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Threshold dividend strategies for a Markov-additive risk model

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Abstract We consider the following risk reserve model. The premium income is a level dependent Markov-modulated Brownian motion. Claim sizes are iid with a phase-type distribution. The claim arrival process is a Markov-modulated Poisson process. For this model the payment of dividends under a threshold dividend strategy and the time until ruin will be analysed.

Keywords dividends · threshold strategy · Markov-additive risk model

1 Introduction

Threshold dividend strategies are sometimes optimal and therefore a popular object of interest in insurance mathematics, see e.g. [7, 10] for the compound Poisson model or [8] for a Brownian motion model. In a threshold strategy, no dividends are paid when the risk reserve is below a certain threshold, while above this threshold dividends are paid at a rate that is less than the rate of premium income. This has been generalised to more than one threshold with different rates of dividend payment (see e.g. [3]).

In the present paper we consider a Markov-additive risk model (to be specified below) with a finite number of thresholds $0 < b_1 < \ldots < b_N$. We derive the joint distribution (in terms of their joint Laplace transform) of the time until ruin and the time durations ζ_n that the risk reserve is between the thresholds b_{n-1} and b_n. This information suffices to compute the dividend payments in a threshold dividend strategy.

The premium income process shall be modelled by a level dependent Markov-modulated Brownian motion. Claim sizes are iid with a phase-type distribution. The

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claim arrival process is a Markov-modulated Poisson process. For an introduction to Markov-modulated processes, which are special Markov-additive processes, see chapter XI in [1]. We now proceed to specify the risk model to be considered.

Let $\tilde{\mathcal{J}} = (\tilde{J}_t : t \geq 0)$ denote an irreducible Markov process with a finite state space $\tilde{E} = \{1, \ldots, m\}$ and infinitesimal generator matrix $Q = (q_{ij})_{i,j \in \tilde{E}}$. We call $\tilde{J}_t$ the phase at time t. A level dependent Markov-modulated Brownian motion $(\mathcal{B}, \tilde{\mathcal{J}})$ with a finite number of thresholds $b_1, \ldots, b_N$ is defined by the stochastic differential equation

$$dB_t = \begin{cases} \mu_{J_t}^{(1)}\, dt + \sigma_{J_t} dW_t, & X_t \leq b_1 \\ \mu_{J_t}^{(k+1)}\, dt + \sigma_{J_t} dW_t, & b_k < X_t \leq b_{k+1}, 1 \leq k \leq N-1 \\ \mu_{J_t}^{(N+1)}\, dt + \sigma_{J_t} dW_t, & X_t > b_N \end{cases}$$

where $\mu_i^{(k)} \in \mathbb{R}$ and $\sigma_i > 0$ for $i \in \tilde{E}$, and $\mathcal{W} = (W_t : t \geq 0)$ denotes the standard Wiener process. Define the intervals $I_1 :=]-\infty, b_1]$ for $k = 1$, $I_k :=]b_{k-1}, b_k]$ for $k \in \{2, \ldots, N\}$, and $I_{N+1} :=]b_N, \infty[$ for $k = N+1$. We call I_k together with the parameters $\left(\mu_i^{(k)}, \sigma_i\right)$, $i \in \tilde{E}$, the kth regime of $(\mathcal{B}, \tilde{\mathcal{J}})$.

The process $(\mathcal{B}, \tilde{\mathcal{J}})$ shall serve as our model for the premium income. Typically, there is a constant rate $c_i\ dt$ of premium income, together with a perturbation $\sigma_i dW_t$. Above the threshold b_1, dividend payments would commence with a constant rate $c_i^{(1)} < c_i$. In a multi-threshold model, other rates $c_i^{(n)}$ of dividend payments would become effective as soon as the risk reserve surpasses the threshold b_n. This is typically constrained by $c_i^{(1)} < \ldots < c_i^{(N)} < c_i$, although this property is not a necessary assumption for the analysis to follow. We now define $\mu_i^{(1)} := c_i$ and $\mu_i^{(k+1)} := c_i - c_i^{(k)}$ for $k = 1, \ldots, N$ to arrive at the notation above.

We assume that claim sizes C_n, $n \in \mathbb{N}$, are iid with a phase-type distribution of order m_C and parameters (α, T). The methods presented in this paper would allow for claim size distributions to depend on the phase process $\tilde{\mathcal{J}}$. This, however, would complicate notations which are on the abundant side already. Thus we shall confine our analysis to iid claim sizes. We assume further that a claim occurs with a constant rate $\lambda_i dt$ when $\tilde{J}_t = i$. This means that the claim arrival process is a Markov-modulated Poisson process $(\mathcal{N}, \tilde{\mathcal{J}})$ with parameters $D_0 = Q - \Lambda$ and $D_1 = \Lambda$ where $\Lambda = diag(\lambda_i : i \in \tilde{E})$ is the diagonal matrix containing the rates λ_i.

Altogether our model for the risk reserve $\tilde{X}_t$ at time t is given by

$$\tilde{X}_t = u + B_t - \sum_{n=1}^{N_t} C_n$$

where $u = \tilde{X}_0$ denotes the initial risk reserve and $\mathcal{N} = (N_t : t \geq 0)$, i.e. N_t denotes the number of claims received until time t.

The process $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ is a level dependent Markov-additive process (MAP) with a generator matrix Q for the phase process $\tilde{\mathcal{J}}$ that is independent of the level. The parameters for the level process $\tilde{\mathcal{X}}$ in the kth regime are $(\tilde{\mu}_i^{(k)}, \tilde{\sigma}_i, \tilde{\nu}_i : i \in \tilde{E})$, where the Lévy measures $\tilde{\nu}_i(dx) = \lambda_i \mathbb{I}_{\{x<0\}}\ \alpha e^{-Tx} \eta\ dx$ are independent of the level. If

$N = 0$, i.e. if there is only one regime, we call the MAP homogeneous (in space). For literature on homogeneous MAPs see [1], chapter XI, and [5,9,6]. The non-perturbed case $\sigma_i = 0$ for $i \in \tilde{E}$ has been analysed in [2,3,11].

In the following section some useful results for homogeneous MAPs will be collected for ease of reference. Section 3 contains the analysis for the case $N = 1$, i.e. two regimes. In the last section, the results will be generalised to the case of a finite N.

2 Results for homogeneous MAPs

2.1 Markov–additive Processes with phase–type Jumps

In this section we construct a new MAP $(\mathcal{X}, \mathcal{J})$ from the given MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ without losing any information. This new MAP will have continuous paths which considerably simplifies the one- and two-sided exit problems (cf. sections 2.2 and 2.3).

Denote the indicator function of a set A by $\mathbb{I}_A$. Our assumption that the claim sizes have a phase-type distribution with parameters (α, T) leads to Lévy measures $\tilde{\nu}_i$ of the form

$$\tilde{\nu}_i(dx) = \lambda_i \mathbb{I}_{\{x<0\}} \; \alpha e^{-Tx} \eta \; dx \tag{1}$$

for all $i \in \tilde{E}$, where $\lambda_i \geq 0$. The column vector $\eta := -T\mathbf{1}$ is called the exit vectors, where $\mathbf{1}$ denotes the column vector of dimension m with all entries being 1.

The main advantage of the phase–type restriction on the jump distributions is the possibility of transforming the jumps into a succession of linear pieces of exponential duration (each with slope 1 or -1), which yields a modified MAP with continuous paths. We can then retrieve the original process via a simple time change. This is explained in detail in sections 2.1 and 2.2 of [6]. Here we shall present only the pertinent information to make the present paper self-contained.

