero in \( \tilde{M} \).
Theorem 3 Let $\mu \in C$ be such that $E(\mu) = 0$, but $E'(\mu) \neq 0$ and $
abla C = \{ f \in C : f \ast e^{\mu t} = 0 \}$. Then

$$\frac{1}{(\nabla S - \mu)^k} = \{ e^{\mu t} A_k(t) \}^\ast, \quad k \in \mathbb{N},$$

(6)

where $A_k(t)$ is a polynomial of degree $k$. The polynomials $A_k(t)$ form an Appell set of polynomials $\{ A_k(t) \}_{k=1}^\infty$, determined by $A_1(t) = \frac{1}{E'(\lambda_j)} t$, $A'_k(t) = A_{k-1}(t)$, $\Phi_\tau \{ e^{\mu \tau} A_k(\tau) \} = 0$, $k = 2, 3, \ldots$.

Theorem 1 and Theorem 3 give interpretation formulae for practical application of the considered approach.

Some examples were considered. It was convenient to perform formulae derivations as well as computation of the explicit solutions of some more involved equations using a computer algebra system (Mathematica, in our case).
A part of the derived interpretation formulae

\[
\begin{align*}
\frac{1}{S - \mu} &= \frac{e^{t_\mu T_\mu}}{-1 + e^{T_\mu}} \\
\frac{1}{S + \mu} &= -\frac{e^{-t_\mu T_\mu}}{-1 + e^{-T_\mu}} \\
\frac{S}{S - \mu} &= \frac{e^{t_\mu T_\mu^2}}{-1 + e^{T_\mu}} \\
\frac{S}{S + \mu} &= \frac{e^{-t_\mu T_\mu^2}}{-1 + e^{-T_\mu}} \\
\frac{S}{(S - \mu)^k} &= \frac{\partial_{\{\mu, -1-k\}} \frac{e^{t_\mu T_\mu^2}}{-1 + e^{T_\mu}}}{(-1 - k)!} \\
\frac{S}{(S + \mu)^k} &= \frac{\partial_{\{-\mu, -1-k\}} \frac{e^{-t_\mu T_\mu^2}}{-1 + e^{-T_\mu}}}{(-1 - k)!} \\
\frac{1}{S^2 - \mu^2} &= \frac{1}{2} T \mathrm{csch}\left[\frac{T \mu}{2}\right] \sinh\left[t \mu - \frac{T \mu}{2}\right] \\
\frac{1}{S^2 + \mu^2} &= \frac{1}{2} T \csc\left[\frac{T \mu}{2}\right] \sin\left[t \mu - \frac{T \mu}{2}\right] \\
\frac{S}{S^2 + \mu^2} &= \frac{1}{2} T \mu \cos\left[t \mu - \frac{T \mu}{2}\right] \csc\left[\frac{T \mu}{2}\right]
\end{align*}
\]
Some conclusions

– In the classical methods for finding periodic solutions, originally the general solution is found and after that the periodicity conditions are used for determining the unknown constants in it. In our approach the periodicity conditions are taken into account at the level of the algebraization of the problem.

– In case of use the Laplace transformation for finding periodic solutions, the existence of Laplace transform of the right-hand side of the equation is needed.

– We find that the presented approach is more efficient (especially in the resonance cases) than the well known (and published) approaches.

– We suggest a more general approach for the case of mean-periodic solutions.

– All proposed algorithms are convenient for use in the environment of a computer algebra system.
Thank you for your attention.