The Primitive p-Frobenius Groups

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Abstract

Let p be a fixed prime. A finite primitive permutation group G with every two-point stabilizer $G_{\alpha,\beta}$ being a p-group is called a *primitive p-Frobenius group*. Using our earlier results on *p-intersection subgroups*, we give a complete classification of the primitive p-Frobenius groups.

1 Introduction

Let Σ_Ω denote the symmetric group on a finite set Ω and $G \leq \Sigma_\Omega$ a transitive permutation group. Suppose that every two-point stabilizer $G_{\alpha,\beta}$ is trivial; then a classical result of Frobenius states that G is a semidirect product $K : G_\alpha$ where the normal subgroup K consists precisely of the identity together with all elements $g \in G$ that do not fix any point of Ω. Moreover the point stabilizer G_α acts on K (via conjugation) in such a way that no nontrivial element $g \in G_\alpha$ has a nontrivial fixed point in K. A permutation group with these properties is called a *Frobenius group* with *Frobenius kernel* K.

As a consequence of a celebrated theorem of J. G. Thompson, the Frobenius kernel K is nilpotent and G_α is a *semiregular group*, i.e. G_α has a faithful action on a suitable vector space V such that $C_V(g) = 0$ for every $g \in G_\alpha \backslash \{id\}$ (notice that V can be chosen as a characteristically simple subgroup of a Sylow group of K). If moreover G is primitive, then $K \cong \mathbb{Z}_p^\ell =: V$ and G_α acts semiregularly on V.
In the course of his investigation of finite near fields, Zassenhaus [9] obtained a complete classification of finite semiregular groups. Putting all these results together one has a complete picture of primitive Frobenius groups.

In this paper we finish our investigation of the following p-local variant of the situation above:

Definition 1.1 Let p be a fixed prime. We define $\mathcal{F}(p)$ to be the set of all (faithful) finite primitive permutation groups with every 2-point stabilizer $G_{\alpha,\beta}$ being a p-group. An element of $\mathcal{F}(p)$ will be called a *primitive p-Frobenius group.*
Furthermore $\mathcal{F}^a(p)$ denotes the subset of $\mathcal{F}(p)$ consisting of those groups with abelian socle; moreover we set $\mathcal{F}^{na}(p) := \mathcal{F}(p) \backslash \mathcal{F}^a(p)$.

In [3], we defined a proper subgroup $X < G$ to be a p - *intersection subgroup* of G if and only if X is not a p-group but for any $g \in G \backslash X$ the intersection $X \cap X^g$ is a p-group. The set of p-intersection subgroups of G will be denoted by $\mathcal{I}_p(G)$. The main result of [3] is a classification of the elements in $\mathcal{I}_p(G)$ for all almost simple groups. Now we will apply this result to obtain a complete classification of the primitive p-Frobenius groups.

Remarks: (i) It is clear that $\mathcal{F}(p)$ includes the class $\mathcal{F}^\infty$ of all finite primitive Frobenius groups, which is the intersection $\bigcap_{p \text{ prime}} \mathcal{F}(p)$.
(ii) Every finite group G with $O_p(G) = 1$ admits a faithful transitive permutation representation with every $G_{\alpha,\beta}$ being a p-group: take the action of G on G/P, $P \in Syl_p(G)$. Thus it is the primitivity condition that makes the class $\mathcal{F}(p)$ interesting.
(iii) It is immediate from the definition, that $G \in \mathcal{F}(p)$ implies that either G_α is a p-group or $G_\alpha \in \mathcal{I}_p(G)$.

Our notation will be as follows: Ω always denotes a finite set and G a finite group with $\pi(G)$ the set of prime divisors of $|G|$. For $\alpha \in \Omega$ and $G \leq \Sigma_\Omega$ we write G_α for $Stab_G(\alpha)$. For any subgroup $S \leq G$ we set $Aut_G(S) := N_G(S)/C_G(S)$. Also, $X < \cdot Y$ means that X is a maximal subgroup of a group Y. The socle of G, $soc(G)$, is the product of all minimal normal subgroups of G. A subgroup $G \leq \Sigma_\Omega$ is called *primitive* (resp. *regular*) if it is transitive and $G_\alpha < \cdot G$ (resp. $G_\alpha = 1$). The symmetric, resp. alternating, group on n symbols is denoted by Σ_n, resp. $\mathcal{A}_n$.

