

Cardinals and the size of infinite sets

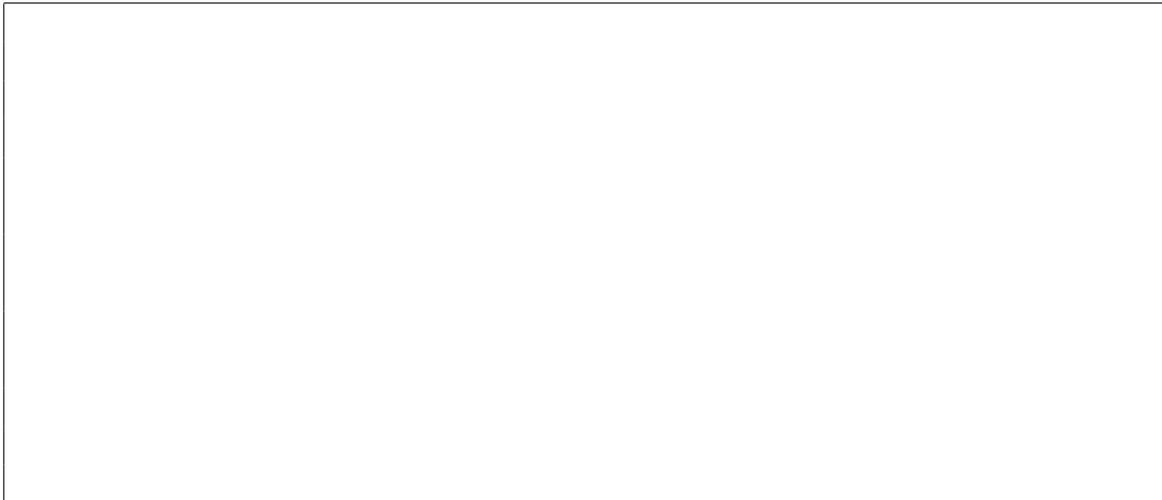
These notes revise some lectured material and have some questions for you to complete yourself. (You might find that some of the boxes are too small—just staple extra sheets at the back.) Note that these are not complete lecture notes—they cover most but not all things mentioned in the lectures—and they do not replace the notes you take in lectures.

1 Review of bijections

Question 1 *Recall the definition of a bijection between two sets.*

Question 2 *(a) Find a bijection between the natural numbers \mathbb{N} and the set of even natural numbers $\{2n \mid n \in \mathbb{N}\}$. (b) Do the same for \mathbb{N} and the set of integers \mathbb{Z} .*

Question 3 *Find a bijection $(0, 1) \rightarrow \mathbb{R}$ between the open interval $(0, 1)$ (which is a subset of \mathbb{R}) and \mathbb{R} itself. [Hint: the function $\tan(x)$ can help by giving a bijection from some interval to \mathbb{R} —that's not quite the right interval, but you'll be able to fix that problem somehow.]*



2 The size of finite sets

A set A is finite with n elements if $A = \{a_1, \dots, a_n\}$ where the elements a_i are all distinct. We often denote the number of elements of such sets A by $|A| = n$. It is clear in this case that there is a bijection

$$\{1, 2, \dots, n\} \rightarrow A \quad \text{given by} \quad i \mapsto a_i.$$

For finite sets, this is a simple observation. But for infinite sets, it will be used as a definition that will provide the starting point of our study.

If A, B are finite sets with $|A| = n$ and $|B| = m$, then the following things are clear (perhaps after a little thought):

- $n \leq m$ if and only if there is an injection $A \rightarrow B$.
- if A and B are disjoint, then $|A \cup B| = |A| + |B|$.
- $|A \times B| = |A| \cdot |B|$, where \times denotes the usual Cartesian product.

Again, these are simple observations for finite sets, but they are the inspiration for our ideas of order and arithmetic in the infinite case.

Question 4 Let $A = \{5, 6\}$ and $B = \{1, 2, 3\}$. Write out the elements of the set $A \times B$. What is $|A \times B|$?



3 Cardinals and their arithmetic

Any set A has a *cardinal*, or *cardinal number*, denoted $|A|$. Such cardinals satisfy the following property: for two sets A, B ,

$$|A| = |B| \iff \text{there is a bijection } A \rightarrow B.$$

This is the fundamental rule governing cardinals. You don't need to worry what kind of objects cardinals are or what they mean—although it's normal to think of $|A|$ as denoting the size of the set A —you just need to keep this characterising property of them in mind at all times.

If A is a finite set with exactly n elements, we often write $|A|$ as the natural number n . If $A = \emptyset$ is the empty set, we often write $|A|$ as the integer 0.

Question 5 *What is the cardinal of the set $A = \{5, 6\}$? And of $B = \{1, 2, 3\}$?*

Explain why there cannot be a bijection between A and B .

