```
> #Maple code for example 1 of paper Parameter Redundancy and Identifiability in Hidden Markov
      Models by D.J. Cole
> with(LinearAlgebra) :
> Dmat :=proc(se, pars)
  local DD1, i, j;
  description "Form the derivative matrix";
  with(LinearAlgebra) :
  DD1 := Matrix(1 ..Dimension(pars), 1 ..Dimension(se)) :
  for i from 1 to Dimension(pars)  do
      for j from 1 to Dimension(se) do
          DD1[i, j] := diff(se[j], pars[i])
      end do
   end do;
  DD1;
   end proc:
> Hybrid :=proc(D1, pars, minpars, maxpars, ret)
   local results, j, numpars, D1rand, ans, roll :
  description    "This procedure finds the rank and alpha for the hybrid-symbolic-numeric method.
       If ret = 1 returns full results. Otherwise returns model rank.";
    results := Matrix(5, 2) :
   for j from 1 to 5 do
       roll := rand(minpars ..maxpars) :
      numpars := seq(pars[i] = evalf(roll( )), i = 1 ..Dimension(pars)) :
      D1rand := eval(D1, {numpars});
      results[j, 1] := Rank( D1rand);
     results[j, 2] := NullSpace(Transpose(D1rand)) :
   end do:
    if ret = 1 then
          ans := results :
   else
         ans := max(results[1 ..5, 1]) :
   end if:
   ans :
   end proc:
> Estpar :=proc(DD1, pars, ret)
  local r, d, alphapre, alpha, PDE, FF, i, ans;
  description "Finds the estimable set of parameters for derivative matrix DD1. If ret = 1 returns
       alpha, PDEs, estimable parameter combinations. Otherwise returns estimable parameter
       combinations";
  with(LinearAlgebra) :
  r := Rank(DD1); d := Dimension(pars) - r :
  alphapre := NullSpace(Transpose(DD1)) : α := Matrix(d, Dimension(pars)) : PDE :=
       Vector(d) :
  FF := f(seq(pars[i], i = 1 ..Dimension(pars))) :
  for i from 1 to d do
        α[i, 1 ..Dimension(pars)] := alphapre[i] :
        PDE[i] := add(diff(FF, pars[j])·α[i, j], j = 1 ..Dimension(pars)) :
  end do:
   if ret = 1 then
          ans := ⟨pdsolve({seq(PDE[i] = 0, i = 1 ..d)}), {alpha}, {PDE}⟩ :
   elif ret = 2 then
```

```
        ans := ⟨{alpha}, {PDE}⟩ :
    else
        ans := pdsolve( {seq(PDE[i] = 0, i = 1 ..d)} ) :
    end if:
    ans :
  end proc:
> HMMexsum :=proc(deltavector, Gammamatrix, Pmatrix, st, T)
  local i, m, Prodkappa, kappa

      description   "This procedure finds the exhaustive summary for a HMM. Inputs: vector and
      matrices of HMM, st = 1 if stationary, T no. data points";
  m := Dimension(deltavector) :
  kappa := Vector(T) :
  if st = 1 then
      Prodkappa := deltavector · eval(Gammamatrix, t = 1).eval(Pmatrix, t = 1);
  else
      Prodkappa := deltavector · eval(Pmatrix, t = 1);
  end if:
  kappa[1] := (Prodkappa · Matrix(m, 1, 1))[1];
  for i from 2 to T do
     Prodkappa := Prodkappa.eval(Gammamatrix, t = i).eval(Pmatrix, t = i) :
     kappa[i] := (Prodkappa · Matrix(m, 1, 1))[1];
  end do:
  kappa
  end proc:
> # Setting up matrices and vector that define the model
> m := 2 :
> deltavector := ⟨delta[1]|1 − delta[1]⟩;
```