Observe that if $G \in \mathcal{F}^a(p)$ then $soc(G) = V = \mathbb{F}_\ell^n$ is elementary abelian (the prime ℓ may differ from p) and G is the semi-direct product of V and any point stabilizer G_α, $a \in \Omega$. In this case, one can identify Ω with V, $a \in \Omega$ with the zero vector of V, then embed G_0 in $GL(V)$ such that the action of $G_\alpha = G_0$ on Ω and the linear action of G_0 on V are compatible.

Let us recall the following definition of [2]:
A pair (G, V) consisting of a finite group G and a finite-dimensional $\mathbb{F}G$ -module V over some field $\mathbb{F}$ is called *p'-semiregular* if every nontrivial p'-element of G acts without any fixed points on $V \setminus \{0\}$. G is called *p'-semiregular* if (G, V) is p'-semiregular for a suitable V.

Now the following statement is immediate:

Proposition 1.2 *Suppose* $soc(G) = V$ *is elementary abelian. Then* $G \in \mathcal{F}^a(p)$ *if and only if the pair* (G_0, V) *is* p'*-semiregular, and* V *is a faithful irreducible* G_0*-module.* $\circ$

In [2] all p'-semiregular pairs (G_0, V) have been determined.

The present paper provides a proof of the following complete classification of the groups in $\mathcal{F}(p)$:

Theorem 1.3 *Let* G *be an element of* $\mathcal{F}(p)$ *and put* $S := soc(G)$*. Then precisely one of the following three cases occurs:*

(i) *G has a regular normal subgroup V. In this case $V = S$ is elementary abelian, $G \in \mathcal{F}^a(p)$ and $G = V : G_0$ with p'-semiregular pair (G_0, V) as described in [2] (and V is a faithful irreducible G_0-module).*

(ii) *G has no regular normal subgroup and G_α is nilpotent. In this case $p = 2$, $G \in \mathcal{F}^{na}(2)$, $G_\alpha \in Syl_2(G)$ and $S = O^2(G) = S_1 \times \ldots \times S_k$ is the unique minimal normal subgroup of G as well as $S_1 \cong \ldots \cong S_k \cong L_2(q)$ with $q = 2^n \pm 1 > 5$ a prime or $q = 9$. Moreover, $[G : S]$ and k are powers of 2.*

(iii) *G has no regular normal subgroup and a point stabilizer G_α is an element of $\mathcal{I}_p(G)$. In this case S is simple, i.e. G is almost simple. Furthermore, the tuples (S, G, p, G_α) are as listed in Table* I *below.*

Conversely, each of the groups mentioned in (i), (ii) *and* (iii) *is a member of* $\mathcal{F}(p)$*, where in case* (ii) *we have to assume in addition that* $Aut_G(S_i) \cong PGL_2(7)$ *if* $q = 7$ *and* $Aut_G(S_i) \cong PGL_2(9), M_{10}$ *or* $Aut(\mathcal{A}_6)$ *if* $q = 9$.

