4TH CHANNEL NETWORK CONFERENCE

St Andrews | Scotland | 3rd July – 5th July 2013

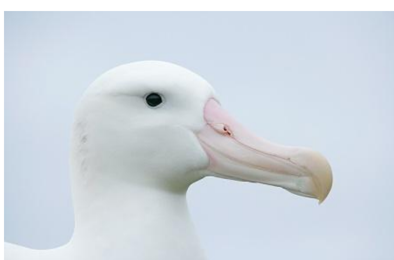
INTERNATIONAL BIOMETRIC SOCIETY

BRITISH AND IRISH REGION

Parameter Redundancy Workshop

Diana Cole and Ben Hubbard

(NCSE, University of Kent)



ncse

National Centre for Statistical Ecology

University of Kent

Workshop Outline

- 9.00 – 9.40 Workshop Part I
  - Introductory Example
  - Definitions and practical implications
  - Symbolic Methods for detecting PR
  - Examples I
- 9.40 – 10.30 Maple Practical I
- 10.30 – 11.00 Tea and Coffee Break
- 11.00 – 11.30 Workshop Part II
  - Problems with Complex Models
  - Hybrid-symbolic Numeric Method
  - Extended Symbolic Method
  - Examples II
- 11.30 – 12.00 Maple Practical II

Introductory Example – CJS Model

- In capture-recapture studies animals are captured, marked and then recaptured again.

Introductory Example – CJS Model

- Herring Gulls (*Larus argentatus*) capture-recapture data for 1983 to 1986 (Lebreton, *et al* 1995):

$$\boldsymbol{R} = \begin{bmatrix} 78 \\ 123 \\ 111 \end{bmatrix}, \boldsymbol{N} = \begin{bmatrix} 67 & 4 & 2 \\ 0 & 103 & 3 \\ 0 & 0 & 91 \end{bmatrix}$$

- Cormark, Jolly Seber (CJS) model for capture-recapture data.
- ϕ_i – probability animals survives from occasion i to occasion i +1.
- p_i – probability an animal is recaptured on occasion i.
- Probabilities of being captured in year i then next captured in year j:

$$\boldsymbol{P} = \begin{bmatrix} \phi_1 p_2 & \phi_1 \bar{p}_2 \phi_2 p_3 & \phi_1 \bar{p}_2 \phi_2 \bar{p}_3 \phi_3 p_4 \\ 0 & \phi_2 p_3 & \phi_2 \bar{p}_3 \phi_3 p_4 \\ 0 & 0 & \phi_3 p_4 \end{bmatrix}$$

- Parameters ϕ_3 and p_4 are confounded, will only ever be able to estimate $\phi_3 \times p_4$.
- Model is parameter redundant.

Definitions

- Let $M(\boldsymbol{\theta})$ be the function that defines a model, which has unknown parameters $\boldsymbol{\theta}$.
- **Parameter redundancy:** A model is parameter redundant if we can write $M(\boldsymbol{\theta})$ as a function just of $\boldsymbol{\beta}$, where $\boldsymbol{\beta} = f(\boldsymbol{\theta})$ and dimension $\boldsymbol{\beta} <$ dimension $\boldsymbol{\theta}$.
- **Identifiability:** A model is globally identifiable if $M(\boldsymbol{\theta}_1) = M(\boldsymbol{\theta}_2)$ implies that $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$. A model is locally identifiable if there exists an open neighbourhood of any $\boldsymbol{\theta}$ such that this is true. Otherwise a model is non-identifiable.
- A parameter redundant model will be non-identifiable. A model that is not parameter redundant will be at least locally identifiable.
- The Bayesian equivalent of parameter redundancy is weak identifiability. Weak identifiability is measured by an overlap of the prior and posterior distributions (Garrett and Zeger, 2000, Gimenez et al 2009).

Practical Implications

- A parameter redundant model will have a ridge in the likelihood, so that there can be more than one maximum likelihood estimate.
- In a parameter redundant model the Hessian matrix and information will be singular, so that the standard errors for parameter estimates do not exist.
- In reality when numerical methods are used to obtain standard errors, the approximate Hessian / information matrix could be non-singular and the standard errors may be just very large.
- It is better to check for parameter redundancy before fitting a model.
- To fit a parameter redundant model a constraint is needed.

Definitions

- **Exhaustive Summary:** A parameter vector $\boldsymbol{\kappa}(\boldsymbol{\theta})$ is an exhaustive summary if knowledge of $\boldsymbol{\kappa}(\boldsymbol{\theta})$ uniquely determines $M(\boldsymbol{\theta})$.
- Examples of exhaustive summaries:
  - The means of an exponential family model (Catchpole and Morgan, 1997);
  - The log-likelihood terms $\kappa_i = l_i, l = \sum_{i=1}^{n} l_i$ (Cole et al, 2010);
  - The capture probabilities, **P** (Catchpole and Morgan, 1997).

Symbolic Method
Checking whether a model is PR

- Method has a long history given in Cole *et al*, (2010).
- Start with an exhaustive summary, **κ**, of length n, for a model with q parameters given by vector, **θ**.
- Form a derivative matrix:

$$\boldsymbol{D} = \frac{\partial \boldsymbol{\kappa}}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial \kappa_1}{\partial \theta_1} & \frac{\partial \kappa_2}{\partial \theta_1} & \cdots & \frac{\partial \kappa_n}{\partial \theta_1} \\ \frac{\partial \kappa_1}{\partial \theta_2} & \frac{\partial \kappa_2}{\partial \theta_2} & & \frac{\partial \kappa_n}{\partial \theta_2} \\ \vdots & & \ddots & \vdots \\ \frac{\partial \kappa_1}{\partial \theta_q} & \frac{\partial \kappa_2}{\partial \theta_q} & \cdots & \frac{\partial \kappa_n}{\partial \theta_q} \end{bmatrix}$$

- r = Rank($\boldsymbol{D}$) is the number of estimable parameters or parameter combinations in a model.
- $d = q - r$ is the deficiency of the model (how many parameters you cannot estimate). If $d = 0$ model is full rank (not parameter redundant). If $d > 0$ model is parameter redundant.

