A Retrial BMAP/PH/N System

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A multi-server retrial queueing model with Batch Markovian Arrival Process and phase-type service time distribution is analyzed. The continuous-time multi-dimensional Markov chain describing the behavior of the system is investigated by means of reducing it to the corresponding discrete-time multi-dimensional Markov chain. The latter belongs to the class of multi-dimensional quasitoeplitz Markov chains in the case of a constant retrial rate and to the class of multi-dimensional asymptotically quasitoeplitz Markov chains in the case of an infinitely increasing retrial rate. It allows to obtain the existence conditions for the stationary distribution and to elaborate the algorithms for calculating the stationary state probabilities.

Keywords: $BMAP/PH/N$ retrial model, matrix analytic methods, batch markovian arrival process, PH-distribution, asymptotically quasitoeplitz Markov chains.

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1. Introduction

Retrial queueing models are characterized by the fact that the arrival calls, which find the servers busy, do not line up or leave the system forever, but they try their luck after some random time.

Retrial queueing models accurately describe the operation of many telecommunication networks. So their investigation is very important. Good overviews of the current research on retrial queues are contained in the book of Falin and Templeton [25], the paper of Kulkarni and Liang [31] and in the recent surveys of Artalejo [4, 5]. The analysis of the current situation makes clear that there is a lack of results in the two following directions.

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The first direction is the investigation of retrial models with correlated inputs, e.g. the MAP and the $BMAP$ input. Since the introduction of the $BMAP$ by Lucantoni [33] in 1991, many queues with $BMAP$ input have been investigated in detail. But except the recent papers of Anisimov and Kurtulus [3] Choi and Chang [12], Choi, Chung, Dudin [13], Diamond and Alfa [14, 15], Dudin and Klimenok [20-22], the results do not concern the retrial models.

The second direction is the consideration of multi-server retrial models. Even the simplest retrial model of $M/M/N$ type is not well investigated yet, see [25]. We can refer to the book [36] and the paper [37] of Stepanov and the papers of M.Neuts and coauthors [7, 35] where some results concerning retrial multi-server queues with stationary Poisson input are obtained. In the papers of Anisimov [1,2], the asymptotic results are presented.

The direction of the research on multi-server retrial queues with MAP or the $BMAP$ input is represented only by papers of Choi and Chang [12], Diamond and Alfa [15] and He, Li, Zhao [28]. Note that the first two papers deal only with MAP-inputs. This simplifies the problem because the behavior of the system is described by the Level Dependent Quasi-Birth-and-Death processes which are investigated rather well, see e.g. Bright and Taylor [10-11], Latouche and Ramaswami [32]. The paper of He, Li, Zhao [28] deals with a complicated system of the $BMAP/PH/N/N+K$ type with PH-retrial times. The authors solve a very important problem of finding the stationary distribution existence condition, but they did not touch the problem of calculation of the stationary state distribution. Unfortunately, the stability condition given in [28] for the case of pure retrial model ($K=0$), which is interesting for us in the present paper, is not proven correctly.

In our present paper, we consider the retrial $BMAP/PH/N$ system. This model is suitable for the analysis and optimization of mobile computer communication networks. The assumption about the $BMAP$-like behavior of the input fits well to the real-life flows. The assumption about the presence of a multi-server device is very important because the base station of a mobile network provides usually more than one channel for information transmission. In the case $N=1$, the problem is solved already by Dudin and Klimenok in the case of the more complicated SM-type service process [20, 21]. The assumption about the PH-service (phase-type service time distribution) is some kind of the trade-off between the relative easiness of investigating systems with exponential service-time distribution and the practical importance of considering more general service time distributions.

Concerning the retrial process, we suppose that the inter-retrial times are exponentially distributed with rate α_i, which may depend on the current number i of customers on the orbit. As special cases, we consider the variants of a constant retrial rate ($\alpha_i = \gamma, i \geq 1$) and linear repeated requests ($\alpha_i = i\alpha+\gamma, \alpha > 0, \gamma \geq 0, i \geq 1$). Note that the results could be extended to the case where the parameters α and γ are modulated by some finite-space continuous-time Markov process (like it was done in [22]).

The content of the paper is the following. In Section 2, we formally describe the model under consideration. In Section 3, the behavior of the queueing model under consideration is described in terms of a multi-dimensional continuous-time Markov chain. To investigate this chain, in Section 4 we reduce it to the multi-dimensional discrete-time Markov chain. In Section 5 we consider the case of a constant retrial rate. In Sections 6 and 7, we consider the case when the total retrial rate is infinitely increasing as the number of calls on the orbit tends to infinity.

2. The model

We consider an N-server queueing system. The primary calls arrive to the system according to a $BMAP$ (Batch Markovian Arrival Process). The notion of the $BMAP$ and its detailed description is given by D.Lucantoni in [33]. We denote the directing process of the $BMAP$ by $\nu_t, t \geq 0$. The state space of this irreducible continuous time Markov chain ν_t is $\{0, 1, \dots, W\}$. As follows from [33], the behavior of the $BMAP$ is characterized completely by the matrix generating function $D(z) = \sum\limits_{k=0}^{\infty} D_k z^k$, $|z| < 1$. The matrix D_k characterizes the intensities of transitions of the process ν_t which are accompanied by generating a batch of k calls, $k \geq 0$. The matrix D_0 is stable one. The matrix $D(1)$ represents the generator of the process $\nu_t, t \geq 0$.

The average arrival rate λ is defined as $\lambda = \vec{\theta}D'(1)\mathbf{1}$ where $\vec{\theta}$ is the invariant vector of the stationary distribution of $\nu_t, t \geq 0$. The vector $\vec{\theta}$ is the unique solution to the system $\vec{\theta}D(1) = \vec{0}, \vec{\theta}\mathbf{1} = 1$. Here $\mathbf{1}$ is the column-vector of appropriate size consisting of units and $\vec{0}$ is the row-vector of appropriate size consisting of zeroes.

The servers are identical and independent of each other. The service of a customer by a server is governed by the directing process m_t. The state space of this continuous time Markov chain m_t is $\{1, \dots, M\}$. The initial state of the process m_t at the epoch of starting the service is determined by the probabilistic row-vector $\beta = (\beta_1, \dots, \beta_M)$. The transitions of the process m_t, which do not lead to service completion, are defined by the irreducible matrix S of size $M \times M$. The intensities of transitions, which lead to service completion, are defined by the vector $S_0 = -S\mathbf{1}$. The service time distribution function has the form $B(x) = 1 - \beta e^{Sx}\mathbf{1}$. A more detailed description of the PH-type service time distribution can be found e.g. in the book [34].

If the arriving batch of the primary calls meets several servers being idle, the primary calls occupy the corresponding number of the servers. If the number of the idle servers is insufficient (or all servers are busy) the rest of the batch (or all the batch) goes to the so called orbit. These calls are said to be repeated calls. These calls try their luck later, until they will be served. We assume that the total flow of retrials is a such as the probability of generating the retrial attempt in the interval $(t, t+\Delta t)$ is equal to $\alpha_i \Delta t + o(\Delta t)$ when the orbit size (the number of calls on the orbit) is equal to i, $i > 0$, $\alpha_0 = 0$. The orbit capacity is assumed to be unlimited. We do not fix the explicit dependence of the intensities α_i on i. The following two variants will be dealt with:

- a constant retrial rate : $\alpha_i = \gamma$, $i > 0$;
- an infinitely increasing retrial rate : $\lim_{i \to \infty} \alpha_i = \infty$. This variant includes the classic retrial strategy ($\alpha_i = i\alpha$) and the linear strategy ($\alpha_i = i\alpha + \gamma$).

Our goal is to derive the stationary state distribution of the orbit.

3. Continuous-time Markov chain

Let:

- i_t be the number of calls on the orbit, $i_t \geq 0$,
- n_t be the number of busy servers, $n_t = \overline{0, N}$,
- $m_t^{(j)}$ be the state of the directing process of the service on the j-th busy server, $m_t^{(j)} = \overline{1, M}$, $j = \overline{1, n_t}$ (we assume here that the busy servers are numerated in order of their occupying, i.e. the server, which begins the service, is appointed the maximal number among all busy servers; when some server finishes the service, the servers are correspondingly enumerated),
- ν_t be the state of the directing process of the $BMAP$, $\nu_t = \overline{0, W}$,

at the epoch $t, t \geq 0$.

