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OCCUPATION TIMES FOR MARKOV-MODULATED BROWNIAN MOTION

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Abstract

We determine the distributions of occupation times of a Markov-modulated Brownian motion (MMBM) in separate intervals before a first passage time or an exit from an interval. They will be derived in terms of their Laplace transforms, distinguishing occupation times in different phases too. For MMBMs with strictly positive variation parameters we further propose scale functions.

Keywords: Markov-modulated Brownian motion; occupation time; scale function; Markov-additive process

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1. Introduction

Let $\mathcal{J} = (J_t : t \geq 0)$ denote an irreducible Markov process with a finite state space $E = \{1, \ldots, m\}$ and infinitesimal generator matrix $Q = (q_{ij})_{i,j \in E}$. We call J_t the phase at time t and $\mathcal{J}$ the phase process. Choosing parameters $\mu_i \in \mathbb{R}$ and $\sigma_i \geq 0$ for all $i \in E$, we define the level process $\mathcal{X} = (X_t : t \geq 0)$ by

$$X_t = X_0 + \int_0^t \mu_{J_s}\, ds + \int_0^t \sigma_{J_s} dW_s$$

for all $t \geq 0$, where $\mathcal{W} = (W_t : t \geq 0)$ denotes a standard Wiener process that is independent of $\mathcal{J}$. Then $(\mathcal{X}, \mathcal{J})$ is called a Markov-modulated Brownian motion (MMBM). An MMBM is a Markov-additive process (MAP, see [2], chapter XI) without jumps.

MMBMs have proved to be powerful tools in stochastic modelling, with applications in queueing theory, insurance and finance. This is even more appparent after one considers the fact that exit problems for MAPs with phase-type jumps can be reduced to an analysis of MMBMs by standard transformation techniques (see e.g. [13, 6]). The class of MAPs with phase-type jumps is dense within all MAPs, see proposition 1 in [3]. Thus we are dealing with occupation times for a dense subset of MAPs.

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Some results for MMBMs go back to the 1990s, with [15] investigating Wiener-Hopf factorisation and stationary distributions for the case that $\sigma_i = \varepsilon$ is independent of the phase process. Around the same time, [1] determined hitting probabilities and based on these expressions for the stationary distributions. More recent results are [10, 7], which analyse MMBMs with two reflecting barriers. Some properties of scale functions for MMBMs are derived in [12].

Occupation times for the phase process before a one- or two-sided exit can be determined via the results in [13, 6]. This will be shortly described in section 2. As an afterthought to this, we shall propose a definition of scale functions for MMBMs with strictly positive variation parameters, i.e. $\sigma_i > 0$ for all $i \in E$, in section 3. The more challenging part will be the determination of occupation time distributions for the combined level and phase process. This is the content of section 4, which deals with the case of only two intervals. The generalisation to more than two intervals is then described in the last section. The appendix contains some lemmata that are used in the proof of the main results.

2. Preliminaries: Occupation times of the phase process

2.1. Occupation times before a first passage

Define the first passage times $\tau(x) := \inf\{t \geq 0 : X_t > x\}$ for all $x \geq 0$ and assume that $X_0 = 0$. We are interested in the occupation times $\zeta_j(x) := \int_0^{\tau(x)} \mathbb{I}_{\{J_t = j\}}\, dt$ in a phase $j \in E$ before the first passage over the level $x \geq 0$. We collect these occupation times in the column vector $\zeta(x) := (\zeta_j(x) : j \in E)$. Consider an E-dimensional row vector $\mathbf{r} = (r_i : i \in E)$ with non-negative entries $r_i \geq 0$ for all $i \in E$. Define

$$\mathbb{E}_{ij}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right) := \mathbb{E}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}; J_{\tau(x)} = j | J_0 = i, X_0 = 0\right) \tag{1}$$

for $i, j \in E$ and $\mathbb{E}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right)$ as the $E \times E$-matrix with these entries. Noting that $e^{-\int_0^{\tau(x)} r_{J_s} ds} = e^{-\mathbf{r}\zeta(x)}$ we see that the matrix $\mathbb{E}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right)$ contains the joint Laplace transforms of the occupation times $\zeta_j(x)$.

In order to determine $\mathbb{E}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right)$, we shall distinguish the phases by the subspaces

$$E_a := \{i \in E : \sigma_i > 0 \text{ or } \mu_i > 0\} \quad \text{and} \quad E_d := E \setminus E_a$$

where phases in E_a (resp. E_d) are called ascending (resp. descending). The same arguments as in [5], section 3, yield

$$\mathbb{E}_{(d,d)}\left(e^{-\int_0^{\tau(x)} r_{J_s}\,ds}\right) = \mathbb{E}_{(a,d)}\left(e^{-\int_0^{\tau(x)} r_{J_s}\,ds}\right) = \mathbf{0}$$

as well as

$$\mathbb{E}_{(d,a)}\left(e^{-\int_0^{\tau(x)} r_{J_s}\,ds}\right) = A(\mathbf{r})e^{U(\mathbf{r})x} \quad \text{and} \quad \mathbb{E}_{(a,a)}\left(e^{-\int_0^{\tau(x)} r_{J_s}\,ds}\right) = e^{U(\mathbf{r})x} \tag{2}$$

where the matrices $A(\mathbf{r})$ and $U(\mathbf{r})$ can be computed as follows. For arguments $\beta \geq 0$ define the functions $\phi_i(\beta) := \beta/\mu_i$ for $i \in E_a$, $\sigma_i = 0$, as well as