$$deltavector := \begin{bmatrix} \delta_1 & 1 - \delta_1 \end{bmatrix} \quad \textbf{(1)}$$

```
> Gammamatrix := Matrix(m, m) :
  for i from 1 to m do
     for j from 1 to m − 1 do
         Gammamatrix[i, j] := gamma[i, j] :
      end do:
         Gammamatrix[i, m] := 1 − add(gamma[i, j], j = 1 ..m − 1) :
  end do:
> Gammamatrix
```

$$\begin{bmatrix} \gamma_{1,1} & 1 - \gamma_{1,1} \\ \gamma_{2,1} & 1 - \gamma_{2,1} \end{bmatrix} \quad \textbf{(2)}$$

```
> Pyvector := ⟨seq( exp(−lambda[j]) · lambda[j]^x[t] / x[t]!, j = 1 ..m)⟩ :
> Pmatrix := DiagonalMatrix( Pyvector);
```

$$Pmatrix := \begin{bmatrix} \frac{e^{-\lambda_1} \lambda_1^{x_t}}{x_t!} & 0 \\ 0 & \frac{e^{-\lambda_2} \lambda_2^{x_t}}{x_t!} \end{bmatrix} \quad \textbf{(3)}$$

> # When there are 3 data points:

>

> $T := 3:$

> kappa := $HMMexsum(deltavector, Gammamatrix, Pmatrix, 1, T);$

$$\kappa := \Bigg[\Bigg[\frac{\left(\delta_1 \gamma_{1,1} + (1-\delta_1)\gamma_{2,1}\right) e^{-\lambda_1} \lambda_1^{x_1}}{x_1!} + \frac{\left(\delta_1 (1-\gamma_{1,1}) + (1-\delta_1)(1-\gamma_{2,1})\right) e^{-\lambda_2} \lambda_2^{x_1}}{x_1!} \quad \textbf{(4)}$$

$$\Bigg],$$

$$\Bigg[\frac{1}{x_2!} \Bigg(\Bigg(\frac{\left(\delta_1 \gamma_{1,1} + (1-\delta_1)\gamma_{2,1}\right) e^{-\lambda_1} \lambda_1^{x_1} \gamma_{1,1}}{x_1!}$$

$$+ \frac{\left(\delta_1 (1-\gamma_{1,1}) + (1-\delta_1)(1-\gamma_{2,1})\right) e^{-\lambda_2} \lambda_2^{x_1} \gamma_{2,1}}{x_1!} \Bigg) e^{-\lambda_1} \lambda_1^{x_2} \Bigg)$$

$$+ \frac{1}{x_2!} \Bigg(\Bigg(\frac{\left(\delta_1 \gamma_{1,1} + (1-\delta_1)\gamma_{2,1}\right) e^{-\lambda_1} \lambda_1^{x_1} (1-\gamma_{1,1})}{x_1!}$$

$$+ \frac{\left(\delta_1 (1-\gamma_{1,1}) + (1-\delta_1)(1-\gamma_{2,1})\right) e^{-\lambda_2} \lambda_2^{x_1} (1-\gamma_{2,1})}{x_1!} \Bigg) e^{-\lambda_2} \lambda_2^{x_2} \Bigg) \Bigg],$$

$$\Bigg[\frac{1}{x_3!} \Bigg(\Bigg(\frac{1}{x_2!} \Bigg(\Bigg(\frac{\left(\delta_1 \gamma_{1,1} + (1-\delta_1)\gamma_{2,1}\right) e^{-\lambda_1} \lambda_1^{x_1} \gamma_{1,1}}{x_1!}$$

$$+\frac{\left(\delta_1\left(1-\gamma_{1,1}\right)+\left(1-\delta_1\right)\left(1-\gamma_{2,1}\right)\right)\mathrm{e}^{-\lambda_2}\lambda_2^{x_1}\gamma_{2,1}}{x_1!}\Bigg)\mathrm{e}^{-\lambda_1}\lambda_1^{x_2}\gamma_{1,1}\Bigg)$$