Consider the multi-dimensional process $\xi_t = (i_t,\, n_t,\, \nu_t,\, m_t^{(1)}, \dots, m_t^{(n_t)}),\ t \geq 0$, in continuous time. It is easy to see that this process is an irreducible Markov chain. Denote the stationary probabilities of this process as

$$p(i, n, \nu, m^{(1)}, \dots, m^{(n)}) = \\ = \lim_{t \to \infty} P\{\, i_t = i,\ n_t = n,\ \nu_t = \nu,\ m_t^{(1)} = m^{(1)}, \dots, m_t^{(n)} = m^{(n)}\} \quad (1)$$

for $i \geq 0,\ \nu = \overline{0, W},\ m^{(j)} = \overline{1, M},\ j = \overline{1, n}$, and $n = \overline{0, N}$. The problem of the existence of the limits (1) will be discussed a little bit later.

Enumerate the states of the chain $\xi_t, t \geq 0$, in lexicographic order and form the row-vector $\vec{p_i}$ of the stationary-state probabilities $p\,(i,\, n,\, \nu,\, m^{(1)}, \dots, m^{(n)})$, $i \geq 0$. Note that the dimensionality of these vectors is equal to $K = (W + 1)\frac{1-M^{N+1}}{1-M}$. E.g., if the number N is equal to 5 and the state spaces of the $BMAP$-input and PH-service consist of two elements ($W = 1,\ M = 2$), then $K = 126$. So, the dimension of the vector $\vec{p_i}$ is rather high and the problem of the accuracy of calculations can arise. Note that in case of large K we can use another Markovian process for describing the queueing model under consideration, see e.g. [38, 40].

Define the infinite-dimensional probability vector $\vec{p} = (\,\vec{p}_0,\, \vec{p}_1 \dots)$.

Proposition 1. If the vector $\vec{p}$ exists then it satisfies the equilibrium equation

$$\vec{p}A = \vec{0} \quad (2)$$

where $\vec{0}$ is the infinite row-vector consisting of zeroes and the matrix A is the infinitesimal generator of the chain $\xi_t, t \geq 0$, and has the following structure:

$$A = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} & \dots \\ A_{10} & A_{11} & A_{12} & A_{13} & \dots \\ 0 & A_{21} & A_{22} & A_{23} & \dots \\ 0 & 0 & A_{32} & A_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (3)$$

where the blocks A_{ij} of size $K \times K$ have the following form:

$$A_{i,i-1} = \alpha_i \begin{pmatrix} 0 & I_{\overline{W}} \otimes \beta & 0 & \ldots & 0 \\ 0 & 0 & I_{\overline{W}M} \otimes \beta & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I_{\overline{W}M^{N-1}} \otimes \beta \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{4}$$

$$A_{i,i+k} = \begin{pmatrix} 0 \ldots 0 & D_{k+N} \otimes \beta^{\otimes N} \\ 0 \ldots 0 & D_{k+N-1} \otimes I_M \otimes \beta^{\otimes (N-1)} \\ 0 \ldots 0 & D_{k+N-2} \otimes I_{M^2} \otimes \beta^{\otimes (N-2)} \\ \vdots \ddots \vdots & \\ 0 \ldots 0 & D_k \otimes I_{M^N} \end{pmatrix}, k \geq 1, \tag{5}$$

$$(A_{i,i})_{r,r'} = \begin{cases} 0, \; r' < r-1, \; r = \overline{2,N}, \\ I_{\overline{W}} \otimes S_0^{\oplus r}, \; r' = r-1, \; r = \overline{1,N}, \\ D_0 \oplus S^{\oplus r} - \alpha_i (1 - \delta_{r,N}) I_{\overline{W}M^r}, \; r' = r, \; r = \overline{0,N}, \\ D_l \otimes I_{M^r} \otimes \beta^{\otimes l}, \; r' = r + l, \; l = \overline{1, N-r}, \; r = \overline{0,N}, \end{cases} \tag{6}$$

$i \geq 0$.

Here $\delta_{r,N} = \begin{cases} 1, \; r = N, \\ 0, \; r \neq N \end{cases}$, is Kronecker's symbol, $\otimes$ is the sign of Kronecker's product, and $\oplus$ is the sign of Kronecker's sum,

$$\beta^{\otimes l} \stackrel{def}{=} \underbrace{\beta \otimes \ldots \otimes \beta}_{l}, \quad l \geq 1, \; S^{\oplus l} \stackrel{def}{=} \underbrace{S \oplus \ldots \oplus S}_{l}, \quad l \geq 1, \quad S^{\oplus 0} \stackrel{def}{=} 0,$$

$$S_0^{\oplus l} \stackrel{def}{=} \sum_{m=0}^{l-1} I_{M^m} \otimes S_0 \otimes I_{M^{l-m-1}}, \quad l \geq 1,$$

$\overline{W} = W + 1$, I_L denotes the identity matrix of size $L \times L$, $I_{M^0} = 1$.

The proof of Proposition 1 consists of deriving the equilibrium equations for the Markov chain $\xi_t, t \geq 0$, and rewriting them into the matrix form. These operations are rather long and trivial, so the proof is omitted.

The technique of solving (2) is known only for the cases when the input flow is a MAP (not a $BMAP$) and the matrix A is a three block-diagonal one or when the matrix A is quasitoeplitz i.e. $A_{ij} = A_{j-i}$, $i \geq \tilde{J}$, $\tilde{J}$ is some fixed integer.

Thus, we should offer a way for solving (2). To this end, we reduce the investigation of the continuous-time Markov chain $\xi_t, t \geq 0$, to investigating an embedded at the epochs t_k of its transitions discrete-time Markov chain $\xi_k = (i_{t_k}, n_{t_k}, \nu_{t_k}, m_{t_k}^{(1)}, \ldots, m_{t_k}^{(n_{t_k})}), k \geq 1$.

4. Discrete-time Markov chain

The technique of generating the embedded discrete-time Markov chain $\xi_k, k \geq 1$, is well-known. Every row of the infinitesimal generator A is divided by the module of the diagonal entry of this generator and a unit is added to the diagonal entry. Let us do so.

It is well-known that the diagonal entries of the matrix D_0 are negative: $(D_0)_{\nu,\nu} = -\lambda_\nu$, $\lambda_\nu > 0$, $\nu = \overline{0,W}$, as well as the diagonal entries of the matrix S: $(S)_{m,m} = -s_m$, $s_m > 0$, $m = \overline{1,M}$.

The diagonal blocks of the generator A have the form

$$(A_{ii})_{r,r} = D_0 \oplus S^{\oplus r} - \alpha_i I_{\overline{W}M^r}(1 - \delta_{r,N}),\ r = \overline{0,N},\ i \geq 0.$$

Introduce the diagonal matrix $\hat{R}_i$:

$$\hat{R}_i = C + \alpha_i \hat{I}, \tag{7}$$

$$C = diag\,\{\,diag\,\{\,\lambda_\nu,\ \nu = \overline{0,W}\,\} \oplus [\,diag\,\{\,s_m,\ m = \overline{1,M}\,\}\,]^{\oplus r},\ r = \overline{0,N}\,\}.$$

Here $diag\,\{a_l,\ l = \overline{1,L}\,\}$ denotes the diagonal matrix of size $L \times L$ with the diagonal entries a_l,

$$\hat{I} = \begin{pmatrix} I_{\overline{W}} & 0 & \ldots & 0 & 0 \\ 0 & I_{\overline{W}M} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots I_{\overline{W}M^{N-1}} & & 0 \\ 0 & 0 & \ldots & 0 & O_{\overline{W}M^N} \end{pmatrix},$$

O_L is the zero matrix of size $L \times L$. When the index is omitted, the size of the matrix should be clear from the context.

The matrix $\hat{R}_i$ defines the diagonal entries of the block A_{ii}, $i \geq 0$.