Symbolic Method
Maple code

- Maple procedure `Dmat(kappa,pars)` finds the derivative matrix for a vector of exhaustive summary terms stored in the vector `kappa` and a vector of parameters stored in the vector `pars`.
- `Rank(D1)` is the intrinsic Maple procedure that finds the rank of a matrix `D1`.

```
> D1 := Dmat(kappa, pars) :
> r := Rank(D1); q := Dimension(pars); d := Dimension(pars) − r;
                                        r := 7
                                        q := 7
                                        d := 0
```

- This model has 7 parameters, rank 7 and deficiency 0, so it is not parameter redundant.

```
> D1 := Dmat(kappa, pars) :
> r := Rank(D1); q := Dimension(pars); d := Dimension(pars) − r;
                                        r := 11
                                        q := 13
                                        d := 2
```

- This model has 13 parameters, rank 13 and deficiency 2, so it is parameter redundant.

Symbolic Method
CJS Example

$$\boldsymbol{P} = \begin{bmatrix} \phi_1 p_2 & \phi_1 \bar{p}_2 \phi_2 p_3 & \phi_1 \bar{p}_2 \phi_2 \bar{p}_3 \phi_3 p_4 \\ 0 & \phi_2 p_3 & \phi_2 \bar{p}_3 \phi_3 p_4 \\ 0 & 0 & \phi_3 p_4 \end{bmatrix} \qquad \bar{p}_i = 1 - p_i$$

$$\boldsymbol{\kappa}^T = [\phi_1 p_2, \phi_1 \bar{p}_2 \phi_2 p_3, \phi_1 \bar{p}_2 \phi_2 \bar{p}_3 \phi_3 p_4, \phi_2 p_3, \phi_2 \bar{p}_3 \phi_3 p_4, \phi_3 p_4]$$

$$\boldsymbol{\theta} = [\phi_1 \quad \phi_2 \quad \phi_3 \quad p_2 \quad p_3 \quad p_4]$$

$$\mathbf{D} = \begin{bmatrix} p_2 & \phi_2 \bar{p}_2 p_3 & \phi_2 \phi_3 \bar{p}_2 \bar{p}_3 p_4 & 0 & 0 & 0 \\ 0 & \phi_1 \bar{p}_2 p_3 & \phi_1 \phi_3 \bar{p}_2 \bar{p}_3 p_4 & p_3 & \phi_3 \bar{p}_3 p_4 & 0 \\ 0 & 0 & \phi_1 \phi_2 \bar{p}_2 \bar{p}_3 p_4 & 0 & \phi_2 \bar{p}_3 p_4 & p_4 \\ \phi_1 & -\phi_1 \phi_2 p_3 & -\phi_1 \phi_2 \phi_3 \bar{p}_3 p_4 & 0 & 0 & 0 \\ 0 & \phi_1 \phi_2 \bar{p}_2 & -\phi_1 \phi_2 \phi_3 \bar{p}_2 p_4 & \phi_2 & -\phi_2 \phi_3 p_4 & 0 \\ 0 & 0 & \phi_1 \phi_2 \phi_3 \bar{p}_2 \bar{p}_3 & 0 & \phi_2 \phi_3 \bar{p}_3 & \phi_3 \end{bmatrix}$$

$$r = \text{Rank}(\boldsymbol{D}) = 5$$

$$d = q - r = 6 - 5 = 1$$

Symbolic Method
Parameter Redundant Models

- Consider a model with q parameters, rank r, deficiency $d = p - r > 0$
- There will be d non-zero solutions to $\boldsymbol{\alpha}^T\mathbf{D} = 0$.
- Zeros in $\boldsymbol{\alpha}$s indicate estimable parameters.
- Solve PDEs to find full set of estimable pars.

$$\sum_{i=1}^{q} \alpha_{ij} \frac{\partial f}{\partial \theta_i} = 0, \qquad j = 1, \ldots, d$$

- Maple procedure `Estpars(D1,pars)` will solve the PDEs and find estimable parameter combinations for derivative matrix stored in `D1` found using parameter stored in the vector `pars`.
- Maple procedure `Estpars2(D1,pars)` also returns $\boldsymbol{\alpha}$ and PDEs.

$$> \; Estpars(D1, pars) \qquad \{f(\phi_1, \phi_2, \phi_3, p_2, p_3, p_4) = \_F1(\phi_1, \phi_2, p_2, p_3, \phi_3 p_4)\}$$

Symbolic Method
CJS Example

$$\boldsymbol{\theta} = [\phi_1 \quad \phi_2 \quad \phi_3 \quad p_2 \quad p_3 \quad p_4]$$

$$r = \text{Rank}(\boldsymbol{D}) = 5$$

$$d = q - r = 6 - 5 = 1$$

$$\alpha^T = \begin{bmatrix} 0 & 0 & -\dfrac{\phi_3}{p_4} & 0 & 0 & 1 \end{bmatrix}$$

$$-\frac{\phi_3}{p_4} \times \frac{\partial f}{\partial \phi_3} + \frac{\partial f}{\partial p_4} = 0$$

Can estimate: ϕ_1, ϕ_2, p_2, p_3 and $\phi_3 p_4$

Symbolic Method
Full Rank Models

- Mallard (*Larus argentatus*) mark-recovery data for 1963 to 1965 (Brownie et al., 1985):

$$\boldsymbol{R} = \begin{bmatrix} 962 \\ 702 \\ 1132 \end{bmatrix}, \boldsymbol{N} = \begin{bmatrix} 82 & 35 & 18 \\ 0 & 103 & 21 \\ 0 & 0 & 82 \end{bmatrix}$$