$$+\frac{1}{x_2!}\Bigg(\Bigg(\frac{\left(\delta_1\gamma_{1,1}+\left(1-\delta_1\right)\gamma_{2,1}\right)\mathrm{e}^{-\lambda_1}\lambda_1^{x_1}\left(1-\gamma_{1,1}\right)}{x_1!}$$

$$+\frac{\left(\delta_1\left(1-\gamma_{1,1}\right)+\left(1-\delta_1\right)\left(1-\gamma_{2,1}\right)\right)\mathrm{e}^{-\lambda_2}\lambda_2^{x_1}\left(1-\gamma_{2,1}\right)}{x_1!}\Bigg)\mathrm{e}^{-\lambda_2}\lambda_2^{x_2}\gamma_{2,1}\Bigg)\Bigg)\mathrm{e}^{-\lambda_1}\lambda_1^{x_3}\Bigg)$$

$$+\frac{1}{x_3!}\Bigg(\Bigg(\frac{1}{x_2!}\Bigg(\Bigg(\frac{\left(\delta_1\gamma_{1,1}+\left(1-\delta_1\right)\gamma_{2,1}\right)\mathrm{e}^{-\lambda_1}\lambda_1^{x_1}\gamma_{1,1}}{x_1!}$$

$$+\frac{\left(\delta_1\left(1-\gamma_{1,1}\right)+\left(1-\delta_1\right)\left(1-\gamma_{2,1}\right)\right)\mathrm{e}^{-\lambda_2}\lambda_2^{x_1}\gamma_{2,1}}{x_1!}\Bigg)\mathrm{e}^{-\lambda_1}\lambda_1^{x_2}\left(1-\gamma_{1,1}\right)\Bigg)$$

$$+\frac{1}{x_2!}\Bigg(\Bigg(\frac{\left(\delta_1\gamma_{1,1}+\left(1-\delta_1\right)\gamma_{2,1}\right)\mathrm{e}^{-\lambda_1}\lambda_1^{x_1}\left(1-\gamma_{1,1}\right)}{x_1!}$$

$$+\frac{\left(\delta_1\left(1-\gamma_{1,1}\right)+\left(1-\delta_1\right)\left(1-\gamma_{2,1}\right)\right)\mathrm{e}^{-\lambda_2}\lambda_2^{x_1}\left(1-\gamma_{2,1}\right)}{x_1!}\Bigg)\mathrm{e}^{-\lambda_2}\lambda_2^{x_2}\left(1-\gamma_{2,1}\right)\Bigg)\Bigg)$$

$$\mathrm{e}^{-\lambda_2}\lambda_2^{x_3}\Bigg)\Bigg]\Bigg]$$

\>

\> $pars := \langle \gamma_{1,1}, \gamma_{2,1}, \lambda_1, \lambda_2 \rangle;$

(5)

$$pars := \begin{bmatrix} \gamma_{1,\,1} \\ \gamma_{2,\,1} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{5}$$

> $D1 := Dmat(\text{kappa}, pars)$:

> $r := Rank(D1); d := Dimension(pars) - r;$

$$r := 3$$
$$d := 1 \tag{6}$$

>

> $T := 4$:

> $\text{kappa} := HMMexsum(deltavector, Gammamatrix, Pmatrix, 1, T)$:

> $D1 := Dmat(\text{kappa}, pars)$:

> $r := Rank(D1); d := Dimension(pars) - r;$

$$r := 4$$
$$d := 0 \tag{7}$$

>

>

> *# Local identifiability too complex to solve*

> *# solve*$(\{seq(\text{kappa}[i] = k[i], i = 1..4)\}, \{seq(pars[i], i = 1..4)\});$

>

> $m := 3$:

> $deltavector := \langle \text{delta}[1] | \text{delta}[2] | 1 - \text{delta}[1] - \text{delta}[2] \rangle;$

$$deltavector := \begin{bmatrix} \delta_1 & \delta_2 & 1 - \delta_1 - \delta_2 \end{bmatrix} \tag{8}$$

> $Gammamatrix := Matrix(m, m)$:
> **for** i **from** 1 **to** m **do**
> **for** j **from** 1 **to** $m - 1$ **do**
> $Gammamatrix[i, j] := \text{gamma}[i, j]$:
> **end do**:
> $Gammamatrix[i, m] := 1 - add(\text{gamma}[i, j], j = 1..m - 1)$:
> **end do**:

> $Gammamatrix$

$$\begin{bmatrix} \gamma_{1,\,1} & \gamma_{1,\,2} & 1 - \gamma_{1,\,1} - \gamma_{1,\,2} \\ \gamma_{2,\,1} & \gamma_{2,\,2} & 1 - \gamma_{2,\,1} - \gamma_{2,\,2} \\ \gamma_{3,\,1} & \gamma_{3,\,2} & 1 - \gamma_{3,\,1} - \gamma_{3,\,2} \end{bmatrix} \tag{9}$$

> $Pyvector := \left\langle seq\left(\frac{\exp(-\text{lambda}[j]) \cdot \text{lambda}[j]^{x[t]}}{x[t]!}, j = 1..m \right) \right\rangle$:

> $Pmatrix := DiagonalMatrix(\ Pyvector);$

$$Pmatrix := \begin{bmatrix} \frac{e^{-\lambda_1} \lambda_1^{x_t}}{x_t!} & 0 & 0 \\ 0 & \frac{e^{-\lambda_2} \lambda_2^{x_t}}{x_t!} & 0 \\ 0 & 0 & \frac{e^{-\lambda_3} \lambda_3^{x_t}}{x_t!} \end{bmatrix} \tag{10}$$

> $T := 8 :$

> $\mathrm{kappa} := eval(HMMexsum(deltavector, Gammamatrix, Pmatrix, 1, T), \{x[1]=1, x[2]=2, x[3]=1, x[4]=2, x[5]=1, x[6]=1, x[7]=1, x[8]=1\}) :$

> $indets(\mathrm{kappa})$

$$\left\{\delta_1, \delta_2, \gamma_{1,\,1}, \gamma_{1,\,2}, \gamma_{2,\,1}, \gamma_{2,\,2}, \gamma_{3,\,1}, \gamma_{3,\,2}, \lambda_1, \lambda_2, \lambda_3, e^{-\lambda_1}, e^{-\lambda_2}, e^{-\lambda_3}\right\} \tag{11}$$

> $pars := \langle \gamma_{1,\,1}, \gamma_{1,\,2}, \gamma_{2,\,1}, \gamma_{2,\,2}, \gamma_{3,\,1}, \gamma_{3,\,2}, \lambda_1, \lambda_2, \lambda_3 \rangle :$

> $D1 := Dmat(\mathrm{kappa}, pars) :$

>

> $indetpars := \langle \delta_1, \delta_2, \gamma_{1,\,1}, \gamma_{1,\,2}, \gamma_{2,\,1}, \gamma_{2,\,2}, \gamma_{3,\,1}, \gamma_{3,\,2}, \lambda_1, \lambda_2, \lambda_3 \rangle :$