Thus, we reduce the investigation of the original continuous-time Markov chain with the generator A to the consideration of a discrete-time Markov chain ξ_k, $k \geq 1$, of the same dimensionality having the transition probability matrix Y which is defined as follows:

$$Y = \|Y_{i,l}\|_{i,l \geq 0},$$

$$Y_{i,l} = \begin{cases} 0, \ l < i-1, \\ \\ \hat{R}_i^{-1} A_{i,i-1}, \ l = i-1, \\ \\ \hat{R}_i^{-1} A_{i,i} + I, \ l = i, \\ \\ \hat{R}_i^{-1} A_{i,i+k}, \ l = i+k, \ k \geq 1, \end{cases} \tag{8}$$

$i \geq 0$.

Introduce the notations

$$\tilde{I}_\beta = \begin{pmatrix} 0 & I_{\overline{W}} \otimes \beta & 0 & \dots & 0 \\ 0 & 0 & I_{\overline{W}M} \otimes \beta & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_{\overline{W}M^{N-1}} \otimes \beta \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

and $\bar{I} = I - \hat{I}$. The matrix $D^{(N)}$ shall be defined by its blocks

$$(D^{(N)})_{r,r'} = \begin{cases} 0, \ r' < r-1, \\ \\ I_{\overline{W}} \otimes S_0^{\oplus r}, \ r' = r-1, \\ \\ D_0 \oplus S^{\oplus r}, \ r' = r, \\ \\ D_l \otimes I_{M^r} \otimes \beta^{\otimes l}, \ r' = r+l, \ l = \overline{1, N-r}, \end{cases}$$

$r = \overline{0, N}$.

Rewriting the matrix $\hat{R}_i$ in the form $\hat{R}_i = (C + \alpha_i I)\hat{I} + C\bar{I}$ and taking into account the evident formula $\hat{R}_i^{-1} = (C + \alpha_i I)^{-1}\hat{I} + C^{-1}\bar{I}$ and formulas (4)-(6), we obtain from (8) another expression for the transition probability blocks $Y_{i,l}$.

Theorem 1. The blocks $Y_{i,l}$ of the transition probability matrix Y of the discrete-time Markov chain ξ_k, $k \geq 1$, are defined as follows:

$$Y_{i,l} = 0, \quad l < i-1,$$

$$Y_{i,i-1} = Q_1^{(i)} Y_0^{(1)}, \quad i \geq 1,$$

$$(9)$$

$$Y_{i,i} = Q_1^{(i)} Y_1^{(1)} + Q_2^{(i)} Y_1^{(2)}, \quad i \geq 0,$$

$$Y_{i,i+k} = Q_1^{(i)} Y_{k+1}^{(1)} + Q_2^{(i)} Y_{k+1}^{(2)}, \quad i \geq 0,\ k \geq 1,$$

where

$$Q_1^{(i)} = (C + \alpha_i I)^{-1} \alpha_i I, \quad Q_2^{(i)} = (C + \alpha_i I)^{-1} C,\ i \geq 0, \tag{10}$$

$$Y_0^{(1)} = \tilde{I}_\beta,$$

$$Y_1^{(1)} = C^{-1} \cdot \bar{I} \cdot D^{(N)} + \bar{I}, \quad Y_{k+1}^{(1)} = C^{-1} \bar{I} A_{i,i+k},\ k \geq 1, \tag{11}$$

$$Y_1^{(2)} = C^{-1} \cdot D^{(N)} + I, \quad Y_{k+1}^{(2)} = C^{-1} A_{i,i+k},\ k \geq 1.$$

Enumerate the states of the discrete-time Markov chain ξ_k, $k \geq 1$, in lexicographic order and denote as $\vec{\pi}_i$ the stationary probability row-vector corresponding to the state i of the first component.

In the following Sections, we derive the sufficient conditions for existence of these probabilities. Suppose that these conditions hold.

Proposition 2. When the stationary distribution $\vec{\pi}_i, i \geq 0$, of the embedded Markov chain exists, the stationary distribution $\vec{p}_i, i \geq 0$, of the original continuous-time Markov chain $\xi_t, t \geq 0$, exists as well and the following relations hold true:

$$\vec{p}_i = \bar{c} \vec{\pi}_i \hat{R}_i^{-1}, \quad i \geq 0, \tag{12}$$

where the positive finite constant $\bar{c}$ is defined as:

$$\bar{c} = \left(\sum_{l=0}^{\infty} \vec{\pi}_l \hat{R}_l^{-1} \mathbf{1}\right)^{-1}. \tag{13}$$

Proof. Because the non-negative diagonal matrices $\hat{R}_i^{-1}$ have a form $\hat{R}_i^{-1} = (C + \alpha_i \hat{I})^{-1}$ and we deal the cases of the constant or the infinitely increasing retrial rates $\alpha_i, i \geq 1$, the series $\sum_{i=0}^{\infty} \vec{\pi}_i \hat{R}_i^{-1}$ evidently converges if the series $\sum_{i=0}^{\infty} \vec{\pi}_i$ converges. Thus assuming that the stationary distribution $\vec{\pi}_i, i \geq 0$, exists,we guarantee the existence of the finite positive constant $\bar{c}$ defined by formula (13).

Let $\vec{\pi} = (\vec{\pi}_0, \vec{\pi}_1, \ldots)$. The vector $\vec{\pi}$ satisfies the equilibrium equation

$$\vec{\pi} Y = \vec{\pi} \tag{14}$$

which can be rewritten in the form:

$$\sum_{i=0}^{l+1} \vec{\pi}_i Y_{i,l} = \vec{\pi}_l, \quad l \geq 0. \tag{15}$$

Substituting (8) into (15), we obtain that the probability vectors $\vec{\pi}_i, i \geq 0$, satisfy the equations:

$$\sum_{i=0}^{l+1} \vec{\pi}_i \hat{R}_i^{-1} A_{i,l} = 0, \quad l \geq 0. \tag{16}$$

Thus, the vectors $\vec{p}_i, i \geq 0$, defined by formula (12) represent a positive solution of equilibrium equations

$$\sum_{i=0}^{l+1} \vec{p}_i A_{i,l} = 0, \quad l \geq 0, \tag{17}$$

such that the series $\sum_{i=0}^{\infty} \vec{p}_i$ converges. By Foster's theorem we conclude that the vectors $\vec{p}_i$ in form (12) define the unique stationary distribution of the original continuous-time Markov chain $\xi_t, t \geq 0$. Proposition 2 is proven.

Corollary 1. The sufficient conditions for existence of the embedded Markov chain stationary distribution are also the sufficient conditions for existence of the continuous-time Markov chain $\xi_t, t \geq 0$, stationary distribution.

Thus, if we prove the stationary distribution existence condition for the probabilities $\vec{\pi}_i$, $i \geq 0$, and calculate these probabilities, the problem of calculation of the stationary state probabilities $\vec{p}_i$, $i \geq 0$, can be considered being solved.

Let us investigate the chain ξ_k, $k \geq 1$, which is determined by the transition probability matrix in the form (9)-(11). The investigation depends essentially on the strategy of retrials. We consider sequentially two strategies mentioned above.

5. Constant retrial rate

We assume $\alpha_0 = 0$, $\alpha_i = \gamma$, $i > 0$. This strategy describes the situations where the retrial process is controlled by some decision-maker. Only one call from the orbit is allowed to make the repeated attempts in intervals, which are exponentially distributed with rate γ. Or all calls are allowed to make the retrials while the individual intensity of retrials should be equal to γ/i when i calls stay on the orbit.

In this case, the matrices $Q_1^{(i)}$, $Q_2^{(i)}$, which are defined by formulas (10), do not depend on i. Thus the transition probability blocks $Y_{i,l}$, $i > 0$, depend only on the value $l - i$ and do not depend on i and l separately. This means that the Markov chain ξ_k, $k \geq 1$, belongs to the class of $M/G/1$ type chains (see [34]) or multi-dimensional quasitoeplitz Markov chains [19, 23] and can be investigated easily.