$$\phi_i(\beta) = \frac{1}{\sigma_i}\sqrt{2\beta + \frac{\mu_i^2}{\sigma_i^2}} - \frac{\mu_i}{\sigma_i^2} \qquad \text{and} \qquad \phi_i^*(\beta) = \frac{1}{\sigma_i}\sqrt{2\beta + \frac{\mu_i^2}{\sigma_i^2}} + \frac{\mu_i}{\sigma_i^2}$$

for $i \in E_a$, $\sigma_i > 0$. The iteration to determine $A(\mathbf{r})$ and $U(\mathbf{r})$ is slightly changed from [6], section 2.2, to the following form: We obtain $(A(\mathbf{r}), U(\mathbf{r})) = \lim_{n\to\infty}(A_n, U_n)$ for initial values $A_0 := \mathbf{0}$, $U_0 := -diag(\phi_i(q_i + r_i))_{i\in E_a}$ and iterations

$$e_i'U_{n+1} = -\frac{q_i + r_i}{\mu_i}e_i' + \frac{1}{\mu_i}\sum_{j\in E, j\neq i} q_{ij}\; e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} \tag{3}$$

for $i \in E_a, \sigma_i = 0$,

$$e_i'U_{n+1} = -\phi_i(q_i + r_i)e_i' + \frac{2}{\sigma_i^2}\sum_{j\in E, j\neq i} q_{ij}\; e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} (\phi_i^*(q_i + r_i)I - U_n)^{-1} \tag{4}$$

for $\sigma_i > 0$, and

$$e_i'A_{n+1} = \sum_{j\in E, j\neq i} q_{ij}e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} ((q_i + r_i)I + \mu_i U_n)^{-1} \tag{5}$$

for $i \in E_d$. Here e_i' denotes the ith canonical row base vector, $q_i := -q_{ii}$ for all $i \in E$, and I_a is the identity matrix on E_a. The case $\mathbf{r} = \mathbf{0}$ has been analysed earlier in [1].

Remark 1. Let us add an absorbing phase, say Δ, to the phase space E to obtain $E' = E \cup \{\Delta\}$. Define an MMBM $(\mathcal{X}', \mathcal{J}')$ on E' as follows. The generator matrix Q' of $\mathcal{J}'$ shall be given by

$$q_{ij}' := \begin{cases} q_{ij}, & i, j \in E, j \neq i \\ q_{ii} - r_i, & j = i \in E \\ r_i, & i \in E, j = \Delta \\ 0, & i = \Delta, j \in E' \end{cases}$$

Further let

$$(\mu_i', \sigma_i') := \begin{cases} (\mu_i, \sigma_i), & i \in E \\ (0,0), & i = \Delta \end{cases}$$

which means that the phase Δ governs the zero process. Let $\tau_\Delta := \min\{t \ge 0 : J_t' = \Delta\}$ denote the time until absorption in Δ and $\tau'(x) := \inf\{t \ge 0 : X_t' > x\}$ the first passage time of $\mathcal{X}'$ over the level $x \ge 0$. Then

$$\mathbb{E}_{ij}\left(e^{-\int_0^{\tau(x)} r_{J_s}\, ds}\right) = \mathbb{P}\left(\tau'(x) < \tau_\Delta, J'_{\tau'(x)} = j | J_0' = i, X_0' = 0\right)$$

for $i, j \in E' \setminus \{\Delta\}$, i.e. the generalised Laplace transforms of the first passage times $\tau(x)$ can be seen as transition probabilities among the transient phases $i, j \in E' \setminus \{\Delta\}$ for the phase process $\mathcal{J}'$ which terminates at a constant rate r_i during $\{t \ge 0 : J_t' = i\}$. Thus we call $\mathbf{r}$ the exit rate vector.

From this perspective a phase-type distribution with parameters (α, T) on a phase space E can be translated as follows. Let $\eta := -T\mathbf{1}$ denote the exit vector and t_{ij} the entries of T. Consider a random variable $Z \sim PH(\alpha, T)$. Setting $r_i := \eta_i$, $q_{ij} := t_{ij}$ for $i \neq j \in E$ and $(\mu_i, \sigma_i) = (1, 0)$ for all $i \in E$ yields $U(\mathbf{r}) = T$ and thus

$$\mathbb{P}(Z > x) = \mathbb{P}_\alpha\left(\tau'(x) < \tau_\Delta | X_0' = 0\right) = \alpha e^{Tx}\mathbf{1}$$

where $\mathbb{P}_\alpha$ denotes the conditional probability given that $\mathbb{P}(J_0' = i) = \alpha_i$ for $i \in E'$.

Example 1. A Markov-additive process (MAP) with phase-type jumps can be transformed into a MMBM as shown in detail in [6], section 2.1. The resulting MMBM has a phase space $E = E_+ \cup E_p \cup E_\sigma \cup E_n \cup E_-$, where

$$E_p = \{i \in \tilde{E} : \tilde{\mu}_i > 0, \tilde{\sigma}_i = 0\}, \quad E_n = \{i \in \tilde{E} : \tilde{\mu}_i < 0, \tilde{\sigma}_i = 0\}, \quad E_\sigma = \{i \in \tilde{E} : \tilde{\sigma}_i > 0\}$$

and phases in $E_\pm$ represent parts of the jumps (see [6], section 2.1, for a precise definition). In order to retrieve the Laplace transform of the first passage times of the original MAP (i.e. the one with phase-type jumps) it suffices to set $r_i := 0$ for $i \in E_+ \cup E_-$ and $r_i := \gamma$ for $i \in E_p \cup E_\sigma \cup E_n$. This method is called fluid embedding and has been described in section 3 of [13], in section 2.7 of [11] or in section 2.2 of [6].