> $Hybrid(D1, indetpars, 0.0, 1.0, 1)$

$$
\left[\begin{array}{cc}
8 & \left\{\left[\begin{array}{c} -0.0710036094303968 \\ -0.259613765489724 \\ 0.878434532541952 \\ -0.0525664381999851 \\ 0.213494380567544 \\ -0.321742326441289 \\ -0.0448803938078468 \\ 0.0301987837126429 \\ 0.0335340183250897 \end{array}\right]\right\} \\
7 & \left\{\left[\begin{array}{c} -0.240210944722951 \\ 0.131340349510768 \\ -0.265675348649239 \\ 0.358236409408899 \\ 0.361523646941555 \\ -0.419070196651291 \\ -0.0472510500910235 \\ 0.00105395426216690 \\ 0.646203334211498 \end{array}\right], \left[\begin{array}{c} 0.377739285499986 \\ -0.383169934550079 \\ 0.346694185304037 \\ -0.381742706304043 \\ 0.467834688209981 \\ -0.472801842878501 \\ 0.0414571842441634 \\ 0.0197330669391829 \\ 0.00710713224621105 \end{array}\right]\right\} \\
8 & \left\{\left[\begin{array}{c} 0.325707227015121 \\ -0.276643510834436 \\ 0.449827173127426 \\ -0.292492990172497 \\ -0.343932228786572 \\ 0.635550407030166 \\ -0.0427947743955878 \\ -0.0371410560593432 \\ 0.0637337160722460 \end{array}\right]\right\} \\
8 & \left\{\left[\begin{array}{c} 0.361174493038883 \\ -0.531185076532133 \\ 0.106639486683413 \\ -0.311495563901396 \\ 0.414490819268559 \\ -0.458609301489467 \\ 0.0611094835309849 \\ \ldots \end{array}\right]\right\}
\end{array}\right] \qquad \textbf{(12)}
$$

>

> $T := 9:$

> kappa $:= eval(HMMexsum(deltavector, Gammamatrix, Pmatrix, 1, T), \{x[1]=1, x[2]=2, x[3]=1, x[4]=2, x[5]=1, x[6]=1, x[7]=1, x[8]=1, x[9]=1\}):$

> $indets(\text{kappa})$

$$\left\{\delta_1, \delta_2, \gamma_{1,1}, \gamma_{1,2}, \gamma_{2,1}, \gamma_{2,2}, \gamma_{3,1}, \gamma_{3,2}, \lambda_1, \lambda_2, \lambda_3, e^{-\lambda_1}, e^{-\lambda_2}, e^{-\lambda_3}\right\} \tag{13}$$

> $pars := \langle \gamma_{1,1}, \gamma_{1,2}, \gamma_{2,1}, \gamma_{2,2}, \gamma_{3,1}, \gamma_{3,2}, \lambda_1, \lambda_2, \lambda_3 \rangle:$

> $D1 := Dmat(\text{kappa}, pars):$

>

> $indetpars := \langle \delta_1, \delta_2, \gamma_{1,1}, \gamma_{1,2}, \gamma_{2,1}, \gamma_{2,2}, \gamma_{3,1}, \gamma_{3,2}, \lambda_1, \lambda_2, \lambda_3 \rangle:$

> $Hybrid(D1, indetpars, 0.0, 1.0, 1)$

$$\begin{bmatrix} 8 & \left\{\begin{bmatrix} -0.0302880862737410 \\ -0.293803354961962 \\ -0.194504522870175 \\ -0.742910050099361 \\ -0.126494471217161 \\ -0.488484014945349 \\ 0.0337356773472763 \\ -0.0636263568925426 \\ -0.251417929281599 \end{bmatrix}\right\} \\ 9 & \varnothing \\ 8 & \left\{\begin{bmatrix} 0.0129185307853816 \\ -0.0651981497258576 \\ 0.178687754086673 \\ -0.424313561663699 \\ -0.404233387851155 \\ 0.670964402619706 \\ -0.0496659457670418 \\ -0.0410341591469710 \\ -0.407262361310401 \end{bmatrix}\right\} \\ 9 & \varnothing \\ 9 & \varnothing \end{bmatrix} \tag{14}$$