Denote $Y_{0,k} = V_k, Y_{i,i+k-1} = Y_k, k \geq 0, i > 0,$

$$Y(z) = \sum_{l=0}^{\infty} Y_l z^l, \quad V(z) = \sum_{l=0}^{\infty} V_l z^l, \quad \vec{\Pi}(z) = \sum_{i=0}^{\infty} \vec{\pi}_i z^i, |z| < 1.$$

It is well-known [34, 19, 23], that the vector generating function $\vec{\Pi}(z)$ satisfies the following linear matrix functional equation:

$$\vec{\Pi}(z)(Y(z) - zI) = \vec{\Pi}(0)(Y(z) - zV(z)) \tag{18}$$

that can be rewritten in the form

$$\vec{\Pi}(z) = \vec{\Pi}(0)\Phi(z)$$

where

$$\Phi(z) = (Y(z) - zV(z))(Y(z) - zI)^{-1}. \tag{19}$$

In our case, the matrix generating functions $Y(z)$ and $V(z)$ are calculated as:

$$V(z) = \sum_{k=0}^{\infty} Y_{0,k} z^k, \quad Y(z) = \sum_{k=0}^{\infty} Y_{i,i+k-1} z^k, \; i > 0.$$

Denote

$$D^\star(z) = \sum_{k=0}^{\infty} D^{(N+k)} z^k = \tag{20}$$

$$\begin{bmatrix} D_0 & D_1 \otimes \beta^{\otimes 1} & D_2 \otimes \beta^{\otimes 2} & \dots & D_{N-1} \otimes \beta^{\otimes (N-1)} & \Delta_{N-1}(z,\beta) \\ I_{\overline{W}} \otimes S_0^{\oplus 1} & D_0 \oplus S^{\oplus 1} & D_1 \otimes I_M \otimes \beta^{\otimes 1} & \dots & D_{N-2} \otimes I_M \otimes \beta^{\otimes (N-2)} & \Delta_{N-2}(z,\beta) \\ 0 & I_{\overline{W}} \otimes S_0^{\oplus 2} & D_0 \oplus S^{\oplus 2} & \dots & D_{N-3} \otimes I_{M^2} \otimes \beta^{\otimes (N-3)} & \Delta_{N-3}(z,\beta) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I_{\overline{W}} \otimes S_0^{\oplus N} & D(z) \oplus S^{\oplus N} \end{bmatrix},$$

with $\Delta_m(z,\beta) = z^{-m+1}(D(z) - \sum\limits_{k=0}^{m} D_k z^k) \otimes I_{M^{N-m-1}} \otimes \beta^{\otimes (m+1)}, m = \overline{0, N-1}$.

Then, we easily derive the explicit expressions for the matrix generating functions as:

$$V(z) = I + C^{-1} D^\star(z),$$

$$(21)$$

$$Y(z) = zI + (C + \gamma \hat{I})^{-1}(\gamma \tilde{I}_\beta - \gamma \hat{I} z + z D^\star(z)).$$

By substituting (21) into (18), we finally get the functional equation for the vector generating function $\vec{\Pi}(z)$ of the stationary distribution of ξ_k, $k \geq 1$, in the following form:

$$\vec{\Pi}(z)(C + \gamma \hat{I})^{-1}(\gamma \tilde{I}_\beta - \gamma \hat{I} z + z D^\star(z)) =$$

$$= \vec{\Pi}(0)(C + \gamma \hat{I})^{-1} \gamma \hat{I} (\tilde{I}_\beta - Iz - C^{-1} z D^\star(z)). \quad (22)$$

As follows from [19, 26, 27], the sufficient condition for the stationary distribution existence here has the form

$$(\, det(-\gamma \tilde{I}_\beta + \gamma \hat{I} z - z D^\star(z))\,)'|_{z=1} > 0. \quad (23)$$

A more constructive form of this condition is given in the following statement.

Theorem 2. The stationary distribution existence condition has the form

$$\vec{X}((D^\star(z))'|_{z=1} - \gamma \hat{I})\,\mathbf{1} < 0, \quad (24)$$

where the vector $\vec{X}$ is the unique solution to the following system of linear algebraic equations:

$$\vec{X}(I - Y(1)) = \vec{0}, \quad (25)$$

$$\vec{X}\mathbf{1} = 1.$$

The proof is given in the *Appendix* 1.

It follows from [19, 26, 27, 30], that if condition (24) is fulfilled, then the equation

$$det(\gamma\tilde{I}_\beta - \gamma\hat{I}z + zD^\star(z)) = 0 \qquad (26)$$

has a simple root $z = 1$ and some number K' of roots z_k having multiplicity n_k, $n_k \geq 1$, $\sum_{k=1}^{K'} n_k = K - 1$, in the open unit disc.

Rewriting (22) in the form

$$\vec{\Pi}(z) = \vec{\Pi}(0)\Phi(z) = \vec{\Pi}(0)(C + \gamma\hat{I})^{-1}\gamma\hat{I}(\tilde{I}_\beta - Iz - C^{-1}zD^\star(z))\times$$

$$Adj(\gamma\tilde{I}_\beta - \gamma\hat{I}z + zD^\star(z)))(C + \gamma\hat{I})\left(det(\gamma\tilde{I}_\beta - \gamma\hat{I}z + zD^\star(z))\right)^{-1}$$

and exploiting the analyticity of the vector generating function $\vec{\Pi}(z)$ in the region $|z| < 1$ and the normalization condition

$$\vec{\Pi}(1)\mathbf{1} = 1,$$

we can get the entries of the unknown vector $\vec{\Pi}(0)$ as the solution of the following system of linear algebraic equations:

$$\vec{\Pi}(0)\frac{d^n}{dz^n}[(C + \gamma\hat{I})^{-1}\gamma\hat{I}(\tilde{I}_\beta - Iz - C^{-1}zD^\star(z))\times$$

$$Adj(\gamma\tilde{I}_\beta - \gamma\hat{I}z + zD^\star(z)))(C + \gamma\hat{I})]|_{z=z_k} = 0,$$

$$n = \overline{0, n_k - 1}, k = \overline{0, K'},$$

$$\vec{\Pi}(0)(C + \gamma\hat{I})^{-1}\gamma\hat{I}(\tilde{I}_\beta - Iz - C^{-1}zD^\star(z))\times$$

$$Adj(\gamma\tilde{I}_\beta - \gamma\hat{I}z + zD^\star(z)))(C + \gamma\hat{I})\left(det(\gamma\tilde{I}_\beta - \gamma\hat{I}z + zD^\star(z))\right)^{-1}|_{z=1}\mathbf{1} = 1.$$

It follows directly from [26], that this system has a unique solution when the stationary distribution existence condition is fulfilled.

By substituting the calculated unique value of the vector $\vec{\Pi}(0)$ into equation (22), we get a unique solution $\vec{\Pi}(z)$ to this equation that is analytic in the region $|z| < 1$, continuous on the border of this region and satisfying the normalization condition.

The values of the vector factorial moments $\vec{\Pi}^{(m)}(1)$ are calculated as

$$\vec{\Pi}^{(m)}(1) = \frac{d^m}{dz^m}\vec{\Pi}(z)|_{z=1} = \vec{\Pi}(0)\frac{d^m}{dz^m}\Phi(z)|_{z=1}, \quad m \geq 0.$$

However, the matrix generating function $\Phi(z)$ and its derivatives at the point $z = 1$ can not be calculated from (19) directly because both multipliers in (19) are singular at this point. Thus, the following auxiliary result can be useful.