- Mark-recovery model with 1st year surival dependent on time, constant adult survival and 2 age classes for reporting probability: $\boldsymbol{\theta} = [\phi_{1,1} \quad \phi_{1,2} \quad \phi_{1,3} \quad \phi_a \quad \lambda_1 \quad \lambda_a]$

$$\boldsymbol{P} = \begin{bmatrix} \bar{\phi}_{1,1}\lambda_1 & \phi_{1,1}\bar{\phi}_a\lambda_a & \phi_{1,1}\phi_a\bar{\phi}_a\lambda_a \\ 0 & \bar{\phi}_{1,2}\lambda_1 & \phi_{1,2}\bar{\phi}_a\lambda_a \\ 0 & 0 & \bar{\phi}_{1,3}\lambda_1 \end{bmatrix}$$

- Model is not parameter redundant. However the nested model with $\phi_{1,1} = \phi_{1,2} = \phi_{1,3}$ is parameter redundant, deficiency = 1.
- This information is in the first model's derivative matrix.

Symbolic Method
Full Rank Models

- Modified PLUR decomposition (or Turing factorisation) to determine whether a full rank model is always full rank. (Cole et al, 2010).
- Write derivative matrix which is full rank r as **D** = **PLUR.**
- **P** is a square permutation matrix .
- **L** is a lower diagonal square matrix, with 1's on the diagonal.
- **U** is an upper triangular square matrix (any entry on the diagonal).
- **R** is a matrix (size of **D**) in reduced echelon form.
- If Det(**U**) = 0 at any point, model is parameter redundant at that point (as long as **R** is defined). The deficiency of **U** evaluated at that point is the deficiency of that nested model (Cole et al, 2010).

Symbolic Method
Full Rank Models

- Mark-recovery example:

$$\boldsymbol{D} = \begin{bmatrix} -\lambda_1 & \bar{\phi}_a\lambda_a & \phi_a\bar{\phi}_a\lambda_a & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_1 & \bar{\phi}_a\lambda_a & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda_1 \\ 0 & -\phi_{1,1}\lambda_a & (1-2\phi_a)\phi_{1,1}\lambda_a & 0 & -\phi_{1,2}\lambda_a & 0 \\ \bar{\phi}_{1,1} & 0 & 0 & \bar{\phi}_{1,2} & 0 & \bar{\phi}_{1,3} \\ 0 & \phi_{1,1}\bar{\phi}_a & \phi_{1,1}\phi_a\bar{\phi}_a & 0 & \phi_{1,2}\bar{\phi}_a & 0 \end{bmatrix}$$

$$\text{Rank}(\boldsymbol{D}) = 6$$

$$\text{Det}(\boldsymbol{U}) = \phi_{1,1}\lambda_1\lambda_a^2(1-\phi_a)^3(\phi_{1,1}-\phi_{1,2})$$

$$\text{Rank}\left(\boldsymbol{U}_{\phi_{1,1}=\phi_{1,2}=\phi_{1,3}}\right) = 5$$

- As $\phi_{1,1} = \phi_{1,2}$ $(= \phi_{1,3})$ results in Det($\mathbf{U}$) = 0 the nested model with $\phi_{1,1} = \phi_{1,2} = \phi_{1,3}$ is parameter redundant with deficiency 1.

Symbolic Method
Full Rank Models

- Internal Maple procedure `LUDecomposition(D1,output=['P','L','U1','R'])` performs a PLUR decomposition on `D1`.
- Internal Maple procedure `Determinant(u1)` finds the determinate of matrix `u1`.

```
> (pp, ll, u1, r1) := LUDecomposition( D1, output = ['P','L','U1','R'] ) :
  pp := pp;
  ll := ll;
  u1 := u1;
  r1 := r1;
  DetU := Determinant(u1);
```

Symbolic Method
General Results

- Extension theorem (Catchpole and Morgan, 1997, Cole et al 2010).
- Suppose a model has exhaustive summary $\boldsymbol{\kappa}_1$ and parameters $\boldsymbol{\theta}_1$.

$$\boldsymbol{D}_1 = \left[\frac{\partial \boldsymbol{\kappa}_1}{\partial \boldsymbol{\theta}_1}\right]$$

- Now extend that model by adding extra exhaustive summary terms $\boldsymbol{\kappa}_2$, and extra parameters $\boldsymbol{\theta}_2$ (e.g. add more years of data). New model's exhaustive summary is $\boldsymbol{\kappa} = [\boldsymbol{\kappa}_1^T\ \boldsymbol{\kappa}_2^T]^T$ and parameters are $\boldsymbol{\theta} = [\boldsymbol{\theta}_1\ \boldsymbol{\theta}_2]$.

$$\boldsymbol{D} = \begin{bmatrix} \frac{\partial \boldsymbol{\kappa}_1}{\partial \boldsymbol{\theta}_1} & \frac{\partial \boldsymbol{\kappa}_2}{\partial \boldsymbol{\theta}_1} \\ \boldsymbol{0} & \frac{\partial \boldsymbol{\kappa}_2}{\partial \boldsymbol{\theta}_2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{D}_1 & \frac{\partial \boldsymbol{\kappa}_2}{\partial \boldsymbol{\theta}_1} \\ \boldsymbol{0} & \boldsymbol{D}_2 \end{bmatrix}$$

- If $\mathbf{D}_1$ is full rank and $\mathbf{D}_2$ is full rank, the extended model will be full rank. The result can be further generalised by induction.
- Result is trivially always true, if you add zero or one extra parameter.
- Method can also be used for parameter redundant models by first rewriting the model in terms of its estimable set of parameters.