>

> # Setting up matrices and vector that define the model

> $m := 2:$

> $deltavector := \langle \text{delta}[1] | 1 - \text{delta}[1] \rangle;$

$$deltavector := \begin{bmatrix} \delta_1 & 1-\delta_1 \end{bmatrix} \quad \textbf{(15)}$$

```
> Gammamatrix := Matrix(m, m) :
  for i from 1 to m do
     for j from 1 to m − 1 do
          Gammamatrix[i, j] := gamma[i, j] :
      end do:
          Gammamatrix[i, m] := 1 − add(gamma[i, j], j = 1 ..m − 1) :
   end do:
```

```
> Gammamatrix := eval(Gammamatrix, {γ[1, 1] = 1 − c·pi[1, 2], γ[2, 1] = c·pi[2, 1]});
```

$$Gammamatrix := \begin{bmatrix} -c\,\pi_{1,2}+1 & c\,\pi_{1,2} \\ c\,\pi_{2,1} & -c\,\pi_{2,1}+1 \end{bmatrix} \quad \textbf{(16)}$$

```
> Pyvector := ⟨seq(exp(−lambda[j])·lambda[j]^x[t], j = 1 ..m)⟩ :
> Pmatrix := DiagonalMatrix( Pyvector);
```

$$Pmatrix := \begin{bmatrix} e^{-\lambda_1}\lambda_1^{x_t} & 0 \\ 0 & e^{-\lambda_2}\lambda_2^{x_t} \end{bmatrix} \quad \textbf{(17)}$$

```
> # When there are 3 data points:
>
> T := 3 :
> kappa := HMMexsum(deltavector, Gammamatrix, Pmatrix, 1, T);
```

$$\kappa := \Big[\Big[\big(\delta_1\,(-c\,\pi_{1,2}+1) + (1-\delta_1)\,c\,\pi_{2,1}\big)\,e^{-\lambda_1}\lambda_1^{x_1} + \big(\delta_1\,c\,\pi_{1,2} + (1-\delta_1)\,(-c\,\pi_{2,1} + 1)\big)\,e^{-\lambda_2}\lambda_2^{x_1}\Big], \quad \textbf{(18)}$$

$$\Big[\Big(\big(\big(\delta_1\,(-c\,\pi_{1,2}+1) + (1-\delta_1)\,c\,\pi_{2,1}\big)\,e^{-\lambda_1}\lambda_1^{x_1}\,(-c\,\pi_{1,2}+1) + \big(\delta_1\,c\,\pi_{1,2} + (1-\delta_1)\,(-c\,\pi_{2,1}+1)\big)\,e^{-\lambda_2}\lambda_2^{x_1}\,c\,\pi_{2,1}\Big)\,e^{-\lambda_1}\lambda_1^{x_2} + \Big(\big(\delta_1\,(-c\,\pi_{1,2}+1) + (1-\delta_1)\,c\,\pi_{2,1}\big)\,e^{-\lambda_1}\lambda_1^{x_1}\,c\,\pi_{1,2} + \big(\delta_1\,c\,\pi_{1,2} + (1-\delta_1)\,(-c\,\pi_{2,1}+1)\big)\,e^{-\lambda_2}\lambda_2^{x_1}\,(-c\,\pi_{2,1}+1)\Big)\,e^{-\lambda_2}\lambda_2^{x_2}\Big],$$

$$\Big[\Big(\Big(\big(\big(\delta_1\,(-c\,\pi_{1,2}+1) + (1-\delta_1)\,c\,\pi_{2,1}\big)\,e^{-\lambda_1}\lambda_1^{x_1}\,(-c\,\pi_{1,2}+1) + \big(\delta_1\,c\,\pi_{1,2} + (1-\delta_1)\,(-c\,\pi_{2,1}+1)\big)\,e^{-\lambda_2}\lambda_2^{x_1}\,c\,\pi_{2,1}\Big)\,e^{-\lambda_1}\lambda_1^{x_2}\,(-c\,\pi_{1,2}+1) + \Big(\big(\delta_1\,(-c\,\pi_{1,2}+1) + (1-\delta_1)\,c\,\pi_{2,1}\big)\,e^{-\lambda_1}\lambda_1^{x_1}\,c\,\pi_{1,2} + \big(\delta_1\,c\,\pi_{1,2} + (1-\delta_1)\,(-c\,\pi_{2,1}+1)\big)\,e^{-\lambda_2}\lambda_2^{x_1}\,(-c\,\pi_{2,1}+1)\Big)\,e^{-\lambda_2}\lambda_2^{x_2}\,c\,\pi_{2,1}\Big)\,e^{-\lambda_1}\lambda_1^{x_3} + \Big(\Big(\big(\delta_1\,(-c\,\pi_{1,2}+1) + (1-\delta_1)\,c\,\pi_{2,1}\big)\,e^{-\lambda_1}\lambda_1^{x_1}\,($$