Definition 1 *The cardinal $|\mathbb{N}|$ is denoted by \aleph_0 , which is pronounced “aleph nought” or “aleph null”.*

Question 6 *Show that $\aleph_0 \neq 2$ by showing that there cannot be a bijection between \mathbb{N} and the set $C = \{1, 2\}$.*

4 Schröder–Bernstein and all that

We want to talk about the relative sizes of cardinals, just as we do for the usual natural numbers. In other words, we would like to have a meaningful way of saying what $\alpha \leq \beta$ means for cardinals α and β . We should bear in mind that our intuition for infinity is probably not good and may lead us astray if we try to treat this question too naively; we

will need to work honestly with the few relevant definitions, rather than trying to guess what ought to be happening.

We used bijections as the tool for judging equality of cardinals. Our experience with finite sets and finite cardinals was an influence for taking this point of view. Looking at finite sets again, we see that the usual order on finite cardinals ($0 < 1 < 2 < \dots$, in case there's any doubt) is characterised by injections.

Question 7 *Let A, B be finite sets with $|A| = n$, $|B| = m$. Show that if $n \leq m$ then there is an injection $A \rightarrow B$. Conversely, given an injection $A \rightarrow B$, show that $n \leq m$.*



This fact for finite sets inspires the definition of \leq for arbitrary sets.

Definition 2 *The order \leq is defined on cardinals by*

$$|A| \leq |B| \iff \text{there is an injection } A \rightarrow B$$

for any sets A, B .

We need to be clear here: Definition 2 simply gives a definition of the symbol \leq when used in the context of cardinals; we cannot assume that we know anything about how \leq works, except exactly what it says in the definition. But it is comforting, as we saw in Question 7 above, that when used with finite cardinals, this definition works in the same way as our usual definition of \leq for natural numbers.

To check that this is a good definition, we should check that \leq as defined in Definition 2 satisfies the usual properties that we expect of the relation \leq . Before we check that, we spend a little time discussing what these usual properties are.

Digression: Orders on sets

Forget all the stuff about cardinals for a moment: we're going to talk about what we expect of symbols like \leq in more familiar settings.

The idea of an order on a set is to have some notion of when one element is bigger than another. The usual notion of which number is bigger than another gives an order on \mathbb{R} : you always draw \mathbb{R} with the smaller (or more negative) numbers on the left and the bigger ones on the right, and you rarely deviate from that order. One point to be clear about: we're not saying here that you can write the real numbers in some order x_1, x_2, \dots , just that given any two different ones we know that one will be bigger than the other.

Definition 3 Let X be a set. An order on X is a relation, denoted $x \leq y$ for suitable elements $x, y \in X$, that satisfies three conditions:

- (i) $x \leq x$ for every $x \in X$.
- (ii) for every $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$.
- (iii) for every $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

The common names for these three properties are *reflexivity*, *antisymmetry* and *transitivity* respectively. It is also traditional to define the “strictly less than” symbol $<$ by

$$x < y \iff x \leq y \text{ and } x \neq y.$$

The symbols \geq and $>$ are defined in terms of what we already have: $x \geq y$ if and only if $y \leq x$, and so on.

To understand the notion of order a bit better, we start by checking that the usual notion of \leq on \mathbb{R} is an order: recall that we say $x \leq y$ for $x, y \in \mathbb{R}$ if and only if $y - x$ is positive or zero, and that if you add two positive numbers together the result is again positive. (We are still not thinking of cardinals at all, right?)

Question 8 Show that the usual order \leq on \mathbb{R} satisfies the three conditions of an order on a set.

Here’s the proof of transitivity. (You do the other two checks below.) Suppose $x, y, z \in \mathbb{R}$ are given with $x \leq y$ and $y \leq z$. By definition, this means that $y - x$ is either positive or zero and $z - y$ is also either positive or zero. The sum of these two numbers is certainly either positive or zero. The sum of these two is of course $(y - x) + (z - y) = z - x$, and so this is also either positive or zero. So, by definition, $x \leq z$, as required.

To check reflexivity:

To check antisymmetry:

End of digression: Back to cardinals

Now we can check that our definition of \leq for cardinals, the one given in Definition 2, is really an order (so it makes good sense using the notation \leq for it).

Theorem 4 *The relation \leq defined on cardinals is an order on any set of cardinals.*

To prove this, first of all recall the definition of \leq for cardinals.

Now check reflexivity for \leq . According to the definition, to show that $|A| \leq |A|$ for a set A , we must find an injection $A \rightarrow A$. But that's easy, because the identity map $A \rightarrow A$ is injective so we use that.

Next check transitivity (usually the third property). Suppose $|A| \leq |B|$ and $|B| \leq |C|$. What maps does the definition of \leq in this context give you?

