The Noether Bound in Invariant Theory of Finite Groups

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Let R be a commutative ring, V a finitely generated free R- module and $G \leq GL_R(V)$ a finite group acting naturally on the graded symmetric algebra $A = \mathrm{Sym}(V)$. Let $\beta(A^G)$ denote the minimal number m, such that the ring A^G of invariants can be generated by finitely many elements of degree at most m. Let furthermore $H \triangleleft G$ be a normal subgroup such that the index $|G : H|$ is invertible in R. In this paper we prove the inequality

$$\beta(A^G) \leq \beta(A^H) \cdot |G : H|.$$

For $H = 1$ and $|G|$ invertible in R we obtain Noether's bound $\beta(A^G) \leq |G|$, which so far had been shown for arbitrary groups only under the assumption that the factorial of the group order, $|G|!$, is invertible in R.

1. INTRODUCTION

Let R be a commutative ring, R^* its group of units and A a graded R - algebra generated by finitely many homogeneous elements of degree one. Suppose the finite group G acts on A via graded R - algebra automorphisms. Then the ring of invariants $A^G := \{a \in A \mid g(a) = a \; \forall g \in G\}$ inherits a grading from A. In [9] (1916) Emmy Noether gave two different constructive proofs for the fact that A^G is finitely generated by invariants of degree at most $|G|$, whenever R is a field of characteristic zero. Both proofs can be adapted to work over any ring R having the property that the factorial of the group order, $|G|!$, is invertible in R (e.g. see [1], [15]).
In a second paper [10] (1926), Noether presented a proof for the fact that A^G is finitely generated whenever R is an abstract noetherian ring. No further assumption on invertibility of $|G|$ is needed but the proof is no longer constructive, in particular it does not give an explicit bound for the degrees of generators.

For any finitely generated graded R - algebra B let $\beta(B)$ denote the minimal number $m \in \mathbb{N}$ such that B can be generated as R - algebra by elements of degree less or equal to m. One says that A^G satisfies the **Noether bound** if $\beta(A^G) \leq |G|$.

Both proofs in [9] make essential use of the fact that the trace map

$$t_1^G : A \to A^G, \; a \mapsto \sum_{g \in G} g(a)$$

is a surjective homomorphism of A^G - modules, whenever $1/|G| \in R$. Therefore it comes as no surprise that A^G need not satisfy the Noether bound if the group order $|G|$ is not invertible in R: In fact let R be the binary field $\mathbb{F}_2$ and $G = \langle g \rangle \cong C_2$ the smallest non - trivial group, acting on $A := R[x_1, x_2, \cdots, x_k, y_1, y_2, \cdots, y_k]$ by interchanging the x_i's and y_i's. Then the element $x_1 \cdot x_2 \cdots x_k + y_1 \cdot y_2 \cdots y_k$ is an 'indecomposable' invariant, i.e. it cannot be written as a sum of products of invariants of smaller degree (see [4] for a more general statement).

But the surjectivity of the trace map is not the only instance where the arithmetic of R enters the proofs in [9]: as t_1^G is not multiplicative, certain combinatorial techniques had to be applied in order to 'reduce monomials', which require the factorial of the group order to be invertible in R (see [1] or [15] for more details). For example the first proof in [9] uses a classical result of H. Weyl [17] on vector invariants of the symmetric groups:

Let $B := R[x_{ij}, i = 1, \cdots, n, \; j = 1, \cdots, k]$ be the polynomial ring in the $n \times k$ commuting variables x_{ij}, on which the symmetric group Σ_n acts by the rule

$${}^{\sigma}(x_{i,j}) = x_{\sigma(i),j}.$$

Weyl's theorem states that if $\mathbb{Q} \subseteq R$, then B^{Σ_n} is generated by the coefficients of the y - terms in the 'Galois resolvent':

$$\phi(\underline{X})(\underline{Y}) := \prod_{i=1}^{n} (1 + \sum_{\ell=1}^{k} x_{i,\ell} y_\ell),$$

in particular $\beta(B^G) \leq n$. A new proof for this result, which works over any commutative ring R with $n! \in R^*$ has been given by D. Richman in [11], but in order to construct generators for A^G one has to use this result with $n = |G|$ and therefore ends up with the arithmetic condition $|G|! \in R^*$. Noether's bound for A^G under the less restrictive condition $|G| \in R^*$ has been conjectured for quite some time, but proven so far only for solvable groups ([11], [14]) and for groups with cyclic and alternating composition factors (see [5]) or for hypersurface invariant rings over arbitrary fields (see [2]). In [7] Göbel showed that if $A = R[X_1, \cdots, X_n]$ with G permuting the X_i, then $\beta(A^G) \leq \binom{n}{2}$, which can be better than Noether's bound in special cases. This bound is yet of a different nature, because it depends on the ring A rather than just the group G. For more background information see the very instructive survey [16] on invariant theory of finite groups, which devotes an extra section to this 'forgotten old problem' of degree bounds.