Symbolic Method
General Results

- Ring-recovery example with 1st year survival dependent on time, adult survival dependent on age, and the reporting probability dependent on time.
- Probability matrix for $n = 5$ years of marking and recovery:

$$\boldsymbol{P} = \begin{bmatrix} \bar{\phi}_{1,1}\lambda_1 & \phi_{1,1}\bar{\phi}_2\lambda_2 & \phi_{1,1}\phi_2\bar{\phi}_3\lambda_3 & \phi_{1,1}\phi_2\phi_3\bar{\phi}_4\lambda_4 & \phi_{1,1}\phi_2\phi_3\phi_4\bar{\phi}_5\lambda_5 \\ 0 & \bar{\phi}_{1,2}\lambda_2 & \phi_{1,2}\bar{\phi}_2\lambda_3 & \phi_{1,2}\phi_2\bar{\phi}_3\lambda_4 & \phi_{1,2}\phi_2\phi_3\bar{\phi}_4\lambda_5 \\ 0 & 0 & \bar{\phi}_{1,3}\lambda_3 & \phi_{1,3}\bar{\phi}_2\lambda_4 & \phi_{1,3}\phi_2\bar{\phi}_3\lambda_5 \\ 0 & 0 & 0 & \bar{\phi}_{1,4}\lambda_4 & \phi_{1,4}\bar{\phi}_2\lambda_5 \\ 0 & 0 & 0 & 0 & \bar{\phi}_{1,5}\lambda_5 \end{bmatrix}$$

$(\bar{\phi}_i = 1 - \phi_i)$

- 14 parameters: $\boldsymbol{\theta}_1 = [\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,5}, \phi_2, \dots, \phi_5, \lambda_1, \dots, \lambda_5]$
- The non-zero terms of $\boldsymbol{P}$ form $\boldsymbol{\kappa}_1$.
- Derivative matrix, $\dfrac{\partial \boldsymbol{\kappa}_1}{\partial \boldsymbol{\theta}_1}$, has full rank 14.

Symbolic Method
General Results

- Probability matrix for $n = 6$ years of marking and recovery:

$$\begin{bmatrix} \bar{\phi}_{1,1}\lambda_1 & \phi_{1,1}\bar{\phi}_2\lambda_2 & \phi_{1,1}\phi_2\bar{\phi}_3\lambda_3 & \phi_{1,1}\phi_2\phi_3\bar{\phi}_4\lambda_4 & \phi_{1,1}\phi_2\phi_3\phi_4\bar{\phi}_5\lambda_5 & \phi_{1,1}\phi_2\phi_3\phi_4\phi_5\bar{\phi}_6\lambda_6 \\ 0 & \bar{\phi}_{1,2}\lambda_2 & \phi_{1,2}\bar{\phi}_2\lambda_3 & \phi_{1,2}\phi_2\bar{\phi}_3\lambda_4 & \phi_{1,2}\phi_2\phi_3\bar{\phi}_4\lambda_5 & \phi_{1,2}\phi_2\phi_3\phi_4\bar{\phi}_5\lambda_6 \\ 0 & 0 & \bar{\phi}_{1,3}\lambda_3 & \phi_{1,3}\bar{\phi}_2\lambda_4 & \phi_{1,3}\phi_2\bar{\phi}_3\lambda_5 & \phi_{1,3}\phi_2\phi_3\bar{\phi}_4\lambda_6 \\ 0 & 0 & 0 & \bar{\phi}_{1,4}\lambda_4 & \phi_{1,4}\bar{\phi}_2\lambda_5 & \phi_{1,4}\phi_2\bar{\phi}_3\lambda_6 \\ 0 & 0 & 0 & 0 & \bar{\phi}_{1,5}\lambda_5 & \phi_{1,5}\bar{\phi}_2\lambda_6 \\ 0 & 0 & 0 & 0 & 0 & \bar{\phi}_{1,6}\lambda_6 \end{bmatrix}$$

$$\boldsymbol{\kappa}_2 = \begin{bmatrix} \phi_{1,1}\phi_2\phi_3\phi_4\phi_5\bar{\phi}_6\lambda_6 \\ \phi_{1,2}\phi_2\phi_3\phi_4\bar{\phi}_5\lambda_6 \\ \phi_{1,3}\phi_2\phi_3\bar{\phi}_4\lambda_6 \\ \phi_{1,4}\phi_2\bar{\phi}_3\lambda_6 \\ \phi_{1,5}\bar{\phi}_2\lambda_6 \\ \bar{\phi}_{1,6}\lambda_6 \end{bmatrix}$$

extra parameters: $\boldsymbol{\theta}_2 = [\phi_{1,6} \quad \phi_6 \quad \lambda_6]$

$\dfrac{\partial \boldsymbol{\kappa}_2}{\partial \boldsymbol{\theta}_2}$ has full rank 3.

- Therefore by the extension theorem the model has full rank $3n - 1$ for any $n \geq 5$.
- Further reading *Cole et al* (2012).