You need to show $|A| \leq |C|$. In other words, you must find an injective map $A \rightarrow C$. Using the maps written above, write down a map $A \rightarrow C$ and prove that it is an injection.

Finally, we must check the antisymmetry of \leq as defined. Suppose that for two sets A, B we have $|A| \leq |B|$ and $|B| \leq |A|$. The task is to show that $|A| = |B|$. This is solved by the following famous theorem.

Theorem 5 (Schröder–Bernstein) *Let A, B be sets. If there are injections $A \rightarrow B$ and $B \rightarrow A$, then there is a bijection $A \rightarrow B$.*

We will not prove this theorem; the proof is not difficult—see, for example, P.M. Cohn, *Algebra* 2nd ed., Wiley 1982, page 4—but it is a small detour from our main aim. (Of course, the name *Schröder* is the sometimes transcribed as *Schroeder*.)

Explain how the Schröder–Bernstein Theorem, together with the definitions of $=$ and \leq in this context, show that \leq is antisymmetric.

Big concept: Countability

Definition 6 A set A is called *countable* if either it is finite or there is a bijection $\mathbb{N} \rightarrow A$. (Sometimes a countable set that is not finite is called *infinite countable*.)

The idea of this definition is that for a *countable* set you can write out the elements of A in order as a_1, a_2, a_3, \dots , where this list will stop if A is finite. If A is finite, you are used to being able to say “name the elements of A as $\{a_1, a_2, \dots, a_n\}$ where $n = |A|$ ”. For an infinite countable set A , we can say “name the elements of A as $\{a_1, a_2, a_3, \dots\}$ ”.

We will see later that there are “different infinities” with some being bigger than others (as measured by our order \leq on cardinals). You should think of \aleph_0 as being the smallest infinity; put another way, countable infinite sets are the smallest possible infinite sets. We say this more precisely as follows.

Question 9 (a) Let A be a set that has an injection $f: A \rightarrow \mathbb{N}$. If A is not finite, show that $|A| = \aleph_0$.

(b) Let B be a set with $|B| < \aleph_0$. Show that B is a finite set.

We can use the order \leq on cardinals to rephrase the definition of countable:

Question 10 *Show that a set A is countable if and only if $|A| \leq \aleph_0$.*

Arithmetic of cardinals

We do not have good intuitions about how to add cardinals together. Remember Hilbert's hotel: there we saw that $\aleph_0 + 1 = \aleph_0$. We need a secure definition to work with.

Definition 7 *Let A, B be two sets that satisfy $A \cap B = \emptyset$. Then we define*

$$|A| + |B| := |A \cup B| \quad \text{and} \quad |A| \cdot |B| := |A \times B|.$$

This definition applies to any cardinals. Earlier, we saw that for finite cardinals it corresponds to the usual arithmetic that we are used to. Our interest now is to see what happens when we also involve \aleph_0 in our calculations.

Proposition 8 $\aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 \cdot \aleph_0 = \aleph_0$.

You could think of the first part as saying $2\aleph_0 = \aleph_0$; using induction for example, you could also prove that $n\aleph_0 = \aleph_0$ for any $n \in \mathbb{N}$.

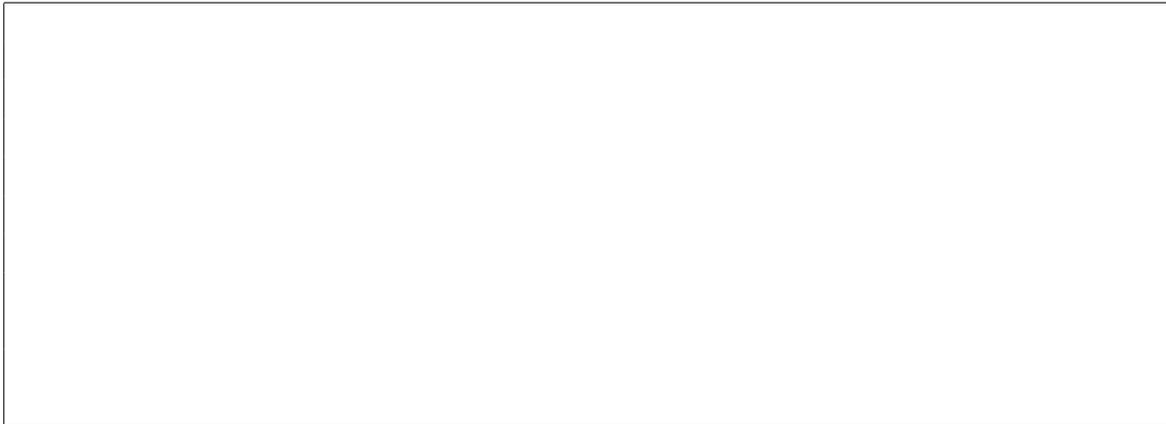
Question 11 *Explain why Proposition 8 holds. (It is OK to be informal here and just explain the idea. You could also give the proof, but complete details are not necessary.)*

5 Cantor's diagonal argument

Recall that $\aleph_0 := |\mathbb{N}|$.