The most recent progress towards the Noether bound has been achieved in a more general context by Harm Derksen [3], where he presents an algorithm to construct invariants for reductive groups. This algorithm goes back to Hilbert's original proof of 1890 for finite generation of invariant rings, which had been considered completely non - constructive at that time. In the case where $R = k$ is a field and G a finite group of order coprime to the characteristic of R, Derksen proves Noether's bound for A^G, provided the following 'subspaces conjecture' holds:

Suppose that W is a k - vector space and $W_1, \cdots, W_d \leq W$ are subspaces. If I is the ideal of $W_1 \cup \cdots \cup W_d$, then I is generated by homogeneous polynomials of degree $\leq d$.

Even though this looks like a very natural statement in algebraic geometry it does not seem to be proven yet.

In this paper I will use a different approach, in modifying and at the same time generalizing the original arguments of Emmy Noether's first proof in [9] to prove her degree bound under the condition $|G| \in R^*$. All this shows that, concerning today's computational algebra, there are lots of buried treasures, hidden in the original papers of the pioneers.

2. REDUCTIONS IN THE HILBERT IDEAL

For a power product

$$a := x_{11}^{a_{11}} \cdot \ldots \cdot x_{n1}^{a_{n1}} \cdot \ldots \cdot x_{1k}^{a_{1k}} \cdot \ldots \cdot x_{nk}^{a_{nk}} \in B$$

with its exponent matrix $(a_{ij}) \in \mathbb{N}_0^{n,k}$, let $a^+ := \sum_{b \in \{^{\sigma}(a) \mid \sigma \in \Sigma_n\}} b \in B^{\Sigma_n}$ denote the corresponding orbit sum.

For any subset S of homogeneous elements in A or B, the corresponding subsets of elements in S of degree (strictly) less than m will be denoted by $S_{\leq m}$ ($S_{<m}$). Furthermore for a graded module N, let N^+ denote its submodule generated by the homogeneous elements of strictly positive degree. In Hilbert's original proof of 1890 an important tool was the ideal in A (or B), generated by all invariants of positive degree. We therefore call the ideal $(A^G)^+ A \triangleleft A$ the **Hilbert ideal** in A.

In [4] the bound $\beta(B) \leq \max\{n, k \cdot (n-1)\}$ has been proved, with equality if $\mathrm{char}\ R = p$, a prime and $n = p^s$. Although invariants of degree larger than n can be indecomposable in the ring B^G, they still can decompose in the Hilbert ideal $(B^G)^+ \cdot B$ in B. In particular for those orbit sums whose exponent matrix consists of exactly one non - zero row, such a decomposition can be written down explicitly. The general form of this reduction is as follows:

Let $m \in R[x_{1,1}, x_{1,2}, \cdots, x_{1,k}]$ be a power product of degree $> n$. Then m is of the form $\bar{T} \cdot x_{1,i_1} x_{1,i_2} \cdots x_{1,i_n}$ with $\bar{T}$ a power product in $R[x_{1,1}, x_{1,2}, \cdots, x_{1,k}]$.

We get

$$m^+ = (\bar{T}\cdot x_{1,i_2}\cdots x_{1,i_n})^+\cdot x_{1,i_1} + (\bar{T}\cdot x_{1,i_1}x_{1,i_3}\cdots x_{1,i_n})^+\cdot x_{2,i_2} + \cdots +$$

$$(\bar{T}\cdot x_{1,i_1}x_{1,i_2}\cdots x_{1,i_{n-1}})^+\cdot x_{n,i_n} - (\bar{T}\cdot x_{1,i_3}\cdots x_{1,i_n})^+\cdot x_{1,i_1}x_{2,i_2}$$

$$-(\bar{T}\cdot x_{1,i_2}x_{1,i_4}\cdots x_{1,i_n})^+\cdot x_{1,i_1}x_{3,i_3} - \cdots$$