Without the jumps, the Lévy process $\tilde{\mathcal{X}}^{(i)}$ during a phase $i \in \tilde{E}$ is simply a Brownian motion with parameters $\tilde{\sigma}_i > 0$ and $\tilde{\mu}_i > 0$. Write $E_\sigma := \tilde{E}$. Now we introduce a new phase space

$$E_- := \{(i,k) : i \in E_\sigma, 1 \leq k \leq m\} \tag{2}$$

to model the jumps. Define now the enlarged phase space $E = E_\sigma \cup E_-$. We define the modified MAP $(\mathcal{X}, \mathcal{J})$ over the phase space E as follows. Set the parameters $(\mu_i, \sigma_i^2, \nu_i)$ for $i \in E$ as

$$(\mu_i, \sigma_i^2, \nu_i) := \begin{cases} (-1, 0, \mathbf{0}), & i \in E_- \\ (\tilde{\mu}_i, \tilde{\sigma}_i, \mathbf{0}), & i \in E_\sigma \end{cases} \tag{3}$$

The modified phase process $\mathcal{J}$ is determined by its generator matrix $Q = (q_{ij})_{i,j \in E}$. For this the construction above yields

$$q_{ih} = \begin{cases} \tilde{q}_{ii} - \lambda_i, & h = i \in E_\sigma \\ \tilde{q}_{ih}, & h \in E_\sigma, h \neq i \\ \lambda_i \alpha_k, & h = (i,k) \in E_- \end{cases} \tag{4}$$

for $i \in E_\sigma$ as well as

$$q_{(i,k),(i,l)} = T_{kl} \qquad \text{and} \qquad q_{(i,k),i} = \eta_k \tag{5}$$

for $i \in E_\sigma$ and $1 \leq k, l \leq m$.

The original level process $\tilde{\mathcal{X}}$ is retrieved via the time change

$$c(t) := \int_0^t \mathbb{I}_{\{J_s \in E_\sigma\}}\, ds \quad \text{and} \quad \tilde{X}_{c(t)} = X_t \tag{6}$$

for all $t \geq 0$. Thus we obtain

$$\tilde{\tau}(a) = c(\tau(a)) \tag{7}$$

for $a \in \mathbb{R}$ and $\tau(a) := \inf\{t \geq 0 : X_t < a\}$. This implies that we can perform an analysis of the MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ in terms of the modified MAP $(\mathcal{X}, \mathcal{J})$ alone.

2.2 First Passage Times

A derivation of the Laplace transforms for the first passage times of MAPs has been given in [5]. Define the first passage times

$$\tilde{\sigma}(x) := \inf\{t \geq 0 : \tilde{X}_t > x\} \quad \text{and} \quad \sigma(x) := \inf\{t \geq 0 : X_t > x\}$$

for all $x \in \mathbb{R}$. Note that $\tilde{\sigma}(x)$ is the first passage time over the level x for the original MAP $\tilde{\mathcal{X}}$, meaning that we do not count the time spent in jump phases $i \in E_-$. This means that

$$\tilde{\sigma}(x) = c(\sigma(x)) = \int_0^{\sigma(x)} \mathbb{I}_{\{J_s \in E_\sigma\}} ds$$

according to (6). In particular, we may compute expectations over $\tilde{\sigma}(x)$ using the distribution of the modified MAP $(\mathcal{X}, \mathcal{J})$ only and without needing to recur to the original MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$. For $\gamma \geq 0$ denote

$$\mathbb{E}_{ij}(e^{-\gamma\tilde{\sigma}(x)}) := \mathbb{E}(e^{-\gamma\tilde{\sigma}(x)}; J_{\tau(x)} = j | J_0 = i, X_0 = 0)$$

for all $i, j \in E$. Let $\mathbb{E}(e^{-\gamma\tilde{\sigma}(x)})$ denote the matrix with these entries and write

$$\mathbb{E}(e^{-\gamma\tilde{\sigma}(x)}) = \begin{pmatrix} \mathbb{E}_{(\sigma,\sigma)}(e^{-\gamma\tilde{\sigma}(x)}) & \mathbb{E}_{(\sigma,-)}(e^{-\gamma\tilde{\sigma}(x)}) \\ \mathbb{E}_{(-,\sigma)}(e^{-\gamma\tilde{\sigma}(x)}) & \mathbb{E}_{(-,-)}(e^{-\gamma\tilde{\sigma}(x)}) \end{pmatrix}$$

in obvious block notation with respect to the subspaces E_σ (ascending phases) and E_- (descending phases). According to section 3 in [5] we can write

$$\mathbb{E}(e^{-\gamma\tilde{\sigma}(x)}) = \begin{pmatrix} I_\sigma \\ A(\gamma) \end{pmatrix} \begin{pmatrix} e^{U(\gamma)x} & \mathbf{0} \end{pmatrix} \tag{8}$$

where I_σ denotes the identity matrix of dimension $E_\sigma \times E_\sigma$, $\mathbf{0}$ the zero matrix of dimension $E_\sigma \times E_-$, $U(\gamma)$ is a sub–generator matrix of dimension $E_\sigma \times E_\sigma$, and $A(\gamma)$ is a sub–transition matrix of dimension $E_- \times E_\sigma$. An iteration to determine

$A(\gamma)$ and $U(\gamma)$ is derived in [5] and further specified to the case of phase-type jumps in [6].

In order to determine the downward first passage times (in particular the time of ruin), we reflect at the original level X_0 and consider upward first passage times for the negative of $\mathcal{X}$. Let $(\mathcal{X}^+, \mathcal{J})$ denote the MAP as constructed in section 2.1 and define the process $\mathcal{X}^- = (X_t^- : t \geq 0)$ by $X_t^- := -X_t^+$ for all $t > 0$ given that $X_0^+ = X_0^- = 0$. Thus $(\mathcal{X}^-, \mathcal{J})$ is the negative of $(\mathcal{X}^+, \mathcal{J})$. The two processes have the same generator matrix Q for $\mathcal{J}$, but the drift parameters are different. Denoting variation and drift parameters for $\mathcal{X}^\pm$ by $\sigma_i^\pm$ and $\mu_i^\pm$, respectively, this means $\sigma_i^+ = \sigma_i^-$ and $\mu_i^- = -\mu_i^+$ for all $i \in E$. This of course implies that phases $i \in E_-$ are ascending phases for $\mathcal{X}^-$.

Let $A^\pm(\gamma)$ and $U^\pm(\gamma)$ denote the matrices that determine the first passage times in (8). We shall write $A^\pm = A^\pm(\gamma)$ and $U^\pm = U^\pm(\gamma)$ except in cases when we wish to underline the dependence on γ. Note that in our case all phases are ascending for $\mathcal{X}^-$ such that A^- vanishes, i.e. has dimension 0. Define the (downward) first passage times

$$\tilde{\tau}(x) := \inf\{t \geq 0 : \tilde{X}_t < x\} \quad \text{and} \quad \tau(x) := \inf\{t \geq 0 : X_t < x\}$$

for all $x \in \mathbb{R}$. We now obtain

$$\mathbb{E}(e^{-\gamma\tilde{\tau}(x)}|X_0 = a) = e^{U^-(\gamma)\cdot(a-x)} \tag{9}$$

for all $x < a$.

2.3 The two-sided Exit Problem

For $l < u$, define the stopping times

$$\sigma(l,u) := \inf\{t \geq 0 : X_t < l \quad \text{or} \quad X_t > u\} \tag{10}$$

and

$$\tilde{\sigma}(l,u) := \int_0^{\sigma(l,u)} \mathbb{I}_{\{J_s \in E_\sigma\}} ds = \inf\{t \geq 0 : \tilde{X}_t < l \quad \text{or} \quad \tilde{X}_t > u\} \tag{11}$$

which are the exit times of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ from the interval $[l, u]$, respectively. Choose any $\gamma \geq 0$. For the main result we need an expression for

$$\Psi_{ij}^+(u-l|x-l) := \mathbb{E}\left(e^{-\gamma\tilde{\sigma}(l,u)}; X_{\sigma(l,u)} = b, J_{\sigma(l,u)} = j | J_0 = i, X_0 = x\right)$$

where $x \in [l, u]$ and $i, j \in E$. Clearly $\Psi_{ij}^+(u-l|x-l) = 0$ for $j \in E_-$ since an exit over the upper boundary can occur only in an ascending phase. Define the matrix $\Psi^+(u-l|x-l) := (\Psi_{ij}^+(u-l|x-l))_{i\in E, j\in E_\sigma}$. A formula for $\Psi^+(u-l|x-l)$ has been derived in [9]. In order to state it we need some additional notation. Define the matrices