TABLE I. p-intersection maximal subgroups in almost simple groups

S	G	p	G_α
$\mathcal{A}_5$	$G = S$	3	$\mathcal{A}_4$
$\mathcal{A}_5$	$\mathcal{A}_5 \leq G \leq \Sigma_5$	2	$N_G(\mathbb{Z}_3)$
$\mathcal{A}_6$	$\mathcal{A}_6 \leq G \leq Aut(\mathcal{A}_6)$	2	$N_G(3^2)$
$\mathcal{A}_6$	$PGL_2(9), M_{10}, Aut(\mathcal{A}_6)$	2	$N_G(\mathbb{Z}_5)$
$\mathcal{A}_n, n$ prime $\notin \{7, 11, 17, 23\}$	$G = S$	p with $\frac{n-1}{2} = p^f$	$\mathbb{Z}_n : \mathbb{Z}_{\frac{n-1}{2}}$
$\mathcal{A}_n, 5 \leq n \in \mathcal{F}$	Σ_n	2	$\mathbb{Z}_n : \mathbb{Z}_{n-1}$
$L_2(7) \cong L_3(2)$	$G = S$	2	$\Sigma_4^{(i)}, i = 1, 2$
	$G = PGL_2(7)$	2	$\mathcal{N}(X), N(T_1) \cong D_{12}$
$L_2(7)$	$G = S$	3	B
$L_2(11)$	$G = S$	2	$N(T_{cox})$
	$PGL_2(11)$	2	$N(T_1) \cong D_{20}$ $N(T_{cox}) \cong D_{24}$
$L_2(11)$	$G = S$	5	B
$L_2(2^a),\ 2^a \geq 4$	$G = S$	$p = 2^a - 1 \in \mathcal{M}$	B
$L_2(q),\ q \notin \mathcal{F} \cup \{4, 7, 9, 11\}$	$\pi(G/S) \subseteq \{2\}$	2	$N(T_1)$
$L_2(q),\ q \notin \mathcal{M} \cup \{4, 5, 9, 11\}$	$\pi(G/S) \subseteq \{2\}$	2	$N(T_{cox})$
$L_2(2^\ell)$	$2 < \lvert G/S \rvert = \ell$ prime	2	$N(T_{cox})$
$L_2(r),\ r = 2p^x + 1$	$G = S$	$p > 2$	B
$L_2(r),\ r \in \mathcal{F}$	$G/S \leq \mathbb{Z}_2$	2	B
$PSU_3(q),\ q \in \{3, 5, 9\}$	$G/S \leq \mathbb{Z}_2$	2	B
${}^2G_2(3)' \cong L_2(8)$	$G/S \leq \mathbb{Z}_3$	2	$B, N_G(B)$
${}^2B_2(2^a),\ 2^a \geq 8$	$G = S$	$p = 2^a - 1 \in \mathcal{M}$	B
$L_3(4)$	$G/S \leq F \cong \mathbb{Z}_2$	2	$N(\mathbb{Z}_3^2 : Q_8)$
$L_3(4)$	$\Delta \leq G/S \leq \mathbb{Z}_2 \times \mathbb{Z}_2$ $\lvert G/S \rvert \in \{2, 4\}$	2	$N(\mathbb{Z}_3^2 : Q_8)$
${}^3D_4(q)$	$\pi(G/S) \subseteq \{2\}$	2	$N(T_5)$
$L_p(q),\ PSU_p(q)$	$\pi(G/S) \subseteq \{p\}$	p	$N(T_{cox})$
M_{11}	$G = S$	2	$3^2 : SD_{16}$
J_1	$G = S$	2	$\Sigma_3 \times D_{10}$
M_{23}	$G = S$	11	$23 : 11$
BM	$G = S$	23	$47 : 23$
M (??)	$G = S$	29	$59 : 29$

In Table I B denotes a Borel subgroup; T_1, T_{cox} denote split- and Coxeter tori, respectively, $L_2(q) := PSL_2(q)$, $N(H) := N_G(H)$; if $G > S$, $\mathcal{N}(X) := \{N_G(X) \mid X \in \mathcal{I}_p(S), \text{maximal in } S\}$. Here ?? means either $59 : 29 < L_2(59) < \cdot\ M$ or $59 : 29 < \cdot\ M$. The existence of $L_2(59)$ in M is not settled yet.
Due to the isomorphisms $\mathcal{A}_5 \cong L_2(4) \cong L_2(5)$ and $\mathcal{A}_6 \cong L_2(9)$ these groups and their automorphic decorations are listed only in the alternating groups' section. For $L_3(4)$, F (resp. $\Delta \cong \mathbb{Z}_2$) is generated by field (resp. graph) automorphisms.

Note that for any $X \in \mathcal{I}_p(G)$ there is a subgroup $H \leq G$ such that $X \in \mathcal{I}_p(H)$ and X is maximal in H. Hence the results above give also a complete classification of p-intersection subgroups occuring in arbitrary finite groups. In particular we obtain the following result which can be viewed as the p-local version of Zassenhaus' classification of Frobenius complements:

Theorem 1.4 *Let G be a finite group, p a prime and $X < G$ a p-intersection subgroup. Then either X is p'-semiregular or X is a solvable group (occuring as G_α in Table I up to a suitable normal p-subgroup of X).*

2 Prerequisites

The reader is referred to [2] for a complete list of p'-semiregular groups; here we restrict ourselves to expose the list of the perfect ones. $\mathcal{R}$ is the set of all primes r such that $r = 2^a \cdot 3^b + 1$ for $a \geq 2$, $b \geq 0$, and $(r+1)/2$ is a prime.
In [5] R. Guralnick and R. Wiegand also classified p'-semiregular groups G such that the underlying field of the corresponding G-modules has characteristic p; however, the groups $SL_2(r)$ with $r \in \mathcal{R}$ for $p = 3$ are missing. The authors' proof in [2] is independent of that in [5] and in addition also describes the relevant G-modules.