$$-(\bar{T}\cdot x_{1,i_1}x_{1,i_2}\cdots x_{1,i_{n-2}})^+\cdot x_{n-1,i_{n-1}}x_{n,i_n}$$

$$+(\bar{T}\cdot x_{1,i_4}\cdots x_{1,i_n})^+\cdot x_{1,i_1}x_{2,i_2}x_{3,i_3} + (\bar{T}\cdot x_{1,i_3}x_{1,i_5}\cdots x_{1,i_n})^+\cdot x_{1,i_1}x_{2,i_2}x_{4,i_4}$$

$$+\cdots-\cdots+(-1)^{n-1}(\bar{T})^+\cdot x_{1,i_1}x_{2,i_2}\cdots x_{n,i_n}.$$

It is easy to see that all terms of m^+ appear exactly once in the expansion, while all the 'superfluous terms' on the right hand side cancel. For $n = 4$ we have

$$m^+ = (\bar{T}\cdot x_{1,j}x_{1,k}x_{1,\ell})^+\cdot x_{1,i} + (\bar{T}\cdot x_{1,i}x_{1,k}x_{1,\ell})^+\cdot x_{2,j}+$$

$$(\bar{T}\cdot x_{1,i}x_{1,j}x_{1,\ell})^+\cdot x_{3,k} + (\bar{T}\cdot x_{1,i}x_{1,j}x_{1,k})^+\cdot x_{4,\ell}$$

$$-(\bar{T}\cdot x_{1,k}x_{1,\ell})^+\cdot x_{1,i}x_{2,j} - (\bar{T}\cdot x_{1,j}x_{1,\ell})^+\cdot x_{1,i}x_{3,k}$$

$$-(\bar{T}\cdot x_{1,j}x_{1,k})^+\cdot x_{1,i}x_{4,\ell} - (\bar{T}\cdot x_{1,i}x_{1,\ell})^+\cdot x_{2,j}x_{3,k}$$

$$-(\bar{T}\cdot x_{1,i}x_{1,k})^+\cdot x_{2,j}x_{4,\ell} - (\bar{T}\cdot x_{1,i}x_{1,j})^+\cdot x_{3,k}x_{4,\ell}$$

$$+(\bar{T}\cdot x_{1,\ell})^+\cdot x_{1,i}x_{2,j}x_{3,k} + (\bar{T}\cdot x_{1,k})^+\cdot x_{1,i}x_{2,j}x_{4,\ell}$$

$$+(\bar{T}\cdot x_{1,j})^+\cdot x_{1,i}x_{3,k}x_{4,\ell} + (\bar{T}\cdot x_{1,i})^+\cdot x_{2,j}x_{3,k}x_{4,\ell}$$

$$-(\bar{T})^+\cdot x_{1,i}x_{2,j}x_{3,k}x_{4,\ell}.$$

Notice that these elements are in $((B^+)^{\Sigma_n}_{<s})B$ if the total degree of m is $s > n$. Notice also that all invariants which occur in these decompositions again consist of 'one - row' invariants only. Hence, by iteration one can generate this type of invariants in the Hilbert ideal by 'one row' - invariants of degree less or equal to n. We summarize this in the following lemma and add a different, very elegant proof, given by Gregor Kemper:

LEMMA 2.1. *Let* $m \in R[x_{1,1}, \cdots, x_{1,k}] \le B$. *Then* $m^+ \in ((B^+)^{\Sigma_n}_{\le n}) \cdot B$.

Proof. Let $m \in R[x_{1,1}, .., x_{1,k}]$ be a power product of degree $\deg(m) > n$. Then there are indices $i_1, .., i_n \in \{1, .., k\}$, not necessarily distinct, such that the product $x_{1,i_1} \cdots x_{1,i_n}$ divides m. Set $m' := m/(x_{1,i_1} \cdot \cdots \cdot x_{1,i_n})$ and $t := x_{1,i_1} \cdot x_{2,i_2} \cdots x_{n,i_n}$. Furthermore let $\pi_l : B \to B$ denote the homomorphism defined by $x_{i,j} \mapsto x_{l,j}$, so $m^+ := \sum_{l=1}^n \pi_l(m)$. Then

$$\sum_{t' \mid t} (-1)^{\deg(t')} \cdot t/t' \cdot \sum_{l=1}^{n} \pi_l(t' \cdot m') = 0.(**)$$