Example 2. We shall derive the joint Laplace transform of the ruin time and the accumulated claims for the classical compound Poisson risk model. Denote the initial risk reserve by $u \ge 0$.

The claim sizes and inter–claim times shall be independent and have exponential distributions with parameters $\beta > 0$ resp. $\lambda > 0$. The rate of premium income is denoted by $c > 0$. This model has been analysed in [8]. The net profit condition is $c/\lambda > 1/\beta$, which is equivalent to $\lambda/(c\beta) < 1$.

We consider a MMBM $(\mathcal{X}, \mathcal{J})$ which is defined as follows. Let the phase space be given by $E = \{1, 2\}$. The parameters are given by $\sigma_1 = \sigma_2 = 0$, $\mu_1 = 1$, $\mu_2 = -c$, and

$$Q = \begin{pmatrix} -\beta & \beta \\ \lambda & -\lambda \end{pmatrix}$$

Then the ruin time $T(u)$ for the compound Poisson model coincides with the occupation time in phase 2 until the first passage time $\tau(u)$ given that we start with $X_0 = 0$. Likewise, the accumulated claims until ruin, denoted by $D(u)$, coincides with the occupation time in phase 1 until $\tau(u)$. The joint Laplace transform of $D(u)$ and $T(u)$ with arguments r_1 and r_2, respectively, is given by

$$\mathbb{E}\left(e^{-r_1 D(u)} e^{-r_2 T(u)} | X_0 = 0, J_0 = 2\right) = e_2'\, \mathbb{E}\left(e^{-\int_0^{\tau(u)} r_{J_s} ds}\right) \mathbf{1} = A(\mathbf{r}) e^{U(\mathbf{r}) u}$$

where $A(\mathbf{r})$ and $U(\mathbf{r})$ are real numbers. They can be computed by formulas (3) and (5) as the fixed points

$$U(\mathbf{r}) = -(\beta + r_1) + \beta A(\mathbf{r}) \quad \text{and} \quad A(\mathbf{r}) = \lambda \cdot (\lambda + r_2 - cU(\mathbf{r}))^{-1}$$

with minimal positive solution

$$A(\mathbf{r}) = \frac{1}{2c\beta}\left(\lambda + r_2 + c \cdot (\beta + r_1) - \sqrt{(\lambda + r_2 + c \cdot (\beta + r_1))^2 - 4\lambda c\beta}\right)$$

from which $U(\mathbf{r})$ can be readily computed. For $r_1 = 0$ we obtain the Laplace transform of the time of ruin, for which the result is the same as equation (5.38) in [8], see example 5 in [5] for the comparison.

2.2. Occupation times before an exit from an interval

For $l < u$, define $\tau(l, u) := \inf\{t \geq 0 : X_t \notin [l, u]\}$ which is the exit time of $\mathcal{X}$ from the interval $[l, u]$. We shall need an expression for

$$\Psi_{ij}^{+}(l, u|x) := \mathbb{E}\left(e^{-\int_0^{\tau(l,u)} r_{J_s} ds}; X_{\tau(l,u)} = u, J_{\tau(l,u)} = j | J_0 = i, X_0 = x\right)$$

where $x \in [l,u]$ and $i,j \in E$. Define the matrix $\Psi^+(l,u|x) := (\Psi^+_{ij}(l,u|x))_{i,j\in E}$. A formula for $\Psi^+(l,u|x)$ has been derived in [13]. In order to state it we need some additional notation. In order to simplify this notation, we shall from now on exclude the case of a phase $i \in E$ with $\mu_i = \sigma_i = 0$.

Let $(\mathcal{X}^+, \mathcal{J})$ denote the original MMBM and define the process $(\mathcal{X}^-, \mathcal{J}) =_d (-\mathcal{X}^+, \mathcal{J})$, where $=_d$ denotes equality in distribution. The two processes have the same generator matrix Q for $\mathcal{J}$, but the drift parameters are different. Denoting variation and drift parameters for $\mathcal{X}^\pm$ by $\sigma_i^\pm$ and $\mu_i^\pm$, respectively, this means $\sigma_i^- = \sigma_i^+$ and $\mu_i^- = -\mu_i^+$ for all $i \in E$.

Let $A^\pm(\mathbf{r})$ and $U^\pm(\mathbf{r})$ denote the matrices that determine the first passage times of $\mathcal{X}^\pm$ in (2). We shall write $A^\pm = A^\pm(\mathbf{r})$ and $U^\pm = U^\pm(\mathbf{r})$ if we do not wish to underline the dependence on $\mathbf{r}$. Distinguish the ascending phases into the spaces $E_s := \{i \in E_a : \sigma_i = 0\}$ and $E_\sigma := \{i \in E_a : \sigma_i > 0\}$ and let I_s resp. I_σ denote the identity matrices on E_s resp. E_σ. We call a phase $i \in E_s$ strictly ascending. Define the matrices

$$C^+ := C^+(\mathbf{r}) := \begin{pmatrix} \mathbf{0} & I_\sigma \\ \multicolumn{2}{c}{A^+(\mathbf{r})} \end{pmatrix} \qquad \text{and} \qquad C^- := C^-(\mathbf{r}) := \begin{pmatrix} \multicolumn{2}{c}{A^-(\mathbf{r})} \\ I_\sigma & \mathbf{0} \end{pmatrix} \tag{6}$$

of dimensions $(E_\sigma \cup E_d) \times E_a$ and $E_a \times (E_\sigma \cup E_d)$, respectively. Further define