Theorem 2.1 *Let G be a perfect finite group and (G, V) a p'-semiregular pair for a faithful irreducible $\mathbb{F}G$- module V. Then one of the following holds:*

(i) $G \cong SL_2(p^a)$ *for some* $a \geq 1$ *with* $p^a > 3$.

(ii) $G \cong {}^2B_2(2^{2a+1})$ *for some* $a \geq 1$ *with* $p = 2$.

(iii) $G \cong {}^2B_2(2^{2a+1}) \times SL_2(2^{2b+1})$ *with* $a, b \geq 1$, $\gcd(2a+1, 2b+1) = 1$ *and* $p = 2$.

(iv) $G \cong SL_2(r)$ *with* $r \in \mathcal{R} \cup \{7, 17\}$ *and* $p = 3$.

(v) $G \cong SL_2(5)$ *and* $p \geq 7$.

(vi) $G = ES$ *where* $E = O_2(G) \cong 2_-^{1+4}$, $S \cong SL_2(5)$, $E \cap S = Z(G) \cong \mathbb{Z}_2$ *and* $p = 2$.

Conversely, if (G, p) satisfies any of the conditions (i) – (vi), *then there exists a faithful absolutely irreducible G-module V such that (G, V) is p'-semiregular.*

Proof: Let (G, V) be p'-semiregular for a faithful irreducible $\mathbb{F}G$-module V and $char\, \mathbb{F} = \ell$. First suppose that ℓ divides $|G|$. Then it is clear that $\ell = p$ and the irreducibility of V forces $O_p(G) = 1$. By Theorem 4.1 [2] (cf. also [5]), (G, p) is as listed in (i) – (iv). Next suppose that ℓ does not divide $|G|$. Then Theorem 5.6 [2] and the irreducibility of V force (G, p) to satisfy one of the conditions (iv) – (vi) or (i) with $p^a = 4, 5, 9$. The existence of p' - semiregular pairs for the groups G listed has also been established in [2]. $\circ$

Example 2.2 (i) (Zassenhaus) $\ell^2 : SL_2(5) \in \mathcal{F}^\infty$ for all primes $\ell \equiv \pm 1 (\bmod 10)$ and $\ell^4 : SL_2(5) \in \mathcal{F}^\infty$ for all primes $\ell \equiv \pm 3 (\bmod 10)$.

(ii) $p^{2a} : SL_2(p^a) \in \mathcal{F}(p)$ for any prime p.

(iii) $3^6 : SL_2(13) \in \mathcal{F}(3)$ (Hering's group).

(iv) $7^8 : (2_-^{1+4} \backslash \mathcal{A}_5) \in \mathcal{F}(2)$ and $7^4 : SL_2(9)$, $5^{12} : SL_2(13) \in \mathcal{F}(3)$.

The following result will be used in the next chapter.

Theorem 2.3 *Let G be a finite group with a nilpotent maximal subgroup H.*
(i) [Thompson; see [4], Thm. 10.3.2] If H has odd order then G is solvable.
(ii) [Baumann; see [1]] If G is non-solvable then $O^2(G/F(G))$ is a direct product of simple groups isomorphic to $L_2(q)$ with primes q of the form $2^n \pm 1$ or $q = 9$.

We will also need the following result, which is an easy consequence of the classification of finite simple groups:

Lemma 2.4 *Let $\mathcal{E}$ be a nonabelian finite simple group and let $\alpha \in \mathrm{Aut}(\mathcal{E})$ be an element whose order is coprime to $|\mathcal{E}|$. Then $C_{\mathcal{E}}(\alpha)$ is not nilpotent.*

Proof: See [3] $\circ$

3 Reduction to the simple socle case

Proposition 3.1 *Let $G \in \mathcal{F}^{na}(p)$. Then G does not contain any regular normal subgroup. In particular $S := soc(G)$ is the unique minimal normal subgroup of G, $C_G(S) = 1$ as well as $G/S \cong G_\alpha / S_\alpha$.*

Moreover for any point stabilizer G_α one of the following is true:

(i) $p = 2$ *and* $G_\alpha \in Syl_2(G)$;
(ii) $G_\alpha \in \mathcal{I}_p(G)$.

Proof: 1) Suppose first that if G_α is a p-group. As G_α is maximal and nilpotent, 2.3 implies $p = 2$ and so $G_\alpha \in Syl_2(G)$.