$$-c\,\pi_{1,2}+1\big)+\Big(\delta_1\,c\,\pi_{1,2}+(1-\delta_1)\,(-c\,\pi_{2,1}+1)\Big)\,e^{-\lambda_2}\lambda_2^{x_1}\,c\,\pi_{2,1}\Big)\,e^{-\lambda_1}\lambda_1^{x_2}\,c\,\pi_{1,2}$$
$$+\bigg(\Big(\delta_1\,(-c\,\pi_{1,2}+1)+(1-\delta_1)\,c\,\pi_{2,1}\Big)\,e^{-\lambda_1}\lambda_1^{x_1}\,c\,\pi_{1,2}+\Big(\delta_1\,c\,\pi_{1,2}+(1-\delta_1)\,(-c\,\pi_{2,1}$$
$$+1)\Big)\,e^{-\lambda_2}\lambda_2^{x_1}\,(-c\,\pi_{2,1}+1)\bigg)\,e^{-\lambda_2}\lambda_2^{x_2}\,(-c\,\pi_{2,1}+1)\bigg)\,e^{-\lambda_2}\lambda_2^{x_3}\bigg]\bigg]$$

> *indets*(kappa)

$$\left\{c,\delta_1,\lambda_1,\lambda_2,\pi_{1,2},\pi_{2,1},x_1,x_2,x_3,\lambda_1^{x_1},\lambda_1^{x_2},\lambda_1^{x_3},\lambda_2^{x_1},\lambda_2^{x_2},\lambda_2^{x_3},e^{-\lambda_1},e^{-\lambda_2}\right\} \tag{19}$$

> *pars* := ⟨c, λ_1, λ_2, pi$_{1,2}$, pi$_{2,1}$⟩;

$$pars := \begin{bmatrix} c \\ \lambda_1 \\ \lambda_2 \\ \pi_{1,2} \\ \pi_{2,1} \end{bmatrix} \tag{20}$$

> *D1* := *Dmat*(kappa, *pars*) :

> *r* := *Rank*(*D1*); *d* := *Dimension*(*pars*) − *r*;

$$r := 3$$
$$d := 2 \tag{21}$$

> *T* := 4 :

> kappa := *eval*$\left(HMMexsum(deltavector, Gammamatrix, Pmatrix, 1, T), \text{delta}[1] = \frac{1}{2}\right)$:

> *D1* := *Dmat*(kappa, *pars*) :

> *r* := *Rank*(*D1*); *d* := *Dimension*(*pars*) − *r*;

$$r := 4$$
$$d := 1 \tag{22}$$

> *Estpar*(*D1*, *pars*, 1);

$$\left[\left[\left\{f\left(c,\lambda_1,\lambda_2,\pi_{1,2},\pi_{2,1}\right)=\_F1\left(\lambda_1,\lambda_2,c\,\pi_{1,2},c\,\pi_{2,1}\right)\right\}\right],\right. \tag{23}$$

$$\left[\left\{\left[\begin{array}{ccccc} -\frac{c}{\pi_{2,1}} & 0 & 0 & \frac{\pi_{1,2}}{\pi_{2,1}} & 1 \end{array}\right]\right\}\right],$$