$$W^+ := W^+(\mathbf{r}) := \begin{pmatrix} I_a \\ A^+(\mathbf{r}) \end{pmatrix} \qquad \text{and} \qquad W^- := W^-(\mathbf{r}) := \begin{pmatrix} \multicolumn{2}{c}{A^-(\mathbf{r})} \\ I_\sigma & \mathbf{0} \\ \mathbf{0} & I_d \end{pmatrix}$$

which are matrices of dimensions $E \times E_a$ and $E \times (E_\sigma \cup E_d)$. Finally, let $Z^\pm := C^\pm e^{U^\pm \cdot (u-l)}$. Then equation (23) in [13] states that

$$\Psi^+(l,u|x) = \left(W^+ e^{U^+ \cdot (u-x)} - W^- e^{U^- \cdot (x-l)} C^+ e^{U^+ \cdot (u-l)} \right) \cdot \left(I - Z^- Z^+ \right)^{-1} \tag{7}$$

for $l \le x \le u$. By reflection at the initial level x, we obtain from (7)

$$\begin{aligned} \Psi^-(l,u|x) &:= \mathbb{E}\left(e^{-\int_0^{\tau(l,u)} r_{J_s}\, ds}; X_{\sigma(l,u)} = l | X_0 = x \right) \\ &= \left(W^- e^{U^- \cdot (x-l)} - W^+ e^{U^+ \cdot (u-x)} C^- e^{U^- \cdot (u-l)} \right) \cdot \left(I - Z^+ Z^- \right)^{-1} \end{aligned} \tag{8}$$

for $l \le x \le u$. Note that the expressions in (7) and (8) depend on a choice of $\mathbf{r}$.

Example 3. To continue example 2, we obtain $A^-(\mathbf{r})$ and $U^-(\mathbf{r})$ by solving

$$U^-(\mathbf{r}) = -\frac{\lambda + r_2}{c} + \frac{\lambda}{c} A^-(\mathbf{r}) \qquad \text{and} \qquad A^-(\mathbf{r}) = \beta \cdot \left(\beta + r_1 - U^-(\mathbf{r}) \right)^{-1}$$

This yields

$$A^{-}(\mathbf{r}) = \frac{1}{2\lambda}\left((c\cdot(\beta+r_1)+\lambda+r_2) - \sqrt{(c\cdot(\beta+r_1)+\lambda+r_2)^2 - 4c\beta}\right)$$

whence $U^{-}(\mathbf{r})$ may be readily obtained.

3. Some remarks on scale functions

Noting that $(I - Z^{-}Z^{+})^{-1} = \sum_{n=0}^{\infty}(Z^{-}Z^{+})^{n}$ and $Z^{-}Z^{+}$ represents a crossing over the interval $[l,u]$ from u to l and back, formula (7) has a clear probabilistic interpretation. The term $W^{+}e^{U^{+}\cdot(u-x)}$ simply yields the event that $\mathcal{X}$ exits from u. The correction term $W^{-}e^{U^{-}\cdot x}Z^{+}$ refers to the event that $\mathcal{X}$ descends below l before exiting from u. Multiplication by $(I - Z^{-}Z^{+})^{-1}$ yields all possible combinations with any number of subsequent (down and up) crossings over the complete interval $[l,u]$.

Since $Z^{+} = C^{+}e^{U^{+}\cdot(u-l)}$ we can write $\Psi^{+}(l,u|x)$ in the form

$$\Psi^{+}(l,u|x) = \left(W^{+}e^{-U^{+}\cdot(x-l)} - W^{-}e^{U^{-}\cdot(x-l)}C^{+}\right)\left(e^{-U^{+}\cdot(u-l)} - C^{-}e^{U^{-}\cdot(u-l)}C^{+}\right)^{-1}$$

This comes closer to the usual expression of the exit time distribution in terms of scale functions. For instance, let $\mathcal{X}$ be a Brownian motion with variation $\sigma > 0$ and drift $\mu \in \mathbb{R}$. We then obtain

$$U^{\pm} = \frac{\pm\mu - \sqrt{\mu^2 + 2\gamma\sigma^2}}{\sigma^2}$$

Denote $r := -U^{+}$ and $s := U^{-}$. Then

$$\Psi^{+}(0,b|x) = \frac{e^{rx} - e^{sx}}{e^{rb} - e^{sb}} \tag{9}$$

cf. [9], (2.12 - 2.15), where the γ-scale function is given as $g(x) = e^{rx} - e^{sx}$.

As we can see from (9), scale functions as solutions to the two-sided exit problem are determined only up to a multiplicative constant. The usual unique definition of the γ-scale function $W^{(\gamma)}(x)$ for a Lévy process with cumulant function ψ is in terms of its Laplace transform

$$\int_0^{\infty} e^{-\beta x}W^{(\gamma)}(x)\,dx = \frac{1}{\psi(\beta)-\gamma} \tag{10}$$

for $\beta > \Phi(\gamma)$, where Φ denotes the right inverse of ψ, see (8.5) in [14].