2) Suppose next that $1 \neq R \triangleleft G$ with $R \cap G_\alpha = 1$. We can assume $R \leq soc(G)$ and R is not solvable (since $G \in \mathcal{F}^{na}(p)$). Suppose in addition that G_α is not a p-group. Then we can find $x \in G_\alpha$ of prime order $q \neq p$. Now for any $1 \neq y \in R$, $G_\alpha \cap G_{y(\alpha)}$ is a p-group. In

particular, $x^y \notin G_\alpha$, so $y^x \neq y$ and x acts fixed-point-freely on $R\backslash\{1\}$. By a well-known theorem of Thompson (cf. [4] Thm. 10.2.1) R must be nilpotent, a contradiction. Thus G_α must be a p-group. By 1), $G_\alpha \in Syl_2(G)$, and $|R| = [G : G_\alpha]$ is odd. So R (and $G = R \cdot G_\alpha$) is solvable, again a contradiction.

3) The claims concerning S now follow immediately. Furthermore, if G_α is a p-group, then (i) is fulfilled due to 1); otherwise one arrives at (ii). $\circ$

Corollary 3.2 *Let $G \in \mathcal{F}(p)$. Then $G \in \mathcal{F}^a(p)$ if and only if G contains a regular normal subgroup.* $\circ$

Proposition 3.3 *Suppose that $G \in \mathcal{F}^{na}(p)$ and $soc(G)$ is not simple. Then conclusion* (ii) *of Theorem* 1.3 *holds.*

Proof: A basic tool for studying finite permutation groups is the reduction theorem first stated by O'Nan and Scott (see [8]). Here we are using an expanded version of this theorem given in [7]. Because of 3.2, the primitive permutation group G under question has no regular normal subgroups (and $soc(G)$ is not simple). In this case, the O'Nan-Scott theorem says that G is either a *simple diagonal action* or a *product action* group, cf. [7]. We shall use the notation given there. In particular, $B = soc(G) = S_1 \times \ldots \times S_k$ with $S_1 \cong \ldots \cong S_k \cong T$ for a non-abelian finite group T.

1) Suppose G is a simple diagonal action group, that is, case III(a) of [7] occurs. Then for some $\alpha \in \Omega$ one has $B_\alpha = \{(a, a, \ldots, a) \mid a \in T\}$. In particular, taking $g := (a, a, \ldots, a)$, $h := (a, 1, \ldots, 1)$ for a non-identity p'-element $a \in T$, one sees that $g = g^h \in B_\alpha \cap B_{h(\alpha)}$. Meanwhile $h \notin B_\alpha$, contrary to the assumption that $G \in \mathcal{F}(p)$.

We have shown that G is a product action group, i.e., case III(b) of [7] occurs. In this case, Ω can be identified with Γ^ℓ for some finite set Γ and some ℓ dividing k. Furthermore, if H denotes $Aut_G(S_1)$ then H acts primitively on Γ, with socle K isomorphic to $T^{k/\ell}$. Moreover, H is of type II or III(a) (in the notation of [7]). Finally, G can be embedded in $W = H \wr \Sigma_\ell$, and the action of G on Ω is induced by the natural product action of W on Ω (cf. [7]).

2) Next we consider the situation when H is of type III(a). Then $\ell < k$. Arguing as in 1), one sees that $K_\gamma \cap K_{\gamma'}$ is not a p-group for some γ, $\gamma' \in \Gamma$, $\gamma \neq \gamma'$. For the distinct points $\alpha := (\gamma, \ldots, \gamma)$, $\alpha' := (\gamma', \ldots, \gamma')$ in Ω, one has $B_\alpha = (K_\gamma)^\ell$, $B_{\alpha'} = (K_{\gamma'})^\ell$. In particular, $B_\alpha \cap B_{\alpha'}$ is not a p-group, a contradiction.

Thus H must be of type II, i.e., $k = \ell$.

3) At this point we show that H_γ is a p-group.

First observe that B_α is a p-group, with $\alpha = (\gamma, \ldots, \gamma)$. Indeed, suppose $s = (s_1, ..., s_k) \in B_\alpha$ is not a p-element. Without loss we may suppose $s_1 \in T$ is not a p-element. Then

$s = s^t \in B_\alpha \cap B_{t(\alpha)}$ with $t := (s_1, 1, \dots, 1)$. This implies that $t \in B_\alpha$. Now taking $u := (1, g_2, \dots, g_k)$ one has $t = t^u \in B_\alpha \cap B_{u(\alpha)}$ for any $g_i \in T$. We conclude that all $(1, g_2, \dots, g_k)$ are contained in B_α, contradicting the equality $B_\alpha = (K_\gamma)^k$.