This follows from swapping the summation:

$$\sum_{l=1}^{n} \sum_{t' \mid t} (-1)^{\deg(t')} \cdot t/t' \cdot \pi_l(t' \cdot m') =$$

$$\sum_{l=1}^{n} \sum_{t' \mid t/x_{l,i_l}} (-1)^{\deg(t')} \cdot [t/t' \cdot \pi_l(t' \cdot m') - t/(x_{l,i_l} \cdot t') \cdot \pi_l(x_{l,i_l} \cdot t' \cdot m')].$$

This is zero, because $\pi_l(x_{l,i_l} \cdot t' \cdot m') = x_{l,i_l} \cdot \pi_l(t' \cdot m')$. For $t' = t$, the summand in (**) is $(-1)^n \cdot \sum_{l=1}^n \pi_l(m) = (-1)^n \cdot m^+$, and for a proper divisor t' we get $(-1)^{\deg(t')} \cdot t/t' \cdot (\pi_1(t' \cdot m'))^+$. Since the elements $\pi_1(t' \cdot m') \in R[x_{1,1}, .., x_{1,k}]$ have smaller degree for $t' \neq t$, the lemma follows by induction. ∎

3. REDUCTION OF G - INVARIANTS

Now let $H \le G$ with $n := |G : H|$ and $G/H := \{H := g_1H, g_2H, \ldots, g_nH\}$ the set of H cosets in G. Consider the Cayley homomorphism

$$c : G \to \Sigma_n, \; g \mapsto (g_iH \mapsto g_jH := gg_iH).$$

Suppose that G acts on A via graded R - algebra automorphisms and $A^H = R[b_1, \ldots, b_k]$ with $b_i \in A$ homogeneous of degree $\le b := \beta(A^H)$. Then

$$\nu : B \to A, \; x_{i\ell} \mapsto g_i(b_\ell)$$

is a G - equivariant R - algebra morphism (see [5]). Since Noether used this for the special case $H = id$ in [9], ν is called the **Noether homomorphism**. Here I combine it with the **relative trace operator**

$$t_H^G : A^H \to A^G, \; a \mapsto \sum_{g_iH \in G/H} g_i(a),$$

which is a well - defined homomorphism of A^G - modules. Notice that $\nu(B) \subseteq A^H$ if H is a normal subgroup of G. One obtains

LEMMA 3.1. *Let $H \leq G$ be a normal subgroup. Then the image $t_H^G((A^+)^H)$ is contained in $A^{+,G}_{\leq b\cdot n} \cdot A^H$.*

Proof. Let $a := t_H^G(f)$ with $f = f(b_1, \dots, b_k) \in A^H$. Define

$$F := f(x_{11}, x_{12}, \dots, x_{1k})$$

and note that $\nu(F^+) = a$. Expanding F as an R - linear combination of power products, one sees that F^+ is an R - linear combination of 'one - row' - vector invariants of Σ_n, considered in the previous lemma. Hence $a = \nu(F^+) \subseteq \nu((B^{+,\Sigma_n})_{\leq n} \cdot B) \subseteq (A^{+,G}_{\leq n\cdot b}) \cdot A^H$. ∎

THEOREM 3.1. *Let $H \lhd G$ with $|G : H| \in R^*$. Then*

$$\beta(A^G) \leq \beta(A^H)|G : H|.$$

In particular $\beta(A^G) \leq |G|$ if $|G| \in R^*$.

Proof.

$$(A^G)^+ = t_H^G((A^H)^+) = t_H^G(t_H^G((A^H)^+)) \subseteq t_H^G(A^{+,G}_{\leq b\cdot n} \cdot A^H) = A^{+,G}_{\leq b\cdot n} \cdot A^G.$$

This implies $A^G = R[A^{+,G}_{\leq b\cdot n}]$ by a standard argument. ∎

Remark 3. 1. Assume $|G : H| \in R^*$ (or $|G| \in R^*$). Since only 'one - row' terms of B are needed to express G - invariants via the Noether homomorphism, the results in 2.1, 3.1 and 3.1 yield an explicit algorithm to rewrite arbitrary G - invariants in terms of those of reduced degree $\leq \beta(A^H) \cdot |G : H|$ (or $\leq |G|$), without knowing a generating set of A^G in the first place.