For the case of a Markov-modulated Brownian motion with $\sigma_i > 0$ for all $i \in E$ we can extend the notion of γ-scale functions. In this case there are no matrices $A^{\pm}$ and thus

$W^{\pm} = C^{\pm} = I$. For a vector $v = (v_1, \ldots, v_m)$ define the diagonal matrix with entries taken from v by $\Delta_v := diag(v_i)_{i \in E}$. With $\sigma^2 := (\sigma_1^2, \ldots, \sigma_m^2)$ and $\mu := (\mu_1, \ldots, \mu_m)$ we obtain by the same arguments as for (7) in [5]

$$\Delta_{\mathbf{r}} = \Delta_{\sigma^2/2} U(\mathbf{r})^2 - \Delta_\mu U(\mathbf{r}) + Q$$

(use the function in (1) instead of $f_{ij}(x)$ as defined in (4) of [5]). Note that there is a typo in equation (8) of [5], where it should state $-\Delta_\mu$ instead of $+\Delta_\mu$. Define the scalar cumulant functions $\psi_i(\beta) := \sigma_i^2/2\,\beta^2 + \mu_i \beta$ for $i \in E$ and write $\psi(\beta) := (\psi_1(\beta), \ldots, \psi_m(\beta))$. Then the (matrix-valued) cumulant function of $(\mathcal{X}^+, \mathcal{J})$ is given as $K(\beta) = \Delta_{\psi(\beta)} + Q$, see proposition XI.2.2 in [2]. This yields

$$\begin{aligned} K(\beta) - \Delta_{\mathbf{r}} &= \Delta_{\sigma^2/2}(\beta^2 I - U_+^2) + \Delta_\mu(\beta I + U_+) \\ &= \left(\Delta_{\sigma^2/2}(\beta I - U_+) + \Delta_\mu\right)(\beta I + U_+) \end{aligned} \tag{11}$$

where we write $U_+ = U^+(\mathbf{r})$. Similarly for the negative process $(\mathcal{X}^-, \mathcal{J})$ we obtain

$$\Delta_{\mathbf{r}} = \Delta_{\sigma^2/2} U^-(\mathbf{r})^2 + \Delta_\mu U^-(\mathbf{r}) + Q$$

and hence

$$\begin{aligned} K(\beta) - \Delta_{\mathbf{r}} &= \Delta_{\sigma^2/2}(\beta^2 I - U_-^2) + \Delta_\mu(\beta I - U_-) \\ &= \left(\Delta_{\sigma^2/2}(\beta I + U_-) + \Delta_\mu\right)(\beta I - U_-) \end{aligned} \tag{12}$$

where we write $U_- = U^-(\mathbf{r})$. We propose

$$W^{(\mathbf{r})}(x) = \left(e^{-U_+ \cdot x} - e^{U_- \cdot x}\right) \cdot C, \qquad x \geq 0$$

to be called the $\mathbf{r}$-scale function of $(\mathcal{X}, \mathcal{J})$, where the constant C remains to be determined. Let $\|U_+\|$ denote the largest absolute value of any eigenvalue of U_+. For $\beta > \|U_+\|$ we evaluate

$$\begin{aligned} (K(\beta) - \Delta_{\mathbf{r}}) \cdot \int_0^\infty & e^{-\beta x} W^{(\mathbf{r})}(x)\, dx \\ &= (K(\beta) - \Delta_{\mathbf{r}}) \cdot \left((\beta I + U_+)^{-1} - (\beta I - U_-)^{-1}\right) \cdot C \\ &= \left(\Delta_{\sigma^2/2}(\beta I - U_+) + \Delta_\mu - \Delta_{\sigma^2/2}(\beta I + U_-) - \Delta_\mu\right) \cdot C \\ &= -\Delta_{\sigma^2/2} \cdot (U_+ + U_-) \cdot C \end{aligned}$$

where the second equality is due to (11) and (12). With $C := -(U_+ + U_-)^{-1} \cdot \Delta_{2/\sigma^2}$ we thus obtain

$$(K(\beta) - \Delta_{\mathbf{r}}) \int_0^\infty e^{-\beta x} W^{(\mathbf{r})}(x)\, dx = I$$

for $\beta > \|U_+\|$, which justifies the name "$\mathbf{r}$-scale function".

Remark 2. In order to compare the above proposal with results obtained in [11], section 7.5, we first translate the notation $U_+ = \Lambda$ and $\mathbf{r} = q \cdot \mathbf{1}$. Further note that $e^{U_- \cdot x} = \mathbb{P}(J(\tau^{\{-x\}}))$ and $\Pi = I$, since $E = E_\sigma$. Thus equation (7.7) in [11] translates as $\tilde{W}(x) = e^{-U_+ \cdot x} - e^{U_- \cdot x}$. Moreover, equation (7.9) in [11] together with the above determination of the matrix C yields the expression $\mathbf{L} = -(U_+ + U_-)^{-1} \cdot \Delta_{2/\sigma^2}$ for the matrix of expected local times at 0. Equation (7.4) in [11] then leads to an expresssion

$$\mathbf{L}^q(x) = -\left(I - e^{U_+ \cdot x} e^{U_- \cdot x}\right) \cdot (U_+ + U_-)^{-1} \cdot \Delta_{2/\sigma^2}$$

for the matrix of expected local times at 0 before the first passage over a level $x \geq 0$.

4. Occupation times for level and phase process in two intervals

While occupation times for the phase process have been obtained in [13, 6] and only needed some translation in section 2, the more interesting (and more difficult) part of our investigation are occupation times of the level process in different intervals. Their distribution will be derived in this section for the case of two contiguous intervals. A general recursion scheme for more than two intervals will be provided in section 5.

4.1. Occupation times before an exit from an interval

Recall the definition of the exit times of $\mathcal{X}$ from an interval $[l, u]$, namely

$$\tau(l, u) := \inf\{t \geq 0 : X_t < l \quad \text{or} \quad X_t > u\} \tag{13}$$

where $X_0 \in [u, l]$. Choose some $b \in]l, u[$ and define

$$\zeta_{1,j}(l, u) := \int_0^{\tau(l,u)} \mathbb{I}_{\{X_t < b, J_t = j\}}\, dt \quad \text{and} \quad \zeta_{2,j}(l, u) := \int_0^{\tau(l,u)} \mathbb{I}_{\{X_t > b, J_t = j\}}\, dt$$

for $j \in E$. Further define the column vectors $\zeta_k(l, u) := (\zeta_{kj}(l, u) : j \in E)$ for $k \in \{1, 2\}$. The random variables $\zeta_{1j}(l, u)$ and $\zeta_{2j}(l, u)$ yield the occupation times of $(\mathcal{X}, \mathcal{J})$ in the sets $[l, b[\times \{j\}$ and $]b, u] \times \{j\}$, $j \in E$, before the level process leaves the interval $[l, u]$.