$$C^+ := \begin{pmatrix} I_\sigma \\ A^+ \end{pmatrix} \qquad \text{and} \qquad C^- := \begin{pmatrix} I_\sigma & \mathbf{0} \end{pmatrix} \tag{12}$$

of dimensions $E \times E_\sigma$ and $E_\sigma \times E$, respectively, where I_σ denotes the identity matrix of dimension $E_\sigma \times E_\sigma$. Further let $Z^\pm := C^\pm e^{U^\pm \cdot (u-l)}$. Then equation (23) in [9] states that

$$\Psi^+(u-l|x-l) = \left(C^+ e^{U^+ \cdot (u-x)} - e^{U^- \cdot (x-l)} Z^+\right) \cdot \left(I - Z^- Z^+\right)^{-1} \qquad (13)$$

for $0 \le x \le b$. This matrix has dimension $E \times E_\sigma$, due to the fact that exit from below can only happen in an ascending phase. By reflection at the initial level x, we obtain further

$$\begin{aligned}\Psi^-(u-l|x-l) &:= \mathbb{E}\left(e^{-\gamma\tilde{\sigma}(l,u)}; X_{\sigma(l,u)} = 0 | X_0 = x\right) \\ &= \left(e^{U^- \cdot (x-l)} - C^+ e^{U^+ \cdot (u-x)} Z^-\right) \cdot \left(I - Z^+ Z^-\right)^{-1} \qquad (14)\end{aligned}$$

for $x \in [l,u]$. This matrix has dimension $E \times E$. Note that the expressions on the right-hand sides of (13) and (14) depend on a choice of $\gamma \ge 0$.

3 The single threshold case

We first consider the time of ruin for the case $N = 1$, i.e. one threshold only. Denote the level of this threshold by $b > 0$. The time of ruin is defined as

$$\tilde{\tau}(0) := \inf\{t \ge 0 : \tilde{X}_t < 0\} \qquad (15)$$

We seek to find an expression for $\mathbb{E}\left(e^{-\gamma\tilde{\tau}(0)}|X_0 = u\right)$ where $\gamma \ge 0$ and u denotes the initial risk reserve. Let $U_i^\pm = U_i^\pm(\gamma)$, $A_i^\pm = A_i^\pm(\gamma)$,and $\Psi_i^\pm = \Psi_i^\pm(\gamma)$ denote the matrices introduced in section 2 for the ith regime, where $i = 1$ means $X_t < b$ and $i = 2$ means $X_t \ge b$.

In the case $u < b$ we obtain

$$\mathbb{E}\left(e^{-\gamma\tilde{\tau}(0)}|X_0 = u\right) = \Psi_1^-(b|u) + \Psi_1^+(b|u)\, \mathbb{E}\left(e^{-\gamma\tilde{\tau}(0)}|X_0 = b\right)$$

while for $u > b$ path continuity of $\mathcal{X}$ yields

$$\mathbb{E}\left(e^{-\gamma\tilde{\tau}(0)}|X_0 = u\right) = e^{U_2^- \cdot (u-b)}\, \mathbb{E}\left(e^{-\gamma\tilde{\tau}(0)}|X_0 = b\right)$$

Thus it suffices to determine $\mathbb{E}\left(e^{-\gamma\tilde{\tau}(0)}|X_0 = b\right)$. Write

$$\mathbb{E}\left(e^{-\gamma\tilde{\tau}(0)}|X_0 = b\right) =: \begin{pmatrix} E_{(\sigma,\sigma)}(b) & E_{(\sigma,-)}(b) \\ E_{(-,\sigma)}(b) & E_{(-,-)}(b) \end{pmatrix} =: \begin{pmatrix} E_{(\sigma,\cdot)}(b) \\ E_{(-,\cdot)}(b) \end{pmatrix} \qquad (16)$$

in obvious block notation. In general we shall use for any matrix M of dimension $E \times E$ the block notation

$$M =: \begin{pmatrix} M_{(\sigma,\sigma)} & M_{(\sigma,-)} \\ M_{(-,\sigma)} & M_{(-,-)} \end{pmatrix} =: \begin{pmatrix} M_{(\sigma,\cdot)} \\ M_{(-,\cdot)} \end{pmatrix} =: \begin{pmatrix} M_{(.,\sigma)} & M_{(.,-)} \end{pmatrix}$$

Then

$$E_{(-,\cdot)}(b) = \Psi_1^-(b|b)_{(-,\cdot)} + \Psi_1^+(b|b)_{(-,\sigma)} E_{(\sigma,\cdot)}(b) \qquad (17)$$

Thus it remains to determine $E_{(\sigma,\sigma)}(b)$ and $E_{(\sigma,-)}(b)$. This will be pursued in theorem 1, for which we state two lemmata first.

Lemma 1 Write

$$\Psi_1^+(b+\varepsilon|b-\varepsilon) = \begin{pmatrix} H^+_{(\sigma,\sigma)}(\varepsilon) & H^+_{(\sigma,-)}(\varepsilon) \\ H^+_{(-,\sigma)}(\varepsilon) & H^+_{(-,-)}(\varepsilon) \end{pmatrix}$$

in block notation. Then

$$\begin{aligned} H_1^+ &:= \frac{d}{d\varepsilon} H^+_{(\sigma,\sigma)}(\varepsilon)\bigg|_{\varepsilon=0} \\ &= 2\left(U_1^+ e^{-U_1^+\cdot b} + C_1^- U_1^- e^{U_1^-\cdot b} C_1^+\right)\left(e^{-U_1^+\cdot b} - C_1^- e^{U_1^-\cdot b} C_1^+\right)^{-1} \end{aligned}$$

Proof: According to (13),

$$\begin{aligned} H^+_{(\sigma,\sigma)}(\varepsilon) &= \left(e^{U_1^+\cdot 2\varepsilon} - C_1^- e^{U_1^-\cdot(b-\varepsilon)} C_1^+ e^{U_1^+\cdot(b+\varepsilon)}\right) \\ &\quad \times \left(I_\sigma - C_1^- e^{U_1^-\cdot(b+\varepsilon)} C_1^+ e^{U_1^+\cdot(b+\varepsilon)}\right)^{-1} \\ &= \left(e^{-U_1^+\cdot(b-\varepsilon)} - C_1^- e^{U_1^-\cdot(b-\varepsilon)} C_1^+\right) \\ &\quad \times \left(e^{-U_1^+\cdot(b+\varepsilon)} - C_1^- e^{U_1^-\cdot(b+\varepsilon)} C_1^+\right)^{-1} \end{aligned}$$

After abbreviating

$$F(\varepsilon) := \left(e^{-U_1^+\cdot(b-\varepsilon)} - C_1^- e^{U_1^-\cdot(b-\varepsilon)} C_1^+\right)$$

and

$$G(\varepsilon) := \left(e^{-U_1^+\cdot(b+\varepsilon)} - C_1^- e^{U_1^-\cdot(b+\varepsilon)} C_1^+\right)$$

we apply the formal rules of derivation for functions of a real variable (see [4], sections I.1.3-4) to obtain

$$H_1^+ = F'(0)G(0)^{-1} - F(0)G(0)^{-1}G'(0)G(0)^{-1}$$

where

$$F(0) = e^{-U_1^+\cdot b} - C_1^- e^{U_1^-\cdot b} C_1^+, \quad F'(0) = U_1^+ e^{-U_1^+\cdot b} + C_1^- U_1^- e^{U_1^-\cdot b} C_1^+$$

and

$$G(0) = e^{-U_1^+\cdot b} - C_1^- e^{U_1^-\cdot b} C_1^+, \quad G'(0) = -U_1^+ e^{-U_1^+\cdot b} - C_1^- U_1^- e^{U_1^-\cdot b} C_1^+$$