Definition 9 The cardinal $|\mathbb{R}|$ is denoted by \aleph_1 , which is pronounced “aleph one”.

Question 12 Prove that $\aleph_0 \leq \aleph_1$. (Remember: you must use Definition 2 for this.)



Our aim is to show that $\aleph_0 \neq \aleph_1$. Given the previous exercise, this means that $\aleph_0 < \aleph_1$. This is truly amazing: it says that **some infinities are bigger than others**.

The idea is very simple. If $\aleph_0 = \aleph_1$, then there must be a bijection $f: \mathbb{N} \rightarrow \mathbb{R}$. Using this, we would be able to write down a list of real numbers $f(1), f(2), f(3), \dots$, and every real number would appear on our list. Cantor's argument imagines that such a list of real numbers exists, and then it finds a real number that has not been mentioned—this is a contradiction, and so the bijection f cannot exist.

The argument uses decimal expansions of real numbers, so before seeing the argument we have a quick reminder of these.

Aside: Decimal expansions of real numbers

Recall that every real number $\alpha \in \mathbb{R}$ has a decimal expansion:

$$\alpha = a_0.a_1a_2a_3 \dots$$

where $a_0 \in \mathbb{Z}$ and all other $a_i \in \{0, 1, 2, \dots, 9\}$. (Notice that I'm using a single full stop $.$ to denote the decimal point—it's a bit invisible, but it's normal, right?) But one must be careful: sometimes two different decimal expansions are equal to the same real number. This is epitomised by the famous and unforgettable

$$1.000\dots = 0.999\dots$$

In fact, this is the only thing that can go wrong: if a decimal expansion ends with repeated 0s, then it can be written in a different way ending with repeated 9s; and conversely if a decimal expansion ends with repeated 9s, then it can be written in a different way ending with repeated 0s. A different example is $0.123999\dots = 0.124$ (where I've omitted the trailing 0s).

Cantor's argument

Suppose there is a bijection $f: \mathbb{N} \rightarrow \mathbb{R}$. List the elements $f(1), f(2), f(3), \dots$ using their decimal expansions—of course, since f is surjective we must be listing all real numbers:

$$\begin{aligned} f(1) &= a_{1,0}.a_{1,1}a_{1,2}a_{1,3}\cdots \\ f(2) &= a_{2,0}.a_{2,1}a_{2,2}a_{2,3}\cdots \\ f(3) &= a_{3,0}.a_{3,1}a_{3,2}a_{3,3}\cdots \\ f(4) &= a_{4,0}.a_{4,1}a_{4,2}a_{4,3}\cdots \\ &\vdots \end{aligned}$$

Now we write down a decimal expansion that is definitely not on this list (even taking into account that business about recurring 0s and 9s above). Let $b_1 \in \{1, 2, 3, \dots, 8\}$ be different from $a_{1,1}$. Let $b_2 \in \{1, 2, 3, \dots, 8\}$ be different from $a_{2,2}$. Let $b_3 \in \{1, 2, 3, \dots, 8\}$ be different from $a_{3,3}$. Let $b_4 \in \{1, 2, 3, \dots, 8\}$ be different from $a_{4,4}$. And so on. Consider the real number $\beta := 0.b_1b_2b_3\cdots$. Certainly $\beta \neq f(1)$ because the two numbers differ in the first place of their decimal expansions. (β does not end with all 0s or all 9s, by our choice of b_i , so even if $f(1)$ could be written in a different form, it could not be equal to β .) Certainly $\beta \neq f(2)$ because the two numbers differ in the second place of their decimal expansions. (Again, this is also true if $f(2)$ could be written in a different form.) Certainly $\beta \neq f(3)$ because the two numbers differ in the third place of their decimal expansions. (Again, this is also true if $f(3)$ could be written in a different form.) In fact, whichever $f(n)$ you choose, it is clear that $\beta \neq f(n)$. But this is a disaster for f : it was meant to be a bijection, so that every real number appeared in the list $f(1), f(2), f(3), \dots$. But we have found a real number not on that list. This is a contradiction, and so the presumed bijection f cannot exist.

Question 13 Suppose $f(1) = 1.234567\cdots$, $f(2) = 2.342531\cdots$, $f(3) = 0.1241235\cdots$, $f(4) = 2.2399999\cdots$, $f(5) = -15.6752356\cdots$. (These decimal expansions might be finite or infinite—I'm not committing myself either way.) Write down the first 5 digits of a decimal expansion of a real number β that is not equal to any of these five real numbers following the recipe in Cantor's argument above.