$$\left[\left\{\left[-\frac{\left(\frac{\partial}{\partial c} f(c, \lambda_1, \lambda_2, \pi_{1,2}, \pi_{2,1})\right) c}{\pi_{2,1}} + \frac{\left(\frac{\partial}{\partial \pi_{1,2}} f(c, \lambda_1, \lambda_2, \pi_{1,2}, \pi_{2,1})\right) \pi_{1,2}}{\pi_{2,1}} + \frac{\partial}{\partial \pi_{2,1}} f(c, \lambda_1, \lambda_2, \pi_{1,2}, \pi_{2,1})\right]\right\}\right]$$

```
> # Reparameterisation:
> m := 2 :
> deltavector := ⟨delta[1]|1 − delta[1]⟩;
```

$$deltavector := \begin{bmatrix} \delta_1 & 1-\delta_1 \end{bmatrix} \quad \textbf{(24)}$$

```
> Gammamatrix := Matrix(m, m) :
  for i from 1 to m do
      for j from 1 to m − 1 do
          Gammamatrix[i, j] := gamma[i, j] :
       end do:
          Gammamatrix[i, m] := 1 − add(gamma[i, j], j = 1 ..m − 1) :
   end do:
> Gammamatrix := eval(Gammamatrix, {γ[1, 1] = 1 − beta[1], γ[2, 1] = beta[2]});
```

$$Gammamatrix := \begin{bmatrix} 1-\beta_1 & \beta_1 \\ \beta_2 & 1-\beta_2 \end{bmatrix} \quad \textbf{(25)}$$

```
> Pyvector := ⟨seq(exp(−lambda[j])·lambda[j]^x[t], j = 1 ..m)⟩ :
> Pmatrix := DiagonalMatrix( Pyvector);
```

$$Pmatrix := \begin{bmatrix} e^{-\lambda_1} \lambda_1^{x_t} & 0 \\ 0 & e^{-\lambda_2} \lambda_2^{x_t} \end{bmatrix} \quad \textbf{(26)}$$

```
> T := 4 :
> kappa := eval(HMMexsum(deltavector, Gammamatrix, Pmatrix, 1, T), delta[1] = 1/2) :
> pars := ⟨β[1], β[2], λ[1], λ[2]⟩ :
> D1 := Dmat(kappa, pars) :
> r := Rank(D1); d := Dimension(pars) − r;
```

$$r := 4$$
$$d := 0 \quad \textbf{(27)}$$

```
> kappa[1]
```

(28)

$$\left(\frac{1}{2}-\frac{\beta_1}{2}+\frac{\beta_2}{2}\right) e^{-\lambda_1} \lambda_1^{x_1}+\left(\frac{\beta_1}{2}+\frac{1}{2}-\frac{\beta_2}{2}\right) e^{-\lambda_2} \lambda_2^{x_1} \tag{28}$$

\> kappa[2]

$$\left(\left(\frac{1}{2}-\frac{\beta_1}{2}+\frac{\beta_2}{2}\right) e^{-\lambda_1} \lambda_1^{x_1}\left(1-\beta_1\right)+\left(\frac{\beta_1}{2}+\frac{1}{2}-\frac{\beta_2}{2}\right) e^{-\lambda_2} \lambda_2^{x_1} \beta_2\right) e^{-\lambda_1} \lambda_1^{x_2}+\left(\left(\frac{1}{2}-\frac{\beta_1}{2}+\frac{\beta_2}{2}\right) e^{-\lambda_1} \lambda_1^{x_1} \beta_1+\left(\frac{\beta_1}{2}+\frac{1}{2}-\frac{\beta_2}{2}\right) e^{-\lambda_2} \lambda_2^{x_1}\left(1-\beta_2\right)\right) e^{-\lambda_2} \lambda_2^{x_2} \tag{29}$$

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