Thus $F(0) = G(0)$ and $G'(0) = -F'(0)$, which yields the statement.
□

Lemma 2 Write

$$\Psi_1^-(b+\varepsilon|b-\varepsilon) = \begin{pmatrix} W^-_{(\sigma,\sigma)}(\varepsilon) & W^-_{(\sigma,-)}(\varepsilon) \\ W^-_{(-,\sigma)}(\varepsilon) & W^-_{(-,-)}(\varepsilon) \end{pmatrix}$$

in block notation. Then

$$W_1^- := \frac{d}{d\varepsilon} W^-_{(\sigma,\cdot)}(\varepsilon)\bigg|_{\varepsilon=0} = -2\left(C_1^- U_1^- + U_1^+ C_1^-\right)\left(e^{-U_1^-\cdot b} - C_1^+ e^{U_1^+\cdot b} C_1^-\right)^{-1}$$

Proof: The proof is analogous to lemma 1. According to (14),

$$\begin{aligned}W^-_{(\sigma,\cdot)}(\varepsilon) &= \left(C_1^- e^{U_1^-\cdot(b-\varepsilon)} - e^{U_1^+\cdot 2\varepsilon} C_1^- e^{U_1^-\cdot(b+\varepsilon)}\right) \\ &\quad \times \left(I - C^+ e^{U_1^+\cdot(b+\varepsilon)} C_1^- e^{U_1^-\cdot(b+\varepsilon)}\right)^{-1} \\ &= \left(C_1^- e^{-U_1^-\cdot 2\varepsilon} - e^{U_1^+\cdot 2\varepsilon} C_1^-\right) \\ &\quad \times \left(e^{-U_1^-\cdot(b+\varepsilon)} - C^+ e^{U_1^+\cdot(b+\varepsilon)} C_1^-\right)^{-1}\end{aligned}$$

Write $F(\varepsilon) = C_1^- e^{-2U_1^-\varepsilon} - e^{2U_1^+\varepsilon} C_1^-$ and $G(\varepsilon) = e^{-U_1^-\cdot(b+\varepsilon)} - C_1^+ e^{U_1^+\cdot(b+\varepsilon)} C_1^-$ to obtain

$$\begin{aligned}\left.\frac{d}{d\varepsilon} W^-_{(\sigma,\cdot)}(\varepsilon)\right|_{\varepsilon=0} &= \left.\frac{d}{d\varepsilon} F(\varepsilon) G(\varepsilon)^{-1}\right|_{\varepsilon=0} \\ &= F'(0)G(0)^{-1} - F(0)G(0)^{-1}G'(0)G(0)^{-1}\end{aligned}$$

according to [4], sections I.1.3-4, where $F(0) = \mathbf{0}$, $F'(0) = -2\left(C_1^- U_1^- + U_1^+ C_1^-\right)$ and further $G(0) = e^{-U_1^-\cdot b} - C_1^+ e^{U_1^+\cdot b} C_1^-$. Altogether this yields the statement.
□

Theorem 1 *Write $E_{(\sigma,\cdot)}(b) := \left(E_{(\sigma,\sigma)}(b)\ E_{(\sigma,-)}(b)\right)$ for the first row in (16). Then*

$$\begin{aligned}E_{(\sigma,\cdot)}(b) &= \left(2\left(U_2^-\right)_{(\sigma,-)} \Psi_1^+(b|b)_{(-,\sigma)} - H_1^+ - 2\left(U_2^-\right)_{(\sigma,\sigma)}\right)^{-1} \\ &\quad \times \left(W_1^- + 2\left(U_2^-\right)_{(\sigma,-)} \Psi_1^-(b|b)_{(-,\cdot)}\right)\end{aligned}$$

Proof: We consider $E(b-\varepsilon) := \mathbb{E}\left(e^{-\gamma\tilde{\tau}(0)}|X_0 = b-\varepsilon\right)$ and assume that the regime changes at $b+\varepsilon$ for upward crossings of b and at $b-\varepsilon$ for downward crossings. Then we let $\varepsilon \downarrow 0$. Due to (17), we need to determine the upper row $E_{(\sigma,\cdot)}(b-\varepsilon)$ only. First we obtain

$$\begin{aligned}E_{(\sigma,\cdot)}(b-\varepsilon) &= W^-_{(\sigma,\cdot)}(\varepsilon) + H^+_{(\sigma,\sigma)}(\varepsilon) C^- e^{U_2^-\cdot 2\varepsilon} E(b-\varepsilon) \\ &= W^-_{(\sigma,\cdot)}(\varepsilon) + H^+_{(\sigma,\sigma)}(\varepsilon) C^- \left(e^{U_2^-\cdot 2\varepsilon}\right)_{(\cdot,-)} E_{(-,\cdot)}(b-\varepsilon) \\ &\quad + H^+_{(\sigma,\sigma)}(\varepsilon) C^- \left(e^{U_2^-\cdot 2\varepsilon}\right)_{(\cdot,\sigma)} E_{(\sigma,\cdot)}(b-\varepsilon)\end{aligned}$$

where $H^+_{(\sigma,\sigma)}(\varepsilon)$ and $W^-_{(\sigma,\cdot)}(\varepsilon)$ are defined in lemmata 1 and 2. This implies

$$\begin{aligned}E_{(\sigma,\cdot)}(b-\varepsilon) &= \left(I_\sigma - H^+_{(\sigma,\sigma)}(\varepsilon) C^- \left(e^{U_2^-\cdot 2\varepsilon}\right)_{(\cdot,\sigma)}\right)^{-1} \cdot \varepsilon \\ &\quad \times \varepsilon^{-1} \left(W^-_{(\sigma,\cdot)}(\varepsilon) + H^+_{(\sigma,\sigma)}(\varepsilon) C^- \left(e^{U_2^-\cdot 2\varepsilon}\right)_{(\cdot,-)} E_{(-,\cdot)}(b-\varepsilon)\right)\end{aligned}$$

We observe that

$$\lim_{\varepsilon\downarrow 0} H^{+}_{(\sigma,\sigma)}(\varepsilon) = \lim_{\varepsilon\downarrow 0} C^{-}\left(e^{U_2^{-}\cdot 2\varepsilon}\right)_{(\cdot,\sigma)} = I_\sigma$$

$$\lim_{\varepsilon\downarrow 0} W^{-}_{(\sigma,\cdot)}(\varepsilon) = \mathbf{0}, \quad \lim_{\varepsilon\downarrow 0} C^{-}\left(e^{U_2^{-}\cdot 2\varepsilon}\right)_{(\cdot,-)} = \mathbf{0}$$

$$\lim_{\varepsilon\downarrow 0} E_{(-,\cdot)}(b-\varepsilon) = E_{(-,\cdot)}(b)$$

where **0** denotes a zero matrix of appropriate dimension. Thus we can write

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \varepsilon \left(I_\sigma - H^{+}_{(\sigma,\sigma)}(\varepsilon) C^{-}\left(e^{U_2^{-}\cdot 2\varepsilon}\right)_{(\cdot,\sigma)}\right)^{-1} \\
&\quad = -\left(\frac{d}{d\varepsilon} H^{+}_{(\sigma,\sigma)}(\varepsilon) C^{-}\left(e^{U_2^{-}\cdot 2\varepsilon}\right)_{(\cdot,\sigma)}\Big|_{\varepsilon=0}\right)^{-1} \\
&\quad = -\left(\frac{d}{d\varepsilon} H^{+}_{(\sigma,\sigma)}(\varepsilon)\Big|_{\varepsilon=0} I_\sigma + I_\sigma \frac{d}{d\varepsilon} C^{-}\left(e^{U_2^{-}\cdot 2\varepsilon}\right)_{(\cdot,\sigma)}\Big|_{\varepsilon=0}\right)^{-1} \\
&\quad = -\left(H_1^{+} + 2\left(U_2^{-}\right)_{(\sigma,\sigma)}\right)^{-1}
\end{aligned}$$

see [4], section I.1.3, and lemma 1 for the last two equalities. In a similar manner,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\, W^{-}_{(\sigma,\cdot)}(\varepsilon) = \frac{d}{d\varepsilon} W^{-}_{(\sigma,\cdot)}(\varepsilon)\Big|_{\varepsilon=0} = W_1^{-}$$

according to lemma 2. Finally,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\, H^{+}_{(\sigma,\sigma)}(\varepsilon) C^{-}\left(e^{U_2^{-}\cdot 2\varepsilon}\right)_{(\cdot,-)} E^{-}_{(-,\cdot)}(b-\varepsilon) \\
&\quad = I_\sigma \left(\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\, C^{-}\left(e^{U_2^{-}\cdot 2\varepsilon}\right)_{(\cdot,-)}\right) E_{(-,\cdot)}(b) \\
&\quad = 2\left(U_2^{-}\right)_{(\sigma,-)} E_{(-,\cdot)}(b)
\end{aligned}$$