Examples I

Naïve Bayesian Network (Whiley and Titterington, 2002)

- The network consists of n observable binary nodes, $y_1,\ldots,y_n$ and a single unobservable binary node, z.
- p is the probability $z = 1$.
- $\pi_{i,j}$ is the probability $y_i = 1$ given $z = j$.
- The probability of an observation $\boldsymbol{y}$ is:

$$\Pr(\boldsymbol{y}) = p\prod_{i=1}^{n}\pi_{i,1}^{y_i}\bar{\pi}_{i,1}^{1-y_i} + \bar{p}\prod_{i=1}^{n}\pi_{i,0}^{y_i}\bar{\pi}_{i,0}^{1-y_i} \quad (\bar{x} = 1 - x)$$

- $n = 2$ $\quad \boldsymbol{\theta} = [p \quad \pi_{1,0} \quad \pi_{1,1} \quad \pi_{2,0} \quad \pi_{2,1}]$

$$\boldsymbol{\kappa} = \begin{bmatrix} p\bar{\pi}_{1,1}\bar{\pi}_{2,1} + \bar{p}\bar{\pi}_{1,0}\bar{\pi}_{2,0} \\ p\bar{\pi}_{1,1}\pi_{2,1} + \bar{p}\bar{\pi}_{1,0}\pi_{2,0} \\ p\pi_{1,1}\bar{\pi}_{2,1} + \bar{p}\pi_{1,0}\bar{\pi}_{2,0} \\ p\pi_{1,1}\pi_{2,1} + \bar{p}\pi_{1,0}\pi_{2,0} \end{bmatrix} \qquad \operatorname{rank}\left(\frac{\partial\boldsymbol{\kappa}}{\partial\boldsymbol{\theta}}\right) = 3$$

- Model is parameter redundant with deficiency 2.
- Is the model when $n = 3$ parameter redundant? Maple Practical I.

Example I

Bio-kinetic models of sludge respiration (Dochain et al, 1995)

- The activated sludge process can be modelled using a non-linear compartment model. For k pollutants S_i, the oxygen uptake is:

$$U = -\sum_{i=1}^{k} \overline{Y}_i \frac{dS_i(t)}{dt} \quad \text{with} \quad \frac{dS_i(t)}{dt} = -\frac{\mu_{\max i} X}{Y_i} \frac{S_i(t)}{K_{mi} + S_i(t)}$$

- An exhaustive summary for compartment models involves a Tayor series expansion of U (Pohjanpalo, 1978). The exhaustive summary terms are $U(0)$, $U^{(1)}(0)$, $U^{(2)}(0)$, $U^{(3)}(0)$, …
- $k = 1$: $\boldsymbol{\theta} = [Y_1 \quad S_1(0) \quad \mu_{\max 1} \quad K_{m1} \quad X]$

$$\boldsymbol{\kappa} = \begin{bmatrix} \frac{\overline{Y}_1 \mu_{\max 1} X S_1(0)}{Y_1 (K_{m1} + S_1(0))} \\ \frac{-\overline{Y}_1 \mu_{\max 1}^2 X^2 S_1(0) K_{m1}}{Y_1^2 (K_{m1} + S_1(0))^3} \\ \frac{\overline{Y}_1 \mu_{\max 1}^3 X^3 S_1(0) K_{m1} (K_{m1} - 2S_1(0))}{Y_1^3 (K_1 + S_1(0))^5} \\ \frac{-\overline{Y}_1 \mu_{\max 1}^2 X^4 S_1(0) K_{m1} (K_{m1}^2 - 8K_{m1} S_1(0) + 6S_1(0)^2)}{Y_1^4 (K_1 + S_1(0))^7} \\ \vdots \end{bmatrix}$$

$$\operatorname{rank}\left(\frac{\partial \boldsymbol{\kappa}}{\partial \boldsymbol{\theta}}\right) = 3$$

Model is parameter redundant with deficiency 2.

9.40 – 10.30   Maple Practical I

10.30 – 11.00   Tea and Coffee Break

Workshop Outline

- 9.00 – 9.40 Workshop Part I
  - Introductory Example
  - Definitions and practical implications
  - Symbolic Methods for detecting PR
  - Examples I
- 9.40 – 10.30 Maple Practical I
- 10.30 – 11.00 Tea and Coffee Break
- 11.00 – 11.30 Workshop Part II
  - Problems with Complex Models
  - Hybrid-symbolic Numeric Method
  - Extended Symbolic Method
  - Examples II
- 11.30 – 12.00 Maple Practical II

Problems with Complex Models

- The key to the symbolic method for detecting parameter redundancy is to find a derivative matrix and its rank.
- Models are getting more complex.
- The derivative matrix is therefore structurally more complex.
- Maple runs out of memory calculating the rank.
- Examples: Hunter and Caswell (2009), Jiang *et al* (2007)

Wandering Albatross

Multi-state models for sea birds

Striped Sea Bass

Age-dependent tag-return models for fish

- How do you proceed?
  - Numerically – can be inaccurate and give wrong results (eg Jiang et al, 2007 corrected in Cole and Morgan, 2010a).
  - Symbolically – involves extending the theory. Again it involves a derivative matrix and its rank, but the derivative matrix is structurally simpler.
  - Hybrid-Symbolic method – combination of both methods.

Hybrid-Symbolic Numeric Method

- Derivative matrix evaluated symbolically, rank is determined at 5 random points. The model rank is equal to the maximum rank of the 5 points (Choquet and Cole, 2012).
- Can also determine which parameters can be estimated in parameter redundant models, but not the estimable parameter combinations.
- Maple procedure `Formnum2(D1,pars)` performs the hybrid method on derivative matrix D1 with parameters `pars`. `Formnum(D1,pars)` just gives the rank and deficiency.
- CJS example:

$$\boldsymbol{\theta} = [\phi_1 \quad \phi_2 \quad \phi_3 \quad p_2 \quad p_3 \quad p_4]$$

$$\left[\begin{array}{ccc} 5 & 1 & \left\{\left[\begin{array}{c} 1.35020038119425\ 10^{-14} \\ -2.05477459444842\ 10^{-11} \\ -0.0266029321290482 \\ -2.73025433657781\ 10^{-14} \\ 4.54859290147031\ 10^{-11} \\ 0.999646079371163 \end{array}\right]\right\} \\ 5 & 1 & \left\{\left[\begin{array}{c} -5.30449660836035\ 10^{-13} \\ 1.86679928663388\ 10^{-11} \\ -0.467520481293457 \\ 8.93700630009506\ 10^{-13} \\ 1.44316265719187\ 10^{-11} \end{array}\right.\right. \end{array}\right.$$