4. APPLICATIONS TO MODULAR INVARIANTS

A conjecture of Harm Derksen and Gregor Kemper states that the Hilbert ideal $A^{+,G} \cdot A$ is always generated by invariants of degree less or equal to the group order. Notice that Noether's bound in case $|G| \in R^*$ would also immediately follow from this conjecture. On the other hand our results at least confirm this conjecture for a special class of invariant rings:

THEOREM 4.1. *Let G act linearly on the R - submodule of elements of degree one of $A = R[X_1, \cdots, X_d]$ in such a way that $H \leq G$ permutes variables. Assume furthermore that $|G : H| \in R^*$. Then*

$$A^{+,G} \cdot A = A^{+,G}_{\leq |G|} \cdot A.$$

Proof. Let $m = t_H^G(b) \in A^G$ with $b \in A^H$. We can assume b to be of the form $t_S^H(a)$, where a is a power product with stabilizer $S := H_a \leq H$ of index n in G. Let furthermore $o_1, o_2, \cdots, o_k$ denote the 'orbit - products' under S, i.e. o_i is the product of all variables in the i - th S - orbit. Obviously the o_i are of degree less or equal to $|S|$. Let ν be the above algebra homomorphism $B \to R[o_1, o_2, \cdots, o_k] \leq A$ (with b_i replaced by o_i). As above we see that $m = t_S^G(a) \in \nu((B^{+,\Sigma_n})_{\leq n} \cdot B) \subseteq (A^{+,G}_{\leq n \cdot |S|}) \cdot A$ ▮

Remark 4. 1.

1. If R is a field of characteristic p dividing $|G|$, P a Sylow p - group of G and V a direct summand of a finitely generated permutation module (a so - called 'trivial source module'), then it is well known that $V_{|P}$ is a permutation module (see e.g. [13], chapter II, 12.5). Hence 4.1 applies to the symmetric algebra $A := \mathrm{Sym}(V)$ with its canonical action of G.
2. In particular the Hilbert ideal $(\mathrm{Sym}(V)^G)^+ \cdot \mathrm{Sym}(V)$ is generated in degrees less or equal to $|G|$, whenever V is a projective module (e.g. the regular representation).

Now let again P be a fixed Sylow p - group of G, $R = \mathbb{F}$ a field of characteristic p, V a finite dimensional $\mathbb{F}G$ - module and $A := \mathrm{Sym}(V^*)$ the symmetric algebra of the dual module V^* with canonical G - action. We define $A^G_{<P} := \sum_{Q<P} t_Q^G(A^Q) \lhd A^G$, the largest proper ideal of A^G which can be constructed using relative trace operations from proper subgroups Q of P. Ideals of this form have been studied in [6], where among other things it was shown that the radical ideal $\sqrt{A^G_{<P}}$ is always a prime ideal of height $\mathrm{codim}_{\mathbb{F}}(V^P)$. Here V^P denotes the subspace of P - fixed points in V. If $\mathbb{F}$ is algebraically closed, the ideal $A^G_{<P}$ has a very natural geometric interpretation: Its variety $\mathcal{V}$ in the orbit space V/G consists of all those G - orbits such that a suitable stabilizer contains P, or in other words

$$\mathcal{V} = \{v^G \in V/G \mid p \text{ does not divide } |v^G|\}.$$

Moreover it was shown that the quotient ring $A^G/\sqrt{A^G_{<P}}$ is a Cohen - Macaulay ring of Krull dimension $\dim_{\mathbb{F}}(V^P)$, provided the action of P on V is defined over

the prime field. This is satisfied for example, if the restricted $\mathbb{F}$ - module $V_{|P}$ is a permutation module, or, in other words, if V is a trivial source module. In this case it also follows from [6] 5.18 that $A^G_{<P} = \sqrt{A^G_{<P}}$, but this is not true for arbitrary modules V (see Example 12.2 in [6]).

The ideal $A^G_{<P}$ is clearly homogeneous and therefore the quotient algebra $\overline{A^G} := A^G/A^G_{<P}$ is graded in a canonical way.

We will now derive a degree bound for this quotient ring, which generalizes our earlier result in coprime characteristic (notice that $A^G_{<P} = 0$ if p does not divide $|G|$.) We start with the following elementary observation:

Let B and C be graded subalgebras of A and I a homogeneous ideal of A such that $B + I = C + I$. Then $B/B \cap I$ and $C/C \cap I$ are isomorphic as graded algebras and in particular $\beta(B/B \cap I) = \beta(C/C \cap I)$.

Let $N := N_G(P)$ and $\bar{N} := N_G(P)/P$, which acts naturally on A^P and $\overline{A^P} := A^P/A^P_{<P}$.