Choose any exit rate vectors $\mathbf{r}_k = (r_{kj} : j \in E)$ for $k \in \{1, 2\}$. We shall derive an expression for

$$E^+(l, u|a) := \mathbb{E}\left(e^{-\mathbf{r}_1\zeta_1(l,u)}e^{-\mathbf{r}_2\zeta_2(l,u)}; X_{\tau(l,u)} = u|X_0 = a\right)$$

where $l < a < u$. This provides the joint Laplace transform of the occupation times $\zeta_{kj}(l, u)$ before the first exit of $[l, u]$, restricted to the exit occurring at u.

There are some simple cases. For $l < a < b < u$ we obtain

$$E^+(l, u|a) = \Psi^+_{\mathbf{r}_1}(l, b|a)\ E^+(l, u|b)$$

by path continuity, and similarly, for $l < b < a < u$, we observe that

$$E^+(l, u|a) = \Psi^+_{\mathbf{r}_2}(b, u|a) + \Psi^-_{\mathbf{r}_2}(b, u|a)\ E^+(l, u|b)$$

Thus it suffices to determine $E^+(l, u|b)$. Write for any matrix M of dimension $E \times E$ the block notation

$$M =: \begin{pmatrix} M_{(a,a)} & M_{(a,d)} \\ M_{(d,a)} & M_{(d,d)} \end{pmatrix} =: \begin{pmatrix} M_{(.,a)} & M_{(.,d)} \end{pmatrix}$$

according to ascending (E_a) or descending phases (E_d). Clearly $E^+_{(.,d)}(l, u|b) = \mathbf{0}$, since u cannot be passed from below in a descending phase. Discerning between initial phases, we find that

$$E^+_{(d,a)}(l, u|b) = \Psi^+_{\mathbf{r}_1}(l, b|b)_{(d,a)}\ E^+_{(a,a)}(l, u|b)$$

such that it remains to determine $E^+_{(a,a)}(l, u|b)$. Write for a matrix M of dimension $E_a \times E_a$

$$M =: \begin{pmatrix} M_{(s,s)} & M_{(s,\sigma)} \\ M_{(\sigma,s)} & M_{(\sigma,\sigma)} \end{pmatrix} =: \begin{pmatrix} M_{(s,.)} \\ M_{(\sigma,.)} \end{pmatrix}$$

in obvious block notation. Conditioning on the number n of possible returns to the level b in a strictly ascending phase (i.e. one from E_s) before exiting the interval $[l, u]$ at u, we observe that

$$\begin{aligned} E^+_{(s,.)}(l, u|b) &= \sum_{n=0}^{\infty} \left(\Psi^-_{\mathbf{r}_2}\Psi^+_{\mathbf{r}_1}\right)^n_{(s,s)} \left(\left(\Psi^+_{\mathbf{r}_2}\right)_{(s,.)} + \left(\Psi^-_{\mathbf{r}_2}\Psi^+_{\mathbf{r}_1}\right)_{(s,\sigma)} E^+_{(\sigma,.)}(l, u|b)\right) \\ &= \left(I_s - \left(\Psi^-_{\mathbf{r}_2}\Psi^+_{\mathbf{r}_1}\right)_{(s,s)}\right)^{-1} \left(\left(\Psi^+_{\mathbf{r}_2}\right)_{(s,.)} + \left(\Psi^-_{\mathbf{r}_2}\Psi^+_{\mathbf{r}_1}\right)_{(s,\sigma)} E^+_{(\sigma,.)}(l, u|b)\right) \end{aligned}$$

where I_s indicates the identity matrix on E_s, $\Psi^\pm_{\mathbf{r}_2} = \Psi^\pm_{\mathbf{r}_2}(b, u|b)$, and $\Psi^+_{\mathbf{r}_1}$ shall denote the $(E_\sigma \cup E_d) \times E_a$ block of $\Psi^+_{\mathbf{r}_1}(l, b|b)$. We have thus reduced the problem to the determination of $E^+_{(\sigma,.)}(l, u|b)$.

Theorem 1. *Write $U_k^\pm := U^\pm(\mathbf{r}_k)$ for $k \in \{1,2\}$. For $l < b < u$,*

$$E^+_{(\sigma,.)}(l,u|b) = 2\left((D_1)_{(\sigma,\sigma)} + (D_2C_1^+)_{(\sigma,\sigma)}\right)^{-1} \\ \times \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(U_2^+ + C_2^- U_2^- C_2^+\right)\left(e^{-U_2^+\cdot(u-b)} - C_2^- e^{U_2^-\cdot(u-b)} C_2^+\right)^{-1}$$

where the constant matrices D_1 and D_2 are given in lemmata 1 and 2.

Proof: We employ the following approximation: Assume that the exit rate vector changes from $\mathbf{r}_2$ to $\mathbf{r}_1$ at $b-\varepsilon$ for downward crossings of b and from $\mathbf{r}_1$ to $\mathbf{r}_2$ at $b+\varepsilon$ for upward crossings. Then we let $\varepsilon \downarrow 0$.