Extended Symbolic Method

Finding simpler exhaustive summaries

Cole *et al* (2010)

1. Choose a reparameterisation, **s**, that simplifies the model structure.

CJS Model (revisited):

$$\boldsymbol{P} = \begin{bmatrix} \phi_1 p_2 & \phi_1 \bar{p}_2 \phi_2 p_3 & \phi_1 \bar{p}_2 \phi_2 \bar{p}_3 \phi_3 p_4 \\ 0 & \phi_2 p_3 & \phi_2 \bar{p}_3 \phi_3 p_4 \\ 0 & 0 & \phi_3 p_4 \end{bmatrix} \qquad \boldsymbol{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} = \begin{bmatrix} \phi_1 p_2 \\ \phi_1 \bar{p}_2 \\ \phi_2 p_3 \\ \phi_2 \bar{p}_3 \\ \phi_3 p_4 \end{bmatrix}$$

2. Reparameterise the exhaustive summary. Rewrite the exhaustive summary, $\kappa(\theta)$, in terms of the reparameterisation, $\kappa(\mathbf{s})$.

$$\boldsymbol{\kappa}(\boldsymbol{\theta}) = \begin{bmatrix} \pi_1 p_2 \\ \phi_1 \bar{p}_2 \phi_2 p_3 \\ \phi_1 \bar{p}_2 \phi_2 \bar{p}_3 \phi_3 p_4 \\ \phi_2 p_3 \\ \phi_2 \bar{p}_3 \phi_3 p_4 \\ \phi_3 p_4 \end{bmatrix} \longrightarrow \boldsymbol{\kappa}(\boldsymbol{s}) = \begin{bmatrix} s_1 \\ s_2 s_3 \\ s_2 s_4 s_5 \\ s_3 \\ s_4 s_5 \\ s_5 \end{bmatrix}$$

Extended Symbolic Method
Finding simpler exhaustive summaries

3. Calculate the derivative matrix $\mathbf{D}_s$.

$$\boldsymbol{D}_s = \frac{\partial \kappa(\boldsymbol{s})}{\partial \boldsymbol{s}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_3 & s_4 s_5 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & s_2 s_5 & 0 & s_5 & 0 \\ 0 & 0 & s_2 s_4 & 0 & s_4 & 1 \end{bmatrix}$$

4. The no. of estimable parameters = Rank($\mathbf{D}_s$) $\left(\text{if Rank}\left(\frac{\partial \boldsymbol{s}}{\partial \boldsymbol{\theta}}\right) - \text{Dim}(\boldsymbol{s})\right)$

   rank($\mathbf{D}_s$) = 5, no. est. pars = 5

5. If $\mathbf{D}_s$ is full rank (Rank($\mathbf{D}_s$) = Dim($\mathbf{s}$)) $\mathbf{s} = \mathbf{s}^{re}$ is a reduced-form exhaustive summary. If $\mathbf{D}_s$ is not full rank solve set of PDE to find a reduced-form exhaustive summary, $\mathbf{s}^{re}$.

   There are 5 s_i and the Rank($\mathbf{D}_s$) = 5, so $\mathbf{D}_s$ is full rank. $\mathbf{s}$ is a reduced-form exhaustive summary.

Extended Symbolic Method
Finding simpler exhaustive summaries

6. Use $\mathbf{s}^{re}$ as an exhaustive summary.

A reduced-form exhaustive summary is $\boldsymbol{s}^{re} = \begin{bmatrix} \phi_1 p_2 \\ \phi_1 \bar{p}_2 \\ \phi_2 p_3 \\ \phi_2 \bar{p}_3 \\ \phi_3 p_4 \end{bmatrix}$

$$\boldsymbol{D}_2 = \frac{\partial \boldsymbol{s}^{re}}{\partial \boldsymbol{\theta}} = \begin{bmatrix} p_2 & \bar{p}_2 & 0 & 0 & 0 \\ 0 & 0 & p_3 & \bar{p}_3 & 0 \\ 0 & 0 & 0 & 0 & p_4 \\ \phi_1 & -\phi_1 & 0 & 0 & 0 \\ 0 & 0 & \phi_2 & -\phi_2 & 0 \\ 0 & 0 & 0 & 0 & \phi_3 \end{bmatrix}$$

Rank($\mathbf{D}_2$) = 5; 5 estimable parameters.
Solve PDEs: estimable parameters are ϕ_1, ϕ_2, p_2, p_3 and $\phi_3 p_4$

See Hubbard et al (2012) for simpler exhaustive summaries for all capture-recapture models.

Examples II

Occupancy Models

- Rather than mark animals, surveys can just record whether or not an animal was present.
- ψ_i – whether a site was occupied in a particular season
- p – probability of detection
- eg h = [1 1 | 1 1] $\mathrm{P}(h) = \psi_c \times p^2 \times \psi_{1,1} \times p^2$
- eg h = [1 1 | 0 0] $\mathrm{P}(h) = \psi_c \times p^2 \times(\psi_{1,1} \times (1 - \mathrm{p})^2 + 1 - \psi_{1,1}\ )$
- Models are all theoretically full rank as long as there are more than one survey each season.
- However parameter redundancy could also be caused by the data.
- Using occupancy data on House Finch, if ψ_i and p are both season dependent this data set is parameter redundant with deficiency 1, which can be shown using the hybrid method.