LEMMA 4.1. *We have*

$$A^G + A^P_{<P} = t^N_P(A^P) + A^P_{<P},$$

$A^G_{<P} = A^P_{<P} \cap A^G$ *and* $\sqrt{A^G_{<P}} = \sqrt{A^P_{<P}} \cap A^G = t^G_P(\sqrt{A^P_{<P}})$.

Proof. We use the well known 'Mackey - formula' for trace operators:

$$t^G_P(b) = \sum_{g \in P\backslash G/P} t^P_{P\cap {}^gP}({}^gb)$$

where $b \in A^P$ and $P\backslash G/P$ denotes a set of double coset representatives for P in G. Each element of A^G is of the form $t^G_P(b)$ for a suitable $b \in A^P$ and for each $g \in G\backslash N$, the P - invariant $t^P_{P\cap {}^gP}({}^gb)$ is contained in $A^P_{<P}$. Hence we get

$$t^G_P(b) \equiv \sum_{g \in N/P} {}^gb \equiv t^N_P(b) \mod A^P_{<P}$$

and therefore

$$A^G + A^P_{<P} = t^G_P(A^P) + A^P_{<P} = t^N_P(A^P) + A^P_{<P}.$$

If $b = t^P_Q(b') \in A^P_{<P}$ for some $Q < P$ and $b' \in A^Q$, then ${}^nb = t^P_{{}^nQ}({}^nb') \in A^P_{<P}$ for each $n \in N$, which therefore stabilizes $A^P_{<P}$ and its radical ideal. Hence

$$A^G_{<P} = t^G_P(A^P_{<P}) \subseteq (t^N_P(A^P_{<P}) + A^P_{<P}) \cap A^G \subseteq A^P_{<P} \cap A^G \text{ and}$$

$$t_P^G(\sqrt{A^P_{<P}}) \subseteq (t_P^N(\sqrt{A^P_{<P}}) + A^P_{<P}) \cap A^G \subseteq \sqrt{A^P_{<P}} \cap A^G = \sqrt{A^P_{<P} \cap A^G}.$$

The remaining inclusions follow easily from the fact that the index $|G : P|$ is invertible in $\mathbb{F}$ and therefore each G - invariant is a relative trace from P. ▌

As a consequence of 4.1 and its proof we get the equality

$$\overline{A^P}^{\bar{N}} = t_1^{\bar{N}}(\overline{A^P}) = \overline{t_P^N(A^P)} = A^N/(A^P_{<P} \cap A^N).$$

For a homogeneous ideal I of a graded Noetherian ring, let $\beta(I)$ denote the minimal number m, such that I is generated by homogeneous elements of degree less or equal to m. We can now summarize the previous results:

THEOREM 4.2. *Let* $X = A^G/A^G_{<P}, Y = A^P/A^P_{<P}$ *or* $X = A^G/\sqrt{A^G_{<P}}$ *and* $Y = A^P/\sqrt{A^P_{<P}}$ *and* $\bar{N} := N_G(P)/P$. *Then*

$$X = Y^{\bar{N}} \text{ and } \beta(X) = \beta(Y^{\bar{N}}) \leq \beta(Y) \cdot |\bar{N}|.$$

In particular

$$\beta(A^G) \leq \max\{\beta(A^G_{<P}), \beta(A^P/A^P_{<P}) \cdot |\bar{N}|\} \leq \max\{\beta(A^G_{<P}), \beta(A^P) \cdot |\bar{N}|\}.$$

Proof. From 4.1 we see

$$A^G/A^G_{<P} = A^G/(A^P_{<P} \cap A^G) \cong A^N/(A^P_{<P} \cap A^N) \cong$$

$$t_1^{\bar{N}}(A^P/A^P_{<P}) \cong Y^{\bar{N}}$$

and the anaologue with $\sqrt{A^G_{<P}}$ instead of $A^G_{<P}$. The statements about the degree bounds of these quotient rings now follow easily from 3.1. If $(A^G_{<P})^+ = (a_1, \cdots, a_m)A^G$ and $A^G/A^G_{<P} = \mathbb{F}[\bar{a}_{m+1}, \cdots, \bar{a}_s]$, with homogeneous elements $a_i \in A^G$ of positive degree, then a standard argument, using induction on degrees, shows that $A^G = \mathbb{F}[a_1, \cdots, a_s]$. As $\beta(Y) \leq \beta(A^P)$ we obtain the stated estimate for $\beta(A^G)$. ▌

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