Let

$$\Delta_k = \frac{d^k D^\star(z)}{dz^k}|_{z=1}, \quad Y^{(k)} = \frac{d^k Y(z)}{dz^k}|_{z=1},$$

$$V^{(k)} = \frac{d^k V(z)}{dz^k}|_{z=1}, \quad \Phi_k = \frac{d^k \Phi(z)}{dz^k}|_{z=1}, \quad k \geq 0.$$

Proposition 3. The matrices $\Phi_l, l \geq 0$, are recurrently calculated as follows:

$$\Phi_l = Z_l T_0^{-1},\ l \geq 0, \tag{27}$$

where the matrix T_0 is obtained by replacing the first column in the matrix $Y^{(0)} - I$ with the column $(Y^{(1)} - I)\mathbf{1}$ and the matrices Z_l are obtained by replacing the first column of the matrix

$$Y^{(l)} - V^{(l)} - lV^{(l-1)} + l\Phi_{l-1} - \sum_{m=0}^{l-1} C_l^m \Phi_m Y^{(l-m)}$$

with the column

$$\frac{Y^{(l+1)} - V^{(l+1)}}{l+1}\mathbf{1} - V^{(l)}\mathbf{1} - (l+1)^{-1}\sum_{m=0}^{l} C_{l+1}^m \Phi_m Y^{(l+1-m)}\mathbf{1}(1 - \delta_{l,0}).$$

Proof. Rewrite formula (19) into the form:

$$\Phi(z)(Y(z) - zI) = (Y(z) - zV(z)). \tag{28}$$

Expand both sides of relation (28) into the series at the point $z = 1$ and consecutively equate the coefficients under the corresponding degrees of $(z - 1)$. The replacement of one column in both sides (the new columns are obtained from relations for the higher degree of $(z - 1)$) is caused by the fact that the matrix $Y^{(0)} - I$, which is a multiplier of the matrix Φ_l on the $(l + 1)$th step of the recurrent procedure, $l \geq 0$, is singular one while the modified matrix T_0 is non-singular (for proof see [16]).

Corollary 2. The matrices Ψ_m defining the expansion:

$$\sum_{m=0}^{\infty} \Psi_m \frac{(z-1)^m}{m!} = \Phi^{-1}(z)$$

are recurrently calculated as follows:

$$\Psi_0 = \Phi_0^{-1}, \quad \Psi_l = -\Phi_0^{-1} \sum_{k=1}^{l} C_l^k \, \Phi_k \, \Psi_{l-k}, \quad l \geq 1. \tag{29}$$

Note that using the formulas (21) we can obtain the explicit expressions for the matrices $Y^{(k)}$, $V^{(k)}$ in the form:

$$Y^{(0)} = I + (C + \gamma\hat{I})^{-1}(\gamma\tilde{I}_\beta - \gamma\hat{I} + \Delta_0),$$

$$V^{(0)} = I + C^{-1}\Delta_0,$$

$$Y^{(1)} = I + (C + \gamma\hat{I})^{-1}(-\gamma\hat{I} + \Delta_0 + \Delta_1),$$

$$Y^{(k)} = (C + \gamma\hat{I})^{-1}(\Delta_{k-1} + \Delta_k), \tag{30}$$

$$V^{(k)} = C^{-1}\Delta_k, \; k \geq 1.$$

Matrices Δ_k are easily calculated directly from (20). Procedure defined by relations (27), which include the matrices defined by (30), allows to calculate any desired number of matrices $\Phi_l, l \geq 0$.

Note that the matrix Φ_0 defines the relation between the vectors $\vec{\Pi}(1)$ and $\vec{\pi}_0 = \vec{\Pi}(0)$:

$$\vec{\Pi}(1) = \vec{\pi}_0 \Phi_0.$$

The average number of calls on the orbit at the embedded epochs is calculated as:

$$\vec{\Pi}'(1)\mathbf{1} = \vec{\pi}_0 \Phi_1 \mathbf{1}. \tag{31}$$

Having known the stationary distribution of the embedded Markov chain $\xi_k, k \geq 1$, we can calculate the distribution of the original continuous-time Markov chain.

Taking into account the formulas

$$\hat{R}_i = \hat{R} = (C + \gamma I)\hat{I} + C\bar{I}, \; i > 0, \quad \hat{R}_0 = C,$$

we get the constant $\bar{c}$ in relation (12) as:

$$\bar{c} = ((\vec{\Pi}(1)\hat{R}^{-1} - \gamma^{-1}\vec{\pi}_0\hat{I})\mathbf{1})^{-1}. \tag{32}$$

The vector generating function $\vec{P}(z) = \sum_{i=0}^{\infty} \vec{p}_i z^i$ of the stationary distribution of the original continuous-time Markov chain is calculated now as

$$\vec{P}(z) = \bar{c}(\vec{\Pi}(z)\hat{R}^{-1} - \gamma^{-1}\vec{\pi}_0\hat{I}), \tag{33}$$

and the stationary probabilities $\vec{p}_i$ are calculated as:

$$\vec{p}_i = \bar{c}\vec{\pi}_i\hat{R}^{-1},\ i > 0, \quad \vec{p}_0 = \bar{c}\vec{\pi}_0 C^{-1}. \tag{34}$$

The vectors $\vec{\pi}_i$, $i > 0$, can be calculated according to Ramaswami's recurrent scheme, see [34, p.143]:

$$\vec{\pi}_i = (\vec{\pi}_0\bar{V}_i + \sum_{j=1}^{i-1} \vec{\pi}_j \bar{Y}_{i-j+1})(1 - \bar{Y}_1)^{-1},\ i \geq 1, \tag{35}$$

where the matrices $\bar{V}_i$, $\bar{Y}_i$ are defined by the formulas

$$\bar{V}_i = \sum_{j=i}^{\infty} V_j G^{j-i}, \quad \bar{Y}_i = \sum_{j=i}^{\infty} Y_j G^{j-i},$$

and the matrix G satisfies the equation

$$G = \sum_{i=0}^{\infty} Y_i G^i.$$

The probabilistic meaning on the matrices $\bar{V}_i$, $\bar{Y}_i, G$ for a more complicated level-dependent case is explained in Section 7.

Although this Ramaswami's scheme seems being hardly implemented because it contains the infinite sums, it is recommended as a very reliable one e.g. by M. Neuts in [34]. Our own numerical experience also confirms a good quality of this scheme.

The factorial moments $\vec{P}^{(m)}(1) = \frac{d^m}{dz^m}\vec{P}(z)|_{z=1}$, $m > 0$, are calculated as:

$$\vec{P}^{(m)}(1) = \bar{c}\vec{\Pi}^{(m)}(1)\hat{R}^{-1}, \quad m \geq 1.$$

Thus, the problem of calculating the stationary state distribution for the retrial $BMAP/PH/N$ system in the case of a constant retrial rate is solved.

Note that M.Neuts' approach for calculating the probability $\vec{\pi}_0$, which is based on the using the matrix G, could be easily applied here instead of the analytic one as well.

6. The case of infinitely increasing retrial rate

We assume that $\alpha_i \to \infty$ when $i \to \infty$. This case includes the classic strategy of retrials ($\alpha_i = i\alpha$) and the strategy of linear repeated requests ($\alpha_i = i\alpha + \gamma$, $i > 0$, see [6]).

In this case we see that

$$Q_1^{(i)} \to I \quad \text{and} \quad Q_2^{(i)} \to 0 \quad \text{as} \quad i \to \infty.$$

Comparing the expressions (9)-(11), which characterize the transition probabilities of the Markov chain ξ_k, $k \geq 1$, with the definition of the asymptotically quasitoeplitz Markov chain ($AQTMC$) in [20], we conclude that the chain ξ_k, $k \geq 1$, belongs to the class of $AQTMC$. So we can use the results of [20] for investigating the Markov chain ξ_k, $k \geq 1$.

As it is stated in [20], the stationary distribution existence condition for the $AQTMC$ coincides with the corresponding condition for the limiting quasitoeplitz chain for a given $AQTMC$.