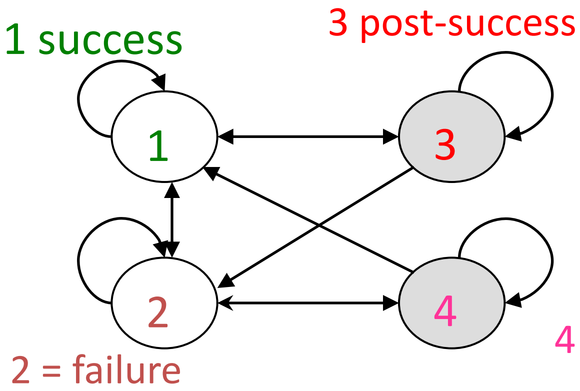
Examples II

Wandering Albatross

Multi-state models with unobservable states

(Cole *et al* 2010)

- Hunter and Caswell (2009) examine parameter redundancy of multi-state mark-recapture models, but cannot evaluate the symbolic rank of the derivative matrix (used a numerical method).
- 4 state breeding success model:



$$\log L = \sum_{r=1}^{N-1} \sum_{c=r+1}^{N} \sum_{i=1}^{4} \sum_{j=1}^{4} m_{i,j}^{(r,c)} \log \Psi_{ij}^{(r,c)}$$

$$\Psi^{(r,c)} = \begin{cases} \left(\Pi_{r+1}\Phi_r\right)^T & c = r+1 \\ \left\{\Pi_c \Phi_{c-1}(\mathbf{I} - \Pi_{c-1})\Phi_{c-2} \ldots (\mathbf{I} - \Pi_{r+1})\Phi_r\right\}^T & c > r+1 \end{cases}$$

survival

breeding given survival

successful breeding

recapture

$$\Phi = \begin{bmatrix} \sigma_1\beta_1\gamma_1 & \sigma_2\beta_2\gamma_2 & \sigma_3\beta_3\gamma_3 & \sigma_4\beta_4\gamma_4 \\ \sigma_1\beta_1(1-\gamma_1) & \sigma_2\beta_2(1-\gamma_2) & \sigma_3\beta_3(1-\gamma_3) & \sigma_4\beta_4(1-\gamma_4) \\ \sigma_1(1-\beta_1) & 0 & \sigma_3(1-\beta_3) & 0 \\ 0 & \sigma_2(1-\beta_2) & 0 & \sigma_4(1-\beta_4) \end{bmatrix} \qquad \Pi = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Examples II

Multi-state models with unobservable states

(Cole *et al* 2010)

1. Choose a reparameterisation, **s**, that simplifies the model structure.

$$s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{13} \\ s_{14} \end{bmatrix} = \begin{bmatrix} \sigma_1\beta_1\gamma_1 \\ \sigma_2\beta_2\gamma_2 \\ \sigma_3\beta_3\gamma_3 \\ \vdots \\ p_1 \\ p_2 \end{bmatrix}$$

2. Rewrite the exhaustive summary, $\boldsymbol{\kappa}(\boldsymbol{\theta})$, in terms of the reparameterisation - $\boldsymbol{\kappa}(\mathbf{s})$.

$$\kappa(\boldsymbol{\theta}) = \begin{bmatrix} p_1\sigma_1\beta_1\gamma_1 \\ p_2\sigma_1\beta_1(1-\gamma_1) \\ p_1\sigma_2\beta_2\gamma_2 \\ p_2\sigma_2\beta_2(1-\gamma_2) \\ p_1\sigma_1^2\beta_1^2\gamma_1^2(1-p_1)+\ldots \\ \vdots \end{bmatrix} \longrightarrow \kappa(\mathbf{s}) = \begin{bmatrix} s_1 s_{13} \\ s_5 s_{14} \\ s_2 s_{13} \\ s_6 s_{14} \\ s_1^2 s_{13}(1-s_{13})+\ldots \\ \vdots \end{bmatrix}$$

Examples II

Multi-state models with unobservable states

(Cole *et al* 2010)

3. Calculate the derivative matrix $\mathbf{D}_s$.

$$D_s = \left[\frac{\partial \kappa_j(\mathbf{s})}{\partial s_i}\right] = \begin{bmatrix} s_{13} & 0 & 0 & 0 & (2s_1 - 2s_1 s_{13})s_{13} & \cdots \\ 0 & 0 & s_{13} & 0 & (s_5 - s_5 s_{14})s_{13} & \\ 0 & 0 & 0 & 0 & s_9 s_{13} & \\ \vdots & & & & & \end{bmatrix}$$

4. The no. of estimable parameters =rank($\mathbf{D}_s$) $\left(\text{if Rank}\left(\frac{\partial \mathbf{s}}{\partial \boldsymbol{\theta}}\right) = \text{Dim}(\mathbf{s})\right)$.

   rank($\mathbf{D}_s$) = 12, no. est. pars = 12, deficiency = 14 – 12 = 2.

5. If $\mathbf{D}_s$ is full rank $\mathbf{s} = \mathbf{s}^{re}$ is a reduced-form exhaustive summary. If $\mathbf{D}_s$ is not full rank solve set of PDE to find a reduced-form exhaustive summary, $\mathbf{s}^{re}$.

$$s^{re} = \begin{bmatrix} s_1 & s_2 & s_5 & s_6 & s_{11} & s_{12} & s_{13} & s_{14} & s_7/s_3 & s_8/s_4 & s_3 s_9 & s_4 s_{10} \end{bmatrix}^T$$

Also see Cole (2012).