Suppose there is an element $x \in H_\gamma$ whose order is a prime r different from p. Every element in W can be canonically written in the form $(g_1, \dots, g_k)\pi$ for $g_i \in \Sigma_\Gamma$ and $\pi \in \Sigma_k$. Then G_α contains an element $g = (x, g_2, \dots, g_k)\pi$ with $\pi(1) = 1$. For any $y \in C_K(x)$ and $h := (y, 1, \dots, 1)$, we have $g = g^h \in G_\alpha \cap G_{h(\alpha)}$, yielding $h \in B_\alpha$. Due to the above observation $C_K(x)$ is a p-group. Choose $Q \in Syl_r(H)$ with $x \in Q$. Then $Q \cap K \lhd Q$ and $Z(Q) \cap (Q \cap K) \leq C_K(x)$. So $1 = Z(Q) \cap (Q \cap K)$, which implies that $1 = Q \cap K$. But $Q \cap K \in Syl_r(K)$, hence $(r, |K|) = 1$. Now 2.4 applied to K and x provides a contradiction.

4) We have proved that H_γ is a p-group. If p is odd, then the maximality of H_γ in H together with 2.3 implies that H is solvable, a contradiction. So we conclude that $p = 2$. Claim that G_α is a 2-group. For, suppose that $g = (g_1, \dots, g_k)\pi \in G_\alpha$ has order r, an odd prime, and $\alpha = (\gamma, \dots, \gamma)$. Observe that $G_\alpha \leq W_\alpha = H_\gamma \wr \Sigma_k$. Since $g_i \in H_\gamma$ has order a power of 2 we conclude that π has order r. In particular, we may suppose that π permutes the groups $S_1, \dots, S_r$ cyclically. Choose $c \in K \backslash K_\gamma$. Then for $y := (c, 1, \dots, 1) \in B$ we have $[y^{g^i}, y^{g^j}] = 1$ for all $i, j = 1, 2, \dots, r$. Hence $\tilde{y} := y y^g y^{g^2} \dots y^{g^{r-1}} \in C_B(g)$. From this it follows that $g = g^{\tilde{y}} \in G_\alpha \cap G_{\tilde{y}(\alpha)}$, yielding $\tilde{y} \in B_\alpha$. But in this case c belongs to K_γ, contrary to the choice of c. Consequently, G_α is a 2-group and so $G_\alpha \in Syl_2(G)$.

Applying 2.3 to the maximal subgroup G_α of G, we come to conclusion (ii) of Theorem 1.3. $\circ$

The following is one of the main results in [3]; here it classifies the elements in $\mathcal{F}^{na}(p)$ with simple socle:

Theorem 3.4 *Let $S \trianglelefteq G \leq Aut(S)$ with S a finite nonabelian simple group and suppose that $X \in \mathcal{I}_p(G)$ is maximal in G. Then $(S, G, p, X = G_\alpha)$ is one of the tuples listed in Table* I.

Proof: See [3]. $\circ$

4 Proof of Theorem 1.3

By 3.2 and the remarks in the introduction we can assume that $G \in \mathcal{F}^{na}(p)$.

First we suppose that G_α is nilpotent. Applying 2.3, we see that $F(G) = 1$ (as $G \in \mathcal{F}^{na}(p)$) and $S := soc(G) = O^2(G)$ is the direct product of k isomorphic (since S is characteristically simple) simple groups $S_1 \cong \dots \cong S_k \cong L_2(q)$ with $q = 2^n \pm 1$ a prime or $q = 9$. If $k = 1$, then a direct inspection of maximal subgroups of G with $L_2(q) \leq G \leq Aut(L_2(q))$ shows that $p = 2$ and $G_\alpha \in Syl_2(G)$, that is, conclusion (ii) of 1.3 holds. If $k > 1$ then applying 3.3 one again obtains that $p = 2$ and $G_\alpha \in Syl_2(G)$. Moreover, due to 3.1, $G/S \cong G_\alpha/S_\alpha$ and hence $[G : S]$ is a 2-power. But G/S acts transitively on the set $\{S_1, \ldots, S_k\}$, therefore k is a 2-power. Thus (ii) is fulfilled.

Now suppose that (ii) does not hold. Then $G_\alpha \in \mathcal{I}_p(G)$. In this case 3.3 shows that $soc(G)$ is simple. Applying 3.4, we arrive at (iii). $\circ$

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