Pasting the above results together, the limit $\varepsilon \downarrow 0$ yields

$$\begin{aligned}
E_{(\sigma,\cdot)}(b) &= -\left(H_1^{+} + 2\left(U_2^{-}\right)_{(\sigma,\sigma)}\right)^{-1} \left(W_1^{-} + 2\left(U_2^{-}\right)_{(\sigma,-)} E_{(-,\cdot)}(b)\right) \\
&= -\left(H_1^{+} + 2\left(U_2^{-}\right)_{(\sigma,\sigma)}\right)^{-1} \left(W_1^{-} + 2\left(U_2^{-}\right)_{(\sigma,-)} \Psi_1^{-}(b|b)_{(-,\cdot)}\right) \\
&\quad - \left(H_1^{+} + 2\left(U_2^{-}\right)_{(\sigma,\sigma)}\right)^{-1} 2\left(U_2^{-}\right)_{(\sigma,-)} \Psi_1^{+}(b|b)_{(-,\sigma)} E_{(\sigma,\cdot)}(b)
\end{aligned}$$

after using (17). Thus

$$\begin{aligned}
E_{(\sigma,\cdot)}(b) &= -\left(I_\sigma - \left(H_1^{+} + 2\left(U_2^{-}\right)_{(\sigma,\sigma)}\right)^{-1} 2\left(U_2^{-}\right)_{(\sigma,-)} \Psi_1^{+}(b|b)_{(-,\sigma)}\right)^{-1} \\
&\quad \times \left(H_1^{+} + 2\left(U_2^{-}\right)_{(\sigma,\sigma)}\right)^{-1} \left(W_1^{-} + 2\left(U_2^{-}\right)_{(\sigma,-)} \Psi_1^{-}(b|b)_{(-,\cdot)}\right) \\
&= -\left(H_1^{+} + 2\left(U_2^{-}\right)_{(\sigma,\sigma)} - 2\left(U_2^{-}\right)_{(\sigma,-)} \Psi_1^{+}(b|b)_{(-,\sigma)}\right)^{-1} \\
&\quad \times \left(W_1^{-} + 2\left(U_2^{-}\right)_{(\sigma,-)} \Psi_1^{-}(b|b)_{(-,\cdot)}\right)
\end{aligned}$$

which is the expression in the statement.

Considering now $E(b+\varepsilon) = \mathbb{E}\left(e^{-\gamma\tau(u)}|X_0 = b+\varepsilon\right)$ instead of $E(b-\varepsilon)$ as above, we observe that

$$E_{(\sigma,\cdot)}(b+\varepsilon) = C^- e^{U_2^- 2\varepsilon} E(b-\varepsilon)$$

due to path continuity. Since $\lim_{\varepsilon\downarrow 0} e^{U_2^- \cdot 2\varepsilon} = I$, we obtain

$$\lim_{\varepsilon\downarrow 0} E_{(\sigma,\cdot)}(b+\varepsilon) = \lim_{\varepsilon\downarrow 0} \begin{pmatrix} I_\sigma & \mathbf{0} \end{pmatrix} E(b-\varepsilon) = \lim_{\varepsilon\downarrow 0} E_{(\sigma,\cdot)}(b-\varepsilon)$$

such that the limits from both sides coincide.
□

Let $\zeta(b) := l\left(\{t < \tilde{\tau}(0) : \tilde{X}_t > b\}\right)$ denote the Lebesgue measure of the time before ruin that the risk reserve process spends above the threshold b. Then the dividends paid out before ruin amount to $D = c_1\zeta(b)$. We now wish to state the joint distribution of the time to ruin and the time spent above b in terms of their joint Laplace transform.

Corollary 1

$$\begin{aligned}
&\mathbb{E}\left(e^{-\gamma\tilde{\tau}(0)-\delta\zeta(b)}|X_0 = b\right) \\
&\quad = \left(2U_2^-(\gamma+\delta)_{(\sigma,-)}\Psi_1^+(b,\gamma|b)_{(-,\sigma)} - 2U_2^-(\gamma+\delta)_{(\sigma,\sigma)} - H_1^+(\gamma)\right)^{-1} \\
&\qquad \times \left(2U_2^-(\gamma+\delta)_{(\sigma,-)}\Psi_1^-(b,\gamma|b)_{(-,\cdot)} + W_1^-(\gamma)\right)
\end{aligned}$$

Proof: We integrate over the same set of sample paths as in theorem 1, only with the additional integrand function $e^{-\delta t}$ when $\tilde{X}_t > b$. Thus the only difference to the statement in theorem 1 is the Laplace argument $\gamma+\delta$ for U_2^-.
□

4 Multi-threshold strategies

We now consider a Markov-additive risk model $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ with a finite number of thresholds $b_1, \ldots, b_N$. Define

$$E^-(a,l) := \mathbb{E}\left(e^{-\gamma\tilde{\tau}(l)-\sum_{n=1}^{N+1}\delta_n\zeta_n}|X_0 = a\right)$$

for $l < a$, where $\delta_n \geq 0$ for $n = 1, \ldots, N+1$ and

$$\zeta_n := l\left(\{t < \tilde{\tau}(0) : b_{n-1} \leq \tilde{X}_t < b_n\}\right)$$

denotes the Lebesgue measure of the time spent in the interval $[b_{n-1}, b_n[$ before ruin, with $b_0 := 0$ and $b_{N+1} := \infty$. Path continuity of $\mathcal{X}$ yields

$$E^-(u,l) = E^-(u,a)E^-(a,l)$$

for all $l < a < u$. For $u \geq b_N$, $E^-(u, b_{N-1})$ has been determined in section 3 (set $b := b_N - b_{N-1}$). We wish to determine

$$E^-(u, 0) = E^-(u, b_k)E^-(b_k, b_{k-1}) \dots E^-(b_1, 0)$$

where $k := \max\{n \leq N : b_n < u\}$. In order to pursue this, let $U_k^\pm = U_k^\pm(\gamma + \delta_k)$, $A_k^\pm = A_k^\pm(\gamma + \delta_k)$, and $\Psi_k^\pm = \Psi_k^\pm(\gamma + \delta_k)$ denote the matrices introduced in section 2 where the parameters are taken from the kth regime, with $k = 1, \dots, N + 1$.

First note that

$$E^-(u, b_k) = \Psi_{k+1}^-(b_{k+1} - b_k | u - b_k) + \Psi_{k+1}^+(b_{k+1} - b_k | u - b_k)E^-(b_{k+1}, b_k)$$

such that it suffices to determine the matrices $E^-(b_k, b_{k-1})$ for $k = 1, \dots, N - 1$. Recalling the definitions (10) and (11), define the matrices

$$E^+(k) := \mathbb{E}\left(e^{-\gamma\tilde{\sigma}(b_{k-1}, b_{k+1}) - \sum_{n=1}^{N+1} \delta_n \zeta_n}; X_{\sigma(b_{k-1}, b_{k+1})} = u | X_0 = b_k\right)$$

and

$$E^-(k) := \mathbb{E}\left(e^{-\gamma\tilde{\sigma}(b_{k-1}, b_{k+1}) - \sum_{n=1}^{N+1} \delta_n \zeta_n}; X_{\sigma(b_{k-1}, b_{k+1})} = l | X_0 = b_k\right)$$

for $k = 1, \dots, N - 1$. Since

$$E^-(b_k, b_{k-1}) = E^-(k) + E^+(k)E^-(b_{k+1}, b_k)E^-(b_k, b_{k-1})$$

we obtain

$$E^-(b_k, b_{k-1}) = \left(I - E^+(k)E^-(b_{k+1}, b_k)\right)^{-1} E^-(k)$$

for $k \leq N-1$. This will provide a recursion scheme for $E^-(u, 0)$ if we can determine $E^+(k)$ and $E^-(k)$ for $k \leq N - 1$.