To be more precise, assume that $X_0 = b+\varepsilon$ and define $t_0 := 0$ as well as the times $s_n := \min\{t > t_{n-1} : X_t = b-\varepsilon\}$, $t_n := \min\{t > s_n : X_t = b+\varepsilon\}$ for all $n \in \mathbb{N}$, where $\min\emptyset := \infty$. Let $N := \max\{n \in \mathbb{N}_0 : t_n < \tau(l,u)\}$. Note that on $\{X_{\tau(l,u)} = u\}$ there is for each $s_n < \tau(l,u)$ a t_n with $s_n < t_n < \tau(l,u)$ due to path continuity. We consider

$$E(\varepsilon) := \mathbb{E}_{(\sigma,a)}\left(e^{-\sum_{n=1}^N \int_{t_{n-1}}^{s_n} \mathbf{r}_2 e_{J_s} ds - \int_{t_N}^{\tau(l,u)} \mathbf{r}_2 e_{J_s} ds} e^{-\sum_{n=1}^N \int_{s_n}^{t_n} \mathbf{r}_1 e_{J_s} ds}; X_{\tau(l,u)} = u | X_0 = b+\varepsilon\right)$$

This converges towards

$$\mathbb{E}_{(\sigma,a)}\left(e^{-\int_0^{\tau(l,u)} \mathbf{r}_1 e_{J_s} \mathbb{I}_{\{X_s<b\}} ds} e^{-\int_0^{\tau(l,u)} \mathbf{r}_2 e_{J_s} \mathbb{I}_{\{X_s>b\}} ds}; X_{\tau(l,u)} = u | X_0 = b\right) = E^+_{(\sigma,a)}(l,u|b)$$

as $\varepsilon \downarrow 0$, since $\lim_{\varepsilon\downarrow 0} \int_0^{\tau(l,u)} \mathbb{I}_{\{b-\varepsilon<X_t<b+\varepsilon\}} dt = 0$ a.s.

Write $\Psi_2^-(\varepsilon)$ for the $E_a \times (E_\sigma \cup E_d)$-block of $\Psi^-_{\mathbf{r}_2}(b-\varepsilon, u|b+\varepsilon)$ and $\Psi_2^+(\varepsilon)$ for the $E_a \times E_a$-block of $\Psi^+_{\mathbf{r}_2}(b-\varepsilon,u|b+\varepsilon)$. Further write $\Psi_1^+(\varepsilon)$ for the $(E_\sigma \cup E_d) \times E_a$-block of $\Psi^+_{\mathbf{r}_1}(l, b+\varepsilon|b-\varepsilon)$. Summing up over the number of down and up crossings of the interval $[b-\varepsilon, b+\varepsilon]$ before leaving the interval $[l,u]$ at u, we obtain

$$\begin{aligned} E(\varepsilon) &= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \sum_{n=0}^\infty \left(\Psi_2^-(\varepsilon)\Psi_1^+(\varepsilon)\right)^n \Psi_2^+(\varepsilon) \\ &= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(I_a - \Psi_2^-(\varepsilon)\ \Psi_1^+(\varepsilon)\right)^{-1} I(\varepsilon)\ I(\varepsilon)^{-1}\ \Psi_2^+(\varepsilon) \end{aligned}$$

First we consider $\lim_{\varepsilon\downarrow 0} \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(I_a - \Psi_2^-(\varepsilon)\Psi_1^+(\varepsilon)\right)^{-1} I(\varepsilon)$. Since

$$\lim_{\varepsilon\downarrow 0} \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \Psi_2^-(\varepsilon)\Psi_1^+(\varepsilon) = \begin{pmatrix} I_\sigma & \mathbf{0}_{(\sigma,d)} \end{pmatrix} \Psi_1^+(\varepsilon) = \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix}$$

we find that $\lim_{\varepsilon\downarrow 0} \Psi_2^-(\varepsilon)\Psi_1^+(\varepsilon)$ is an upper triagonal block matrix. We thus obtain

$$\lim_{\varepsilon\downarrow 0} \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(I_a - \Psi_2^-(\varepsilon)\Psi_1^+(\varepsilon)\right)^{-1} I(\varepsilon) = \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & D \end{pmatrix}$$

for an $E_\sigma \times E_\sigma$ matrix D that is given by

$$\begin{aligned}
D &= \lim_{\varepsilon \downarrow 0} \varepsilon \left(I_\sigma - \left(\Psi_2^-(\varepsilon) \Psi_1^+(\varepsilon) \right)_{(\sigma,\sigma)} \right)^{-1} \\
&= - \left(\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left(\left(\Psi_2^-(\varepsilon) \Psi_1^+(\varepsilon) \right)_{(\sigma,\sigma)} - I_\sigma \right) \right)^{-1} \\
&= - \left(\frac{d}{d\varepsilon} \left(\Psi_2^-(\varepsilon) \Psi_1^+(\varepsilon) \right)_{(\sigma,\sigma)} \Big|_{\varepsilon=0} \right)^{-1} \\
&= - \left(\left(\frac{d}{d\varepsilon} \Psi_2^-(\varepsilon) \Big|_{\varepsilon=0} \Psi_1^+(0) \right)_{(\sigma,\sigma)} + \left(\Psi_2^-(0) \, \frac{d}{d\varepsilon} \Psi_1^+(\varepsilon) \Big|_{\varepsilon=0} \right)_{(\sigma,\sigma)} \right)^{-1} \\
&= - \left(\left(D_2 C_1^+ \right)_{(\sigma,\sigma)} + (D_1)_{(\sigma,\sigma)} \right)^{-1}
\end{aligned}$$

see [4], section I.1.3, as well as lemmata 1 and 2 for the last two equalities. In a similar manner, since $\lim_{\varepsilon \downarrow 0} \Psi_2^+(\varepsilon)_{(\sigma,a)} = \mathbf{0}$, we obtain

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \, \Psi_2^+(\varepsilon)_{(\sigma,a)} = \frac{d}{d\varepsilon} \Psi_2^+(\varepsilon)_{(\sigma,a)} \Big|_{\varepsilon=0} = D_3$$

according to lemma 3. Altogether this yields the expression in the statement.