Define as $\tilde{Y}(z)$ and $\tilde{V}(z)$ the matrix generating functions characterizing the limiting chain. From the definition of a limiting quasitoeplitz Markov chain and (9)-(11) (or from (21)), we get the expressions

$$\tilde{V}(z) = I + C^{-1}D^{\star}(z),$$

$$\tilde{Y}(z) = \tilde{I}_{\beta} + z\bar{I} + C^{-1}\bar{I}zD^{\star}(z). \tag{36}$$

Substituting the explicit expressions for the matrices $\bar{I}, \tilde{I}_{\beta}$, C and $D^{\star}(z)$ into (36), we see that the matrix $\tilde{Y}(z)$ is reducible. The normal form of the matrix $\tilde{Y}(1)$ contains only one irreducible stochastic diagonal block $\tilde{\tilde{Y}}(1)$. The corresponding block $\tilde{\tilde{Y}}(z)$ of the matrix $\tilde{Y}(z)$ has the form

$$\tilde{\tilde{Y}}(z) = \begin{pmatrix} (\Lambda \oplus S_N)^{-1}(zD(z) \oplus zS^{\oplus N}) + zI & (\Lambda \oplus S_N)^{-1}(I_{\overline{W}} \otimes S_0^{\oplus N})z \\ I_{\overline{W}M^{N-1}} \otimes \beta & O_{\overline{W}M^{N-1}} \end{pmatrix}. \tag{37}$$

Here $\Lambda \oplus S_N \stackrel{def}{=} diag\,\{\, \lambda_\nu,\ \nu = \overline{0, W}\,\} \oplus [\, diag\,\{\, s_m,\ m = \overline{1, M}\,\}\,]^{\oplus N}$.

It follows from *Corollary* 1 in [20], that a sufficient condition for the stationary distribution existence is the fulfillment of the inequality

$$(\, det\,(zI - \tilde{\tilde{Y}}(z)\,)' \,|_{z=1} > 0. \tag{38}$$

A more constructive form of condition (38) in our case is given in the following statement.

Theorem 3. A sufficient condition for the existence of the stationary distribution for the $BMAP/PH/N$ model with infinitely increasing retrial rate is given by

$$\rho = \lambda/\bar{\mu} < 1, \tag{39}$$

where λ is the average arrival rate,

$$\bar{\mu} = \vec{y} S_0^{\oplus N} \mathbf{1}_{M^{N-1}}, \tag{40}$$

$\vec{y}$ is the unique solution of the following system of linear algebraic equations:

$$\begin{aligned} \vec{y}(S^{\oplus N} + S_0^{\oplus N}(I_{M^{N-1}} \otimes \beta)) = 0, \\ \vec{y}\mathbf{1} = 1. \end{aligned} \tag{41}$$

The proof is given in the *Appendix* 2.

Corollary 3. For the $BMAP/PH/1$ retrial system the sufficient condition for the stationary distribution existence has the form

$$\rho = \lambda/\mu < 1,$$

where $\mu = -(\beta S^{-1}\mathbf{1})^{-1}$ is the intensity of the PH-service.

The proof follows from (39)-(41) if we set $\vec{y} = -\mu\beta S^{-1}$.

Corollary 4. For the $BMAP/M/N$ retrial system the sufficient condition for the stationary distribution existence has the form (39) where

$$\bar{\mu} = N\mu, \quad \mu = -(\beta S^{-1}\mathbf{1})^{-1}. \tag{42}$$

The proof follows from (39)-(41) if we set $\vec{y} = 1$ and get the equality $\vec{y}\, S_0^{\oplus N} 1 = N\mu$.

Remark 1. For the $BMAP/PH/N/N+K$ retrial model, $K > 0$ (it means the system has a finite buffer) the sufficient condition also has the form (39), (42). The proof is analogous to the one given in *Appendix* 2 taking into account that in this case the matrix $S^{\oplus N} + S_0^{\oplus N}(I_{M^{N-1}} \otimes \beta)$ in (41) must be changed to the matrix $S^{\oplus N} + (S_0\beta)^{\oplus N}$ and the vector $\vec{y}$ has the form: $\vec{y} = -(\mu\beta S^{-1})^{\otimes N}$.

In case $K = 0$ (i.e. we have the pure retrial model), we get the condition in the form (39)-(41). In opposite to us, in [28] it is stated that in case $K = 0$ the condition has the form (39), (42). But the proof given in [28] is not correct.

Remark 2. Condition (39), (42) holds true also for the $BMAP/PH/N/\infty$ model with the standard enumeration of the busy servers. Condition (39)-(41) is valid for the $BMAP/PH/N/\infty$ models in case we enumerate the busy servers according to the rule described in the beginning of *Section* 3, i.e. the servers are enumerated at the service completion epochs and the maximal number gets the server which last begins the service.

In case of a general dependence α_i of i, to calculate the stationary state distribution of the Markov chain ξ_k, $k \geq 1$, we can develop a numerical algorithm using the steady-state equations (15).

In the case when the dependence has a linear form: $\alpha_i = i\alpha + \gamma$, $i > 0$ we can additionally derive the vector functional-differential equation for the vector generating function $\vec{\Pi}(z)$.

Theorem 4. Let the stationary distribution existence conditions (39)-(41) be fulfilled. Then the vector generating function $\vec{\Pi}(z)$ of the stationary distribution of ξ_k, $k \geq 1$, satisfies the following functional-differential equation:

$$\vec{\Pi}'(z) = \vec{\Pi}(z)S(z) + \vec{\Pi}(0)z^{-1}\gamma\alpha^{-1}\tilde{\Phi}(z), \tag{43}$$

where the matrices $S(z)$ are defined as follows:

$$S(z) = \tilde{\Phi}^{-1}(z)\tilde{\Phi}'(z) - \tilde{\Phi}^{-1}(z)(C + \gamma I)\alpha^{-1}z^{-1}\tilde{\Phi}(z) + \alpha^{-1}z^{-1}C\tilde{\Phi}(z),$$

$$\tilde{\Phi}(z) = (\tilde{I}_\beta - z\hat{I} - C^{-1}\hat{I}zD^\star(z)) \cdot (\tilde{I}_\beta - z\hat{I} - C^{-1}\bar{I}zD^\star(z))^{-1}. \tag{44}$$

Remark 3. The methods for solving equation (43) have not been elaborated yet. However, by sequential differentiation of this equation we can get the exact analytic relations between the factorial moments of any order of the stationary distribution. Thus, this equation has a high practical importance for constructing the effective stop-rules for the numerical procedures for calculating the vectors $\vec{\Pi}(0) = \vec{\pi}_0$, $\vec{\pi}_i$, $i \geq 1$, see, e.g. the procedure presented in *Section* 7.

Remark 4. Expression (44) is equivalent to:

$$\tilde{\Phi}(z) = (\tilde{Y}(z) - z\tilde{V}(z))(\tilde{Y}(z) - zI)^{-1}, \tag{45}$$

where the matrix generating functions $\tilde{Y}(z)$ and $\tilde{V}(z)$ define a limiting quasitoeplitz chain and have form (36).

The proof of *Theorem* 4 is implemented by analogy with the proof of *Theorem* 4 in [20].

Mention that in the proof we get an important intermediate formula, which is valid for the arbitrary form of intensities of retrials:

$$\vec{\Pi}(z) = \sum_{i=0}^{\infty} \vec{\pi}_i z^i (C + \alpha_i I)^{-1} C\Phi(z). \tag{46}$$

This formula is useful for finding the relationship between the function $\vec{P}(z)$, which determines the stationary distribution of the original continuous-time Markov chain, and the vector generating function $\vec{\Pi}(z)$.

The relation (12) has here the form:

$$\vec{p}_i = \bar{c}\vec{\pi}_i((C + \alpha_i I)^{-1}\hat{I} + C^{-1}\bar{I}),\ i \geq 0. \tag{47}$$

Multiplying these relations by corresponding degrees of z and summing up, we get the expression

$$\vec{P}(z) = \bar{c}\sum_{i=0}^{\infty} \vec{\pi}_i z^i ((C + \alpha_i I)^{-1}\hat{I} + C^{-1}\bar{I}).$$

Taking into account (46), we get the relation:

$$\vec{P}(z) = \bar{c}\,\vec{\Pi}(z)C^{-1}(\tilde{\Phi}^{-1}(z)\hat{I} + \bar{I}). \tag{48}$$

Now it is easy to calculate the coefficient $\bar{c}$:

$$\bar{c} = ((\vec{\Pi}(1)C^{-1}(\tilde{\Phi}^{-1}(1)\hat{I} + \bar{I}))\mathbf{1})^{-1}.$$

The values of the matrix $\tilde{\Phi}^{-1}(z)$ in (48) and its derivatives are easily calculated basing on Proposition 3 and its Corollary. Comparing formulas (19) and (45), we see that the coefficients of expanding $\tilde{\Phi}(z)$ are calculated by formulas (27) with replacement of matrices $Y^{(l)}, V^{(l)}$ by the matrices $\tilde{Y}^{(l)}, \tilde{V}^{(l)}$ which are defined by the recurrent relations:

$$\tilde{Y}^{(0)} = \bar{I} + \tilde{I}_\beta + C^{-1}\bar{I}\Delta_0, \quad \tilde{V}^{(0)} = I + C^{-1}\Delta_0,$$

$$\tilde{Y}^{(1)} = \bar{I} + C^{-1}\bar{I}(\Delta_0 + \Delta_1),$$

$$\tilde{Y}^{(k)} = C^{-1}\bar{I}(\Delta_{k-1} + \Delta_k),\ k \geq 2, \quad \tilde{V}^{(k)} = C^{-1}\Delta_k,\ k \geq 1.$$

The coefficients of expanding the matrix function $\tilde{\Phi}^{-1}(z)$ are the direct analogs of those given by formulas (29).