4.1 Determine $E^+(k)$

We first observe that $E^+_{(\cdot,-)}(k) = \mathbf{0}$ as an upward exit from an interval cannot happen in a descending phase. Further,

$$E^+_{(-,\sigma)}(k) = \Psi_k^+(b_k - b_{k-1} | b_k - b_{k-1})_{(-,\sigma)} E^+_{(\sigma,\sigma)}(k)$$

such that it remains to determine $E^+_{(\sigma,\sigma)}(k)$. We shall pursue this in theorem 2 but need some additional lemmata before.

Lemma 3 For $k \leq N - 1$, write

$$\Psi_{k+1}^-(b_{k+1} - b_k + \varepsilon | 2\varepsilon) = \begin{pmatrix} H_{k+1}^-(\varepsilon)_{(\sigma,\sigma)} & H_{k+1}^-(\varepsilon)_{(\sigma,-)} \\ H_{k+1}^-(\varepsilon)_{(-,\sigma)} & H_{k+1}^-(\varepsilon)_{(-,-)} \end{pmatrix}$$

in block notation. Then

$$\begin{aligned} H_{k+1}^- &:= \left.\frac{d}{d\varepsilon} H_{k+1}^-(\varepsilon)_{(\sigma,\cdot)}\right|_{\varepsilon=0} \\ &= 2\Big[\Big(U_{k+1}^- e^{-U_{k+1}^-\cdot(b_{k+1}-b_k)} + C_{k+1}^+ U_{k+1}^+ e^{U_{k+1}^+\cdot(b_{k+1}-b_k)} C^-\Big) \\ &\quad \times \Big(e^{-U_{k+1}^-\cdot(b_{k+1}-b_k)} - C_{k+1}^+ e^{U_{k+1}^+\cdot(b_{k+1}-b_k)} C^-\Big)^{-1}\Big]_{(\sigma,\cdot)} \end{aligned}$$

Proof: According to (14),

$$\begin{aligned} H_{k+1}^-(\varepsilon)_{(\sigma,\cdot)} &= \Big(C^- e^{U_{k+1}^- 2\varepsilon} - e^{U_2^+\cdot(b_{k+1}-b_k-\varepsilon)} C^- e^{U_{k+1}^-\cdot(b_{k+1}-b_k+\varepsilon)}\Big) \\ &\quad \times \Big(I - C_{k+1}^+ e^{U_{k+1}^+\cdot(b_{k+1}-b_k+\varepsilon)} C^- e^{U_{k+1}^-\cdot(b_{k+1}-b_k+\varepsilon)}\Big)^{-1} \\ &= C^- \Big(e^{-U_{k+1}^-\cdot(b_{k+1}-b_k-\varepsilon)} - C_{k+1}^+ e^{U_{k+1}^+\cdot(b_{k+1}-b_k-\varepsilon)} C^-\Big) \\ &\quad \times \Big(e^{-U_{k+1}^-\cdot(b_{k+1}-b_k+\varepsilon)} - C_{k+1}^+ e^{U_{k+1}^+\cdot(b_{k+1}-b_k+\varepsilon)} C^-\Big)^{-1} \end{aligned}$$

After abbreviating $F(\varepsilon) := e^{-U_{k+1}^-\cdot(b_{k+1}-b_k-\varepsilon)} - C_{k+1}^+ e^{U_{k+1}^+\cdot(b_{k+1}-b_k-\varepsilon)} C^-$ as well as $G(\varepsilon) := e^{-U_{k+1}^-\cdot(b_{k+1}-b_k+\varepsilon)} - C_{k+1}^+ e^{U_{k+1}^+\cdot(b_{k+1}-b_k+\varepsilon)} C^-$ we apply the formal rules of derivation for functions of a real variable (see [4], sections I.1.3-4) to obtain

$$H_{k+1}^- = C^- \left(F'(0)G(0)^{-1} - F(0)G(0)^{-1}G'(0)G(0)^{-1}\right)$$

where

$$\begin{aligned} F(0) &= e^{-U_{k+1}^-\cdot(b_{k+1}-b_k)} - C_{k+1}^+ e^{U_{k+1}^+\cdot(b_{k+1}-b_k)} C^- = G(0) \\ F'(0) &= U_{k+1}^- e^{-U_{k+1}^-\cdot(b_{k+1}-b_k)} + C_{k+1}^+ U_{k+1}^+ e^{U_{k+1}^+\cdot(b_{k+1}-b_k)} C^- = -G'(0) \end{aligned}$$

Hence $H_{k+1}^- = 2\left(F'(0)G(0)^{-1}\right)_{(\sigma,\cdot)}$, which is the statement.
□

Lemma 4 For $k \le N-1$, write

$$\Psi_k^+(b_k - b_{k-1} + \varepsilon | b_k - b_{k-1} - \varepsilon) = \begin{pmatrix} H_k^+(\varepsilon)_{(\sigma,\sigma)} & H_k^+(\varepsilon)_{(\sigma,-)} \\ H_k^+(\varepsilon)_{(-,\sigma)} & H_k^+(\varepsilon)_{(-,-)} \end{pmatrix}$$

in block notation. Then

$$\begin{aligned} H_k^+ &:= \left.\frac{d}{d\varepsilon} H_k^+(\varepsilon)_{(\sigma,\sigma)}\right|_{\varepsilon=0} \\ &= 2\Big(U_k^+ e^{-U_k^+\cdot(b_k-b_{k-1})} + C_k^- U_k^- e^{U_k^-\cdot(b_k-b_{k-1})} C_k^+\Big) \\ &\quad \times \Big(e^{-U_k^+\cdot(b_k-b_{k-1})} - C_k^- e^{U_k^-\cdot(b_k-b_{k-1})} C_k^+\Big)^{-1} \end{aligned}$$

Proof: Use exactly the same arguments as in lemma 1.
□

Lemma 5 For $k \leq N-1$, write

$$\Psi^+_{k+1}(b_{k+1}-b_k+\varepsilon|2\varepsilon) = \begin{pmatrix} W^+_{k+1}(\varepsilon)_{(\sigma,\sigma)} & W^+_{k+1}(\varepsilon)_{(\sigma,-)} \\ W^+_{k+1}(\varepsilon)_{(-,\sigma)} & W^+_{k+1}(\varepsilon)_{(-,-)} \end{pmatrix}$$

in block notation. Then

$$\begin{aligned} W^+_{k+1} &:= \frac{d}{d\varepsilon} W^+_{k+1}(\varepsilon)_{(\sigma,\sigma)}\Big|_{\varepsilon=0} \\ &= -2\left(U^+_{k+1} + C^- U^-_{k+1} C^+_{k+1}\right) \\ &\quad \times \left(e^{-U^+_{k+1}\cdot(b_{k+1}-b_k)} - C^- e^{U^-_{k+1}\cdot(b_{k+1}-b_k)} C^+_{k+1}\right)^{-1} \end{aligned}$$

Proof: The proof is analogous to lemma 3. According to (13),

$$\begin{aligned} W^+_{k+1}(\varepsilon)_{(\sigma,\sigma)} &= \left(e^{U^+_{k+1}\cdot(b_{k+1}-b_k-\varepsilon)} - C^- e^{U^-_{k+1}\cdot 2\varepsilon} C^+_{k+1} e^{U^+_{k+1}\cdot(b_{k+1}-b_k+\varepsilon)}\right) \\ &\quad \times \left(I - C^- e^{U^-_{k+1}\cdot(b_{k+1}-b_k+\varepsilon)} C^+_{k+1} e^{U^+_{k+1}\cdot(b_{k+1}-b_k+\varepsilon)}\right)^{-1} \\ &= \left(e^{-U^+_{k+1}\cdot 2\varepsilon} - C^- e^{U^-_{k+1}\cdot 2\varepsilon} C^+_{k+1}\right) \\ &\quad \times \left(e^{-U^+_{k+1}\cdot(b_{k+1}-b_k+\varepsilon)} - C^- e^{U^-_{k+1}\cdot(b_{k+1}-b_k+\varepsilon)} C^+_{k+1}\right)^{-1} \end{aligned}$$