□

Example 4. Consider a finite buffer with capacity $u > 0$. The buffer content is modelled by a Brownian motion with parameters $\mu < 0$ (drift) and $\sigma > 0$ (variation). This corresponds to a phase space $E = E_\sigma = \{1\}$ consisting of one element only. Thus $C^\pm = 1$.

Assume that there is a level $b \in]0, u[$ above which there is a higher cost attached. We wish to compute the Laplace transform of the time spent above the level b along with the probability of a buffer overflow. To shorten considerations, we assume that the initial buffer content is b. Then the Laplace transform we aim for can be computed as $E^+(0, b, u)$ with exit rate vectors $\mathbf{r}_1 = 0$ and $\mathbf{r}_2 = \gamma$, where γ is the argument for the Laplace transform. Thus we need to determine only $U_1^\pm$ and $U_2^\pm$, which are real numbers. We obtain

$$U_k^\pm = -\frac{1}{\sigma} \sqrt{2\mathbf{r}_k + \frac{\mu^2}{\sigma^2}} \pm \frac{\mu}{\sigma^2}$$

for $k \in \{1, 2\}$, according to (4). Since $\mu < 0$, this yields $U_1^+ = 2\mu/\sigma^2$ and $U_1^- = 0$. Hence

$$D_1 = 2 \frac{2\mu/\sigma^2}{1 - e^{2\mu/\sigma^2 \cdot b}}$$

and, after abbreviating $W := \frac{2}{\sigma}\sqrt{2\gamma + \frac{\mu^2}{\sigma^2}}$,

$$\begin{aligned}
E^+(0,u|b) &= \left(\frac{2\mu/\sigma^2}{1-e^{2\mu/\sigma^2\cdot b}} + \frac{U_2^- e^{-U_2^-\cdot(u-b)} + U_2^+ e^{U_2^+\cdot(u-b)}}{e^{-U_2^-\cdot(u-b)} - e^{U_2^+\cdot(u-b)}}\right)^{-1} \\
&\quad\times \frac{U_2^+ + U_2^-}{e^{-U_2^+\cdot(u-b)} - e^{U_2^-\cdot(u-b)}} \\
&= \left(\frac{-2\mu/\sigma^2}{1-e^{2\mu/\sigma^2\cdot b}} + \frac{-U_2^- - U_2^+ e^{-W\cdot(u-b)}}{1-e^{-W\cdot(u-b)}}\right)^{-1} \frac{W}{1-e^{-W\cdot(u-b)}} e^{U_2^+\cdot(u-b)} \\
&= \left(\frac{-2\mu/\sigma^2}{1-e^{2\mu/\sigma^2\cdot b}} + \frac{W}{1-e^{-W\cdot(u-b)}} + U_2^+\right)^{-1} \frac{W}{1-e^{-W\cdot(u-b)}} e^{U_2^+\cdot(u-b)}
\end{aligned}$$

since $U_2^+ + U_2^- = -\frac{2}{\sigma}\sqrt{2\gamma + \frac{\mu^2}{\sigma^2}} = -W$.

Example 5. Considering Brownian motion as in the previous example, but this time with exit rates $\mathbf{r}_1 = \mathbf{r}_2 = \gamma$, we obtain $U_1^\pm = U_2^\pm =: u^\pm$. This yields

$$D_1 = 2\frac{u^+ e^{-u^+\cdot(b-l)} + u^- e^{u^-\cdot(b-l)}}{e^{-u^+\cdot(b-l)} - e^{u^-\cdot(b-l)}}$$

and

$$D_2 = 2\frac{u^- e^{-u^-\cdot(u-b)} + u^+ e^{u^+\cdot(u-b)}}{e^{-u^-\cdot(u-b)} - e^{u^+\cdot(u-b)}} = 2\frac{u^- e^{-u^+\cdot(u-b)} + u^+ e^{u^-\cdot(u-b)}}{e^{-u^+\cdot(u-b)} - e^{u^-\cdot(u-b)}}$$

Thus

$$\begin{aligned}
\mathbb{E}\left(e^{-\gamma\tau(l,u)}; X_{\tau(l,u)} = u | X_0 = b\right) &= \frac{u^+ + u^-}{e^{-u^+\cdot(u-b)} - e^{u^-\cdot(u-b)}} \\
&\times \left(\frac{u^+ e^{-u^+\cdot(b-l)} + u^- e^{u^-\cdot(b-l)}}{e^{-u^+\cdot(b-l)} - e^{u^-\cdot(b-l)}} + \frac{u^- e^{-u^+\cdot(u-b)} + u^+ e^{u^-\cdot(u-b)}}{e^{-u^+\cdot(u-b)} - e^{u^-\cdot(u-b)}}\right)^{-1}
\end{aligned}$$

Extending the fractions by $(e^{-u^+\cdot(b-l)} - e^{u^-\cdot(b-l)})(e^{-u^+\cdot(u-b)} - e^{u^-\cdot(u-b)})$ yields

$$\begin{aligned}
\mathbb{E}\left(e^{-\gamma\tau(l,u)}; X_{\tau(l,u)} = u | X_0 = b\right) &= (u^+ + u^-)(e^{-u^+\cdot(b-l)