Thus, if we succeed to calculate the generating function $\vec{\Pi}(z)$ for the discrete-time asymptotically quasitoeplitz Markov chain, we easily get the distribution of the original chain describing the arbitrary-time distribution of the states of the retrial $BMAP/PH/N$ system.

7. The algorithm for calculating the probability vectors $\vec{\pi}_i$, $i \geq 0$, in case of infinitely increasing retrial rate

In the previous Section, we have got the stationary distribution existence condition for the embedded Markov chain which is the sufficient condition for existence of the stationary distribution of the original continuous-time Markov chain as well. We have got also formulas (47),(48) relating the distributions of these chains. However, the problem of calculation of the stationary state probabilities $\vec{\pi}_i, i \geq 0$ of the embedded chain is not solved yet.

We can not exploit here the algorithm for calculating the stationary distribution of the asymptotically quasitoeplitz Markov chains which is elaborated in [20]. That algorithm is oriented to the case when the subdiagonal blocks $Y_{i,i-1}$ of the transition matrix Y are non-singular. This assumption was not very restrictive in the analysis of the $BMAP/SM/1$ retrial model by means of the asymptotically quasitoeplitz Markov chains [20]. But in case of our $BMAP/PH/N$ retrial systems all subdiagonal blocks are singular.

So, in the present Section we develop two new algorithms for calculating the vector $\vec{\pi}_0$ which are suitable for application for arbitrary asymptotically quasitoeplitz Markov chains in case of singular subdiagonal blocks $Y_{i,i-1}$ of the transition matrix Y. The results of this Section can be considered as the complementary to the theoretical results of paper [20].

Both these new algorithms (as well as the algorithm given in [20]) are based on exploiting the level-dependent analogs to the matrix G defined in Section 5.

Let $G^{(k)}$ be the probabilistic matrix describing the transitions of the finite components of the Markov chain $\xi_n, n \geq 1$, during the first passage time of the denumerable component from the level k to the level $k-1, k \geq 1$. The matrices $G^{(k)}$ satisfy the following recurrent relations:

$$G^{(k)} = Y_{k,k-1} + \sum_{n=0}^{\infty} Y_{k,k+n} \prod_{j=0}^{n} G^{(k+n-j)} \tag{49}$$

(see [29], *Corollary* 5.3 or [20]). In the classical level independent case, there are several algorithms for computing the matrix G, see Breuer et al. [9]. Similarly, we have several possibilities in the level–dependent case. The simplest (but not the worst) way consists of setting the value of $G^{(l)}$ being equal to the constant matrix G for all l greater than some threshold L. The matrix G is defined as a solution to the matrix equation $G = \sum_{n=0}^{\infty} \tilde{Y}_n G^n$ where the matrices $\tilde{Y}_n$ are the coefficients of expanding the generating function $\tilde{Y}(z)$ (see formula (36)) into series at the point $z = 0$. The value of the threshold L depends on the convergence rate of the asymptotically quasitoeplitz Markov chains to the corresponding limiting quasitoeplitz Markov chain and the desired accuracy of calculation of the vector $\vec{\pi}_0$. The rest of the matrices $G^{(l)}$ (those for $l = \overline{1, L-1}$) are easily calculated now from the backward recursion (49).

Extending Ramaswami's idea [39] to the case of a level-dependent Markov chain, we prove the following result.

Proposition 4. Let the stationary distribution existence conditions (39)-(41) be fulfilled. Then the probability vectors $\vec{\pi}_i, i \geq 1$, are calculated as:

$$\vec{\pi}_i = \vec{\pi}_0 F_i, i \geq 1, \tag{50}$$

where the matrices F_i are calculated recurrently:

$$F_0 = I,$$

$$F_k = (\bar{V}_k + \sum_{i=1}^{k-1} F_i \bar{Y}_{k+1-i}^{(i)})(I - \bar{Y}_1^{(k)})^{-1}, k \geq 1, \tag{51}$$

the matrices $\bar{Y}_n^{(k)}$ and $\bar{V}_n$ are defined by formulas

$$\bar{Y}_n^{(k)} = \sum_{i=n}^{\infty} Y_{k,k+i-1} \prod_{j=1}^{i-n} G^{(k+i-j)},$$

$$\bar{V}_n = \sum_{i=n}^{\infty} Y_{0,i} \prod_{j=0}^{i-n-1} G^{(i-j)}. \tag{52}$$

Note that the entries of the matrix $\bar{Y}_n^{(k)}, n \geq k, k \geq 1$, define the transition probabilities of the finite components during the time interval until the state n of the denumerable component will be reached first time conditional that the initial value of this component was k and the first transition was made into the state

$i, i \geq n$. The entries of the matrix $\bar{V}_n, n \geq 0$, define the corresponding transition probabilities for the initial state $k = 0$.

Using recursion (51), we can calculate the matrices F_i until they become negligible with the desired accuracy. Thus, we need to calculate the vector $\vec{\pi}_0$ and the problem of calculating all probabilities $\vec{\pi}_l, l \geq 0$, will be solved.

The first way to calculate the vector $\vec{\pi}_0$ is extremely simple. In the same way as the recursion (51) is derived, we get the equation

$$\vec{\pi}_0(I - \bar{V}_0) = \vec{0}. \tag{53}$$

The rank of the matrix $I - \bar{V}_0$ is equal to $K - 1$. So we need one more equation to get a solvable system of linear algebraic equations for the entries of the vector $\vec{\pi}_0$. Taking into account representation (50) and the normalization condition, we get the additional equation in the form:

$$\vec{\pi}_0 \sum_{k=0}^{\infty} F_k \mathbf{1} = 1. \tag{54}$$

The system (53),(54) has a unique solution, so the probability vector $\vec{\pi}_0$ is found.

Surprisingly, this simple procedure is not described in the literature and should be considered as a novel one. The probable reason of this situation is the following. The procedure of Ramaswami was initially developed for the quasi-toeplitz Markov chains and it is applied for calculating the probability vectors $\vec{\pi}_i, i \geq 1$, by means of analogs of formulas (50), (52). The vector $\vec{\pi}_0$ is assumed being calculated in advance using the well-known correspondence between the stationary probability of the state and the average time until the first return to this state. Approach of M. Neuts is based namely on this correspondence. As the result, M. Neuts has elaborated a simple procedure for calculating the vector $\vec{\pi}_0$. So Ramaswami's procedure was applied only for reliable calculation of the vectors $\vec{\pi}_i, i \geq 1$ and the possibility of calculating the vector $\vec{\pi}_0$ from the same procedure is not described.

The second way to calculate the vector $\vec{\pi}_0$ consists of the direct generalization of M. Neuts' approach to the case of the level-dependent Markov chains. According to the M. Neuts' approach, the vector $\vec{\pi}_0$ is calculated as:

$$\vec{\pi}_0 = \frac{\vec{\kappa}}{(\vec{\kappa}, \vec{c}_B)}, \tag{55}$$

where the probability row-vector $\vec{\kappa}$ is an eigenvector of the matrix which characterizes the transition probabilities of the finite components during the time interval until the first return to the state 0. The column-vector $\vec{c}_B$ consists of conditional expectations of the number of transitions until the first return to the state 0 starting from the corresponding state of the finite components.