We abbreviate $F(\varepsilon) = e^{-U^+_{k+1}\cdot 2\varepsilon} - C^- e^{U^-_{k+1}\cdot 2\varepsilon} C^+_{k+1}$ as well as

$$G(\varepsilon) = e^{-U^+_{k+1}\cdot(b_{k+1}-b_k+\varepsilon)} - C^- e^{U^-_{k+1}\cdot(b_{k+1}-b_k+\varepsilon)} C^+_{k+1}$$

to obtain

$$\begin{aligned} \frac{d}{d\varepsilon} W^+_{k+1}(\varepsilon)_{(\sigma,\sigma)}\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon} F(\varepsilon) G(\varepsilon)^{-1}\Big|_{\varepsilon=0} \\ &= F'(0)G(0)^{-1} - F(0)G(0)^{-1}G'(0)G(0)^{-1} \end{aligned}$$

according to [4], sections I.1.3-4, where

$$F(0) = \mathbf{0} \quad \text{and} \quad F'(0) = -2\left(U^+_{k+1} + C^- U^-_{k+1} C^+_{k+1}\right)$$

and further

$$G(0) = e^{-U^+_{k+1}\cdot(b_{k+1}-b_k)} - C^- e^{U^-_{k+1}\cdot(b_{k+1}-b_k)} C^+_{k+1}$$

Altogether this yields the statement.
□

Theorem 2 *For* $k \leq N-1$,

$$\begin{aligned} E^+_{(\sigma,\sigma)}(k) &= 2\left(H^-_{k+1} + H^+_k\right)^{-1}\left(U^+_{k+1} + C^- U^-_{k+1} C^+_{k+1}\right) \\ &\quad \times \left(e^{-U^+_{k+1}\cdot(b_{k+1}-b_k)} - C^- e^{U^-_{k+1}\cdot(b_{k+1}-b_k)} C^+_{k+1}\right)^{-1} \end{aligned}$$

Proof: The proof is analogous to the one for theorem 1. We consider the matrix

$$E(\varepsilon) := \mathbb{E}\left(e^{-\gamma\tilde{\sigma}(b_{k-1},b_{k+1})-\delta_k\zeta_k-\delta_{k+1}\zeta_{k+1}}; X_{\sigma(b_{k-1},b_{k+1})} = b_{k+1}|X_0 = b_k+\varepsilon\right)$$

and assume that the regime changes at $b_k - \varepsilon$ for downward crossings of b_k and at $b_k + \varepsilon$ for upward crossings. Then we let $\varepsilon \downarrow 0$. We first find that

$$\begin{aligned} E_{(\sigma,\sigma)}(\varepsilon) &= W_{k+1}^{+}(\varepsilon)_{(\sigma,\sigma)} + H_{k+1}^{-}(\varepsilon)_{(\sigma,\cdot)} H_k^{+}(\varepsilon)_{(\cdot,\sigma)} E_{(\sigma,\sigma)}(\varepsilon) \\ &= \left(I_\sigma - H_{k+1}^{-}(\varepsilon)_{(\sigma,\cdot)} H_k^{+}(\varepsilon)_{(\cdot,\sigma)}\right)^{-1} \varepsilon\, \varepsilon^{-1}\, W_{k+1}^{+}(\varepsilon)_{(\sigma,\sigma)} \end{aligned}$$

Since $\lim_{\varepsilon\downarrow 0} H_{k+1}^{-}(\varepsilon)_{(\sigma,\cdot)} = \begin{pmatrix} I_\sigma & \mathbf{0}\end{pmatrix}$ and $\lim_{\varepsilon\downarrow 0} H_k^{+}(\varepsilon)_{(\cdot,\sigma)} = \begin{pmatrix} I_\sigma \\ \mathbf{0}\end{pmatrix}$, we can write

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} \varepsilon \left(I_\sigma - H_{k+1}^{-}(\varepsilon)_{(\sigma,\cdot)} H_k^{+}(\varepsilon)_{(\cdot,\sigma)}\right)^{-1} \\ &\quad = -\left(\frac{d}{d\varepsilon} H_{k+1}^{-}(\varepsilon)_{(\sigma,\cdot)} H_k^{+}(\varepsilon)_{(\cdot,\sigma)}\Big|_{\varepsilon=0}\right)^{-1} \\ &\quad = -\left(\frac{d}{d\varepsilon} H_{k+1}^{-}(\varepsilon)_{(\sigma,\cdot)}\Big|_{\varepsilon=0} \begin{pmatrix} I_\sigma \\ \mathbf{0}\end{pmatrix} + \begin{pmatrix} I_\sigma & \mathbf{0}\end{pmatrix} \frac{d}{d\varepsilon} H_k^{+}(\varepsilon)_{(\cdot,\sigma)}\Big|_{\varepsilon=0}\right)^{-1} \\ &\quad = -\left(\frac{d}{d\varepsilon} H_{k+1}^{-}(\varepsilon)_{(\sigma,\sigma)}\Big|_{\varepsilon=0} + \frac{d}{d\varepsilon} H_k^{+}(\varepsilon)_{(\sigma,\sigma)}\Big|_{\varepsilon=0}\right)^{-1} \\ &\quad = -\left(H_{k+1}^{-} + H_k^{+}\right)^{-1} \end{aligned}$$

where we have used [4], sections I.1.3-4, as well as lemmata 3 and 4. Similarly, since $\lim_{\varepsilon\downarrow 0} W_{k+1}^{+}(\varepsilon)_{(\sigma,\sigma)} = \mathbf{0}$, we obtain

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\, W_{k+1}^{+}(\varepsilon)_{(\sigma,\sigma)} = \frac{d}{d\varepsilon} W_{k+1}^{+}(\varepsilon)_{(\sigma,\sigma)}\Big|_{\varepsilon=0} = W_{k+1}^{+}$$

according to lemma 5. Altogether this yields the expression in the statement.

Considering now

$$E(-\varepsilon) := \mathbb{E}\left(e^{-\gamma\tilde{\sigma}(b_{k-1},b_{k+1})-\delta_k\zeta_k-\delta_{k+1}\zeta_{k+1}}; X_{\sigma(b_{k-1},b_{k+1})} = u|X_0 = b_k-\varepsilon\right)$$

instead of $E(\varepsilon)$ as above, we observe that

$$E_{(\sigma,\sigma)}(-\varepsilon) = H_k^{+}(\varepsilon)_{(\sigma,\sigma)}\; E_{(\sigma,\sigma)}(\varepsilon)$$

due to path continuity. Since $\lim_{\varepsilon\downarrow 0} H_k^{+}(\varepsilon)_{(\sigma,\sigma)} = I_\sigma$, we obtain

$$\lim_{\varepsilon\downarrow 0} E_{(\sigma,\sigma)}(-\varepsilon) = \lim_{\varepsilon\downarrow 0} E_{(\sigma,\sigma)}(\varepsilon)$$

meaning that the limits from both sides coincide.

□

4.2 Determine $E^-(k)$

Regarding the matrix $E^-(k)$, we find that

$$E^-_{(-,\cdot)}(k) = \Psi^-_k(b_k - l|b_k - l)_{(-,\cdot)} + \Psi^+_k(b_k - l|b_k - l)_{(-,\sigma)} E^-_{(\sigma,\cdot)}(k) \tag{18}$$

Thus it suffices to determine $E^-_{(\sigma,\cdot)}(k)$.