Examples II

Naïve Bayesian Network (Whiley and Titterington, 2002 and Cole et al, 2010)

- The network consists of n observable binary nodes, $y_1,\ldots,y_n$ and a single unobservable binary node, z.
- p is the probability $z = 1$.
- $\pi_{i,j}$ is the probability $y_i = 1$ given $z = j$.
- The probability of an observation $\boldsymbol{y}$ is:

$$\Pr(\boldsymbol{y}) = p\prod_{i=1}^{n}\pi_{i,1}^{y_i}\bar{\pi}_{i,1}^{1-y_i} + \bar{p}\prod_{i=1}^{n}\pi_{i,0}^{y_i}\bar{\pi}_{i,0}^{1-y_i} \qquad (\bar{x} = 1 - x)$$

- When $n \geq 4$, standard symbolic method does not work. Instead we start by applying the extended symbolic method to $n = 3$.

$$\boldsymbol{s}^T = [p\pi_{1,1}\pi_{2,1}\pi_{3,1} \quad p\pi_{1,1}\pi_{2,1} \quad p\pi_{1,1} \quad \bar{p}\pi_{1,0}\pi_{2,0}\pi_{3,0} \quad \bar{p}\pi_{1,0}\pi_{2,0} \quad p\pi_{1,0} \quad p]$$

- Using the extension theorem model is always full rank for $n \geq 3$.

Extended Symbolic Method

General Simple Exhaustive Summaries:

- Capture-recapture models – Hubbard *et al* (2012)
- Capture-recapture-recovery models – Hubbard *et al* (2012)
- Multi-state models (including unobservable states) – Cole (2012)
- Memory Models (Cole *et al*, 2013)
- Stop-over models (Matechou, 2010, Matechou and Cole in prep)

Tables of General Results:

- Capture-recapture models – Hubbard *et al* (2012)
- Capture-recapture-recovery models – Hubbard *et al* (2012)
- Ring-recovery – Cole *et al* (2012)
- Age-dependent mixture mark-recovery models – McCrea *et al* (2013)

Parameter Redundancy References

- Website: http://www.kent.ac.uk/smsas/personal/djc24/parameterredundancy.htm

- Catchpole, E. A. and Morgan, B. J. T. (1997) Detecting parameter redundancy. *Biometrika*, **84**, 187-196.
- Catchpole, E. A., Morgan, B. J. T and Freeman, S. N. (1998) Estimation in parameter redundant models. *Biometrika*, **85**, 462-468.
- Choquet, R., and Cole, D. J. (2012) A hybrid symbolic-numerical method for determining model structure. *Mathematical Biosciences*, **236**, 117-125.
- Cole, D. J., Morgan, B. J. T. and Titterington, D. M. (2010) Determining the parametric structure of models. *Mathematical Biosciences*, **228**, 16-30.
- Cole, D. J. and Morgan, B. J. T. (2010a) A note on determining parameter redundancy in age dependent tag return models for estimating fishing mortality, natural mortality and selectivity. *Journal of Agricultural, Biological, and Environmental Statistics*, **15**, 431-434.
- Cole, D. J. and Morgan, B. J. T. (2010b) Parameter redundancy with covariates. *Biometrika*, **97**, 1002-1005.
- Cole, D. J. (2012) Determining parameter redundancy of multi-state mark-recapture models for sea birds. *Journal of Ornithology*, **152** (Suppl 2), *S305-S315*.
- Cole, D.J., Morgan, B.J.T., Catchpole, E.A. and Hubbard, B. A. (2012) Parameter Redundancy in Mark-Recovery Models. *Biometrical Journal*, **54**, 507-523.
- Cole, D. J., Choquet, R. Gimenez, O., McCrea, R. S., Morgan, B. J. T. and Pradel, R. (2013) Does Your Species Have Memory? Analysing Capture-Recapture Data with Memory Models. *Submitted to methods in Ecology and Evolution.*
- Hubbard, B. A., Cole, D.J. and Morgan, B.J.T. (2012) Parameter Redundancy in Capture-Recapture-Recover Models. *Statistical Methodology*, doi: 10.1016/j.stamet.2012.11.005.
- Matechou, E. and Cole, D.J. (in prep) Parameter redundancy in Jolly-Seber models.
- McCrea, R.S., Morgan, B.J.T. and Cole, D.J. (2013) Age-dependent mixture models for recovery data on animals marked at unknown age, *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, **62**, 101-113.

Other References

Weak Identifiability:

- Garrett, E.S. and Zeger, S.L. (2000) Latent class model diagnosis, *Biometrics*, **56**, 1055-1067.
- Gimenez, O., Morgan, B.J.T. Brooks, S.P. (2009) Weak identifiability in models for mark-recapture-recovery data, in: D.L. Thomson, E.G. Cooch, M.J. Conroy (Eds.), Modelling Demographic Processes in Marked Populations: Series: Environmental and Ecological Statistics, vol. 3, pp. 1057–1070.

Applications:

- Brownie, C., Anderson,D. R., Burnham, K. P., and Robson,D. S. (1985). Statistical inference from band recovery data – handbook.Resource PublicationNo. 156.Washington, DC: United StatesDepartment of the Interior, Fish and Wildlife Service.
- Dochain, D., Vanrolleghem, P.A. and Van Daele, M. (1995) Structural identifiability of biokinetic models of activated sludge respiration, *Water Research*, **29**, 2571-2578
- Hunter, C.M. and Caswell, H. (2009). Rank and redundancy of multistate mark-recapture models for seabird populations with unobservable states. *Ecological and Environmental Statistics*, **3**, 797-82
- Jiang, H. Pollock, K. H., Brownie, C., *et al* (2007) Age-dependent tag return models for estimating fishing mortality, natural mortality and selectivity, *JABES*, **12**, 177-194
- Lebreton, J. Morgan, B. J. T., Pradel R. and Freeman, S. N. (1995) A simultaneous survival rate analysis of dead recovery and live recapture data. *Biometrics*, **51**, 1418-1428.
- Matechou, E. (2010) Applications and extensions of capture-recapture stop-over models, University of Kent PhD thesis.
- Pohjanpalo, H. (1978) System identifiability based on the power series of the solution, *Mathematical Biosciences*, **41**, 21-33.
- Whiley, M. and Titterington, D.M. (2002) Model identifiability in naive Bayesian networks, Technical Report 02-1, Department of Statistics, University of Glasgow.