According to the definition of the vector $\vec{\kappa}$ and definition of the matrices $\bar{V}_k, k \geq 0$, we conclude that the vector $\vec{\kappa}$ is the unique solution to the following system of linear algebraic equations:

$$\vec{\kappa}\bar{V}_0 = \vec{\kappa}, \vec{\kappa}\mathbf{1}. \tag{56}$$

The procedure for calculation of the vector $\vec{c}_B$ is analogous to one described in Hofmann ([29], pp. 40-45, p.52).

Proposition 5. The vector $\vec{c}_B$ is defined as

$$\vec{c}_B = \sum_{j=1}^{\infty} Y_{0,j} \sum_{l=0}^{j-1} \left(\prod_{i=0}^{l-1} G^{(j-i)} \right) \vec{c}_F^{(j-l)}$$

with $\vec{c}_F^{(k)}$ denoting the mean number of transitions before the orbit size decreases from k to $k-1$.

The vectors $\vec{c}_F^{(k)}$ can be computed as

$$\begin{pmatrix} \vec{c}_F^{(1)} \\ \vec{c}_F^{(2)} \\ \vec{c}_F^{(3)} \\ \vdots \end{pmatrix} = \mathcal{M}'\mathbf{1}$$

with $\mathcal{M}' = \lim_{n\to\infty} \mathcal{M}_n^{-1}$ being determined by the iteration

$$\mathcal{M}_1^{-1} = \left(M^{(1,0)}\right)^{-1}$$

and

$$\mathcal{M}_{n+1}^{-1} = \left(\begin{array}{c|c} \mathcal{M}_n^{-1} & -\mathcal{M}_n^{-1} \begin{pmatrix} M^{(1,n)} \\ \vdots \\ M^{(n,1)} \end{pmatrix} \left(M^{(n+1,0)}\right)^{-1} \\ \hline & \left(M^{(n+1,0)}\right)^{-1} \end{array} \right)$$

for $n > 0$. Here the matrices $M^{(k,n)}$ are defined by

$$M^{(k,0)} = I - \bar{Y}_1^{(k)}$$

for $k > 0$, and

$$M^{(k,n)} = -\sum_{l=1}^{\infty} Y_{k,k+n+l-1} \prod_{j=1}^{l-1} G^{(k+n+l-j)}$$

for $k, n > 0$.

Thus, two ways for calculating the vector $\vec{\pi}_0$ are described. Recalling relations (50), (52), we conclude that the problem of calculation of the stationary probabilities for the embedded Markov chain is solved.

8. Conclusion

We have considered the $BMAP/PH/N$ retrial queueing system. The behavior of this system is evidently described by a multi-dimensional continuous-time Markov chain. In general, this chain has an infinitesimal generator with level depending blocks. The generator does not have a three-diagonal form and the chain does not belong to the known class of Level Dependent Birth-and-Death Processes. To investigate this Markov chain, we reduce it to the corresponding multi-dimensional discrete-time Markov chain. Depending on the strategy of retrials (constant retrial rate or infinitely increasing retrial rate), this discrete-time Markov chain belongs to the class of multi-dimensional quasitoeplitz Markov chains [19] or asymptotically quasitoeplitz Markov chains [20]. The technique in the first case is well-known. In the second case, we exploit and develop the results of [20] to investigate the chain. The numerical algorithm for approximate calculating the stationary state probabilities is modified essentially comparing to [20].

Our experience in implementation of the similar algorithms (see, e.g. [17,18,20,21,24]) allows to predict the stable work of this algorithm. However, the numerical work and obtaining the graphical dependences of the main performance characteristics of the model on its numerous parameters should be made carefully. So the results are planned to be reported later in some Journal in the field of performance evaluation and capacity planning for the computer communication networks.

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Appendix 1.

Lemma Let the matrix $-B$ be an irreducible infinitesimal generator. Then
• the vector $\vec{\bigtriangledown}$ of the cofactors for the entries of the some column of the determinant $det\, B$ is positive,
• all solutions of the system of linear algebraic equations

$$\vec{x}B = 0$$

can be represented in the form

$$\vec{x} = c\vec{\bigtriangledown}.$$

Proof. The first conclusion follows from the structure of the matrix B. The diagonal entries of this matrix are positive, the non-diagonal entries are non-positive and the row sums are equal to zero. All cofactors of the determinant of such a kind of matrices are non-negative (see, e.g. [8]). This implies $\vec{\bigtriangledown} \geq 0$. The inequality $\vec{\bigtriangledown} > 0$ follows from the irreducibility of the matrix B. The second conclusion of the lemma is a well-known fact from algebra.

The proof of **Theorem 2.**

Denote as $B(z)$ the matrix in (23), i.e.

$$B(z) = -\gamma\tilde{I}_\beta + \gamma\hat{I}z - zD^\star(z).$$

Summing up all columns of the $detB(z)$ to the first column, expanding the determinant along the entries of the first column and using the fact that $-B(1)$ is the infinitesimal generator, we obtain the inequality which is equivalent to inequality (23)

$$\vec{\bigtriangledown}B'(1)\,\mathbf{1} > 0, \qquad (A1.1)$$

where $\vec{\bigtriangledown}$ is the vector of the cofactors for the entries of the first column of the $detB(1)$.

Consider the system of linear algebraic equations

$$\begin{cases} \vec{X}B(1) = 0, \\ \vec{X}\,\mathbf{1} = 1. \end{cases} \qquad (A1.2)$$

It follows from *Lemma* that the unique solution of system (A1.2) can be represented in the form

$$\vec{X} = (\vec{\bigtriangledown}\,\mathbf{1})^{-1}\vec{\bigtriangledown} > 0,$$

where $\vec{\bigtriangledown} > 0$.

Then inequality (A1.1) is equivalent to the following inequality:

$$\vec{X}B'(1)\,\mathbf{1} > 0, \qquad (A1.3)$$

where $\vec{X}$ satisfies the system (A1.2).

Taking into account the form of the matrix $B(z)$, we can easily verify that (A1.2), (A1.3) prove the theorem.

Appendix 2.

The proof of **Theorem 3.**

Using the block structure of the determinant $det\,(zI - \tilde{\tilde{Y}}(z))$ in (38), we can reduce it to the following form:

$$\begin{aligned} det\,(zI - \tilde{\tilde{Y}}(z)) = det\,(\Lambda \oplus S_N)^{-1} z^{\bar{W}M^{N-1}} \times \\ \times det\,(-z(D(z) \oplus S^{\oplus N}) - (I_{\bar{W}} \otimes S_0^{\oplus N})\,(I_{\bar{W}M^{N-1}} \otimes \beta)). \end{aligned} \qquad (A2.1)$$

Taking into account (A2.1), it is easy to show that inequality (38) is equivalent to the following inequality:

$$(det\,T(z))'\,|_{z=1} > 0, \qquad (A2.2)$$

where

$$T(z) = -z(D(z) \oplus S^{\oplus N}) - (I_{\bar{W}} \otimes S_0^{\oplus N})\,(I_{\bar{W}M^{N-1}} \otimes \beta).$$

In the same way as in Appendix 1, we can prove that inequality (A2.2) is equivalent to the inequality

$$\vec{X}\,T'(1)\,\mathbf{1} > 0, \qquad (A2.3)$$

where $\vec{X}$ satisfies the system of linear algebraic equations

$$\begin{cases} \vec{X}\,T(1) = 0, \\ \vec{X}\,\mathbf{1} = 1. \end{cases} \qquad (A2.4)$$

Representing the vector $\vec{X}$ in the form $\vec{X} = \vec{\theta} \otimes \vec{y}$ we can easily verify that $\vec{X}$ satisfies (A2.4) if the vector $\vec{y}$ is a solution of system (41). Since the matrix $S^{\oplus N} + S_0^{\oplus N}(I_{\bar{W}M^{N-1}} \otimes \beta)$ is an irreducible generator, it follows from Lemma in Appendix 1 that system (41) has a unique solution.

Theorem 3 is proven.

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