Lemma 6 Write $\Delta b_k := b_k - b_{k-1}$ and

$$\Psi^-_k(\Delta b_k + \varepsilon|\Delta b_k - \varepsilon) = \begin{pmatrix} W^-_k(\varepsilon)_{(\sigma,\sigma)} & W^-_k(\varepsilon)_{(\sigma,-)} \\ W^-_k(\varepsilon)_{(-,\sigma)} & W^-_k(\varepsilon)_{(-,-)} \end{pmatrix}$$

in block notation. Then

$$\begin{aligned} W^-_k &:= \left.\frac{d}{d\varepsilon} W^-_k(\varepsilon)_{(\sigma,\cdot)}\right|_{\varepsilon=0} \\ &= -2\left(C^- U^-_k + U^+_k C^-\right)\left(e^{-U^-_k \cdot \Delta b_k} - C^+_k e^{U^+_k \cdot \Delta b_k} C^-\right)^{-1} \end{aligned}$$

Proof: Use exactly the same arguments as in lemma 2.
□

Theorem 3 *Write $\Delta b_k := b_k - b_{k-1}$ and $H^-_{k+1} = \left(H^{-,\sigma}_{k+1}\ H^{-,-}_{k+1}\right)$. Then*

$$\begin{aligned} E^-_{(\sigma,\cdot)}(k) &= \left(H^{-,-}_{k+1}\Psi^+_k(\Delta b_k|\Delta b_k)_{(-,\sigma)} - H^+_k - H^{-,\sigma}_{k+1}\right)^{-1} \\ &\quad \times \left(W^-_k + H^{-,-}_{k+1}\Psi^-_k(\Delta b_k|\Delta b_k)_{(-,\cdot)}\right) \end{aligned}$$

for $k \leq N-1$, where the matrices H^-_{k+1}, H^+_k and W^-_k are given in lemmata 3, 4 and 6.

Proof: The proof is almost the same as the proof of theorem 1, with H^-_{k+1} instead of $C^- e^{U^-_2 \cdot 2\varepsilon}$. We consider

$$E(b_k - \varepsilon) := \mathbb{E}\left(e^{-\gamma\tilde{\sigma}(b_{k-1},b_{k+1}) - \delta_k\zeta_k - \delta_{k+1}\zeta_{k+1}}; X_{\sigma(b_{k-1},b_{k+1})} = b_{k-1}|X_0 = b - \varepsilon\right)$$

and assume that the regime changes at $b+\varepsilon$ for upward crossings of b and at $b-\varepsilon$ for downward crossings. Then we let $\varepsilon \downarrow 0$. First we obtain

$$\begin{aligned} E_{(\sigma,\cdot)}(b_k - \varepsilon) &= W^-_k(\varepsilon)_{(\sigma,\cdot)} + H^+_k(\varepsilon)_{(\sigma,\sigma)} H^-_{k+1}(\varepsilon)_{(\sigma,\cdot)} E(b-\varepsilon) \\ &= W^-_k(\varepsilon)_{(\sigma,\cdot)} + H^+_k(\varepsilon)_{(\sigma,\sigma)} H^-_{k+1}(\varepsilon)_{(\sigma,-)} E_{(-,\cdot)}(b_k - \varepsilon) \\ &\quad + H^+_k(\varepsilon)_{(\sigma,\sigma)} H^+_k - H^-_{k+1}(\varepsilon)_{(\sigma,\sigma)} E_{(\sigma,\cdot)}(b_k - \varepsilon) \end{aligned}$$

where $H^+_k(\varepsilon)_{(\sigma,\sigma)}$, $H^-_{k+1}(\varepsilon)_{(\sigma,\cdot)}$ and $W^-_k(\varepsilon)_{(\sigma,\cdot)}$ are defined in lemmata 4, 3 and 6. This implies

$$\begin{aligned} E_{(\sigma,\cdot)}(b-\varepsilon) &= \left(I_\sigma - H^+_k(\varepsilon)_{(\sigma,\sigma)} H^-_{k+1}(\varepsilon)_{(\sigma,\sigma)}\right)^{-1} \cdot \varepsilon \\ &\quad \times \varepsilon^{-1}\left(W^-_k(\varepsilon)_{(\sigma,\cdot)} + H^+_k(\varepsilon)_{(\sigma,\sigma)} H^-_{k+1}(\varepsilon)_{(\sigma,-)} E_{(-,\cdot)}(b-\varepsilon)\right) \end{aligned}$$

We observe that

$$\lim_{\varepsilon\downarrow 0} H_k^+(\varepsilon)_{(\sigma,\sigma)} = \lim_{\varepsilon\downarrow 0} H_{k+1}^-(\varepsilon)_{(\sigma,\sigma)} = I_\sigma$$
$$\lim_{\varepsilon\downarrow 0} W_k^-(\varepsilon)_{(\sigma,\cdot)} = \mathbf{0}, \quad \lim_{\varepsilon\downarrow 0} H_{k+1}^-(\varepsilon)_{(\sigma,-)} = \mathbf{0}$$
$$\lim_{\varepsilon\downarrow 0} E_{(-,\cdot)}(b_k-\varepsilon) = E_{(-,\cdot)}^-(k)$$

where **0** denotes a zero matrix of appropriate dimension. As in the proof to theorem 1 we obtain

$$\lim_{\varepsilon\downarrow 0} \varepsilon \left(I_\sigma - H_k^+(\varepsilon)_{(\sigma,\sigma)} H_{k+1}^-(\varepsilon)_{(\sigma,\sigma)}\right)^{-1} = -\left(H_k^+ + H_{k+1}^{-,\sigma}\right)^{-1}$$

using lemmata 4 and 3. In a similar manner,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\; W_k^-(\varepsilon)_{(\sigma,\cdot)} = \left.\frac{d}{d\varepsilon} W_k^-(\varepsilon)_{(\sigma,\cdot)}\right|_{\varepsilon=0} = W_k^-$$

according to lemma 6. Finally,

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\; H_k^+(\varepsilon)_{(\sigma,\sigma)} H_{k+1}^-(\varepsilon)_{(\sigma,-)} E_{(-,\cdot)}(b_k-\varepsilon) = H_{k+1}^{-,-} E_{(-,\cdot)}(b_k)$$

Pasting the above results together, the limit $\varepsilon \downarrow 0$ yields

$$\begin{aligned} E_{(\sigma,\cdot)}^-(k) &= -\left(H_k^+ + H_{k+1}^{-,\sigma}\right)^{-1} \left(W_k^- + H_{k+1}^{-,-} E_{(-,\cdot)}^-(k)\right) \\ &= -\left(H_k^+ + H_{k+1}^{-,\sigma}\right)^{-1} \left(W_k^- + H_{k+1}^{-,-} \Psi_k^-(\Delta b_k|\Delta b_k)_{(-,\cdot)}\right) \\ &\quad - \left(H_k^+ + H_{k+1}^{-,\sigma}\right)^{-1} H_{k+1}^{-,-} \Psi_k^+(\Delta b_k|\Delta b_k)_{(-,\sigma)} E_{(\sigma,\cdot)}^-(k) \end{aligned}$$

after using (18). Thus

$$\begin{aligned} E_{(\sigma,\cdot)}^-(k) &= -\left(I_\sigma - \left(H_k^+ + H_{k+1}^{-,\sigma}\right)^{-1} H_{k+1}^{-,-} \Psi_k^+(\Delta b_k|\Delta b_k)_{(-,\sigma)}\right)^{-1} \\ &\quad \times \left(H_k^+ + H_{k+1}^{-,\sigma}\right)^{-1} \left(W_k^- + H_{k+1}^{-,-} \Psi_k^-(\Delta b_k|\Delta b_k)_{(-,\cdot)}\right) \\ &= -\left(H_k^+ + H_{k+1}^{-,\sigma} - H_{k+1}^{-,-} \Psi_k^+(\Delta b_k|\Delta b_k)_{(-,\sigma)}\right)^{-1} \\ &\quad \times \left(W_k^- + H_{k+1}^{-,-} \Psi_k^-(\Delta b_k|\Delta b_k)_{(-,\cdot)}\right) \end{aligned}$$

which is the expression in the statement. The same arguments as in the proof to theorem 1 show that $\lim_{\varepsilon\downarrow 0} E_{(\sigma,\cdot)}(b+\varepsilon) = \lim_{\varepsilon\downarrow 0} E_{(\sigma,\cdot)}(b-\varepsilon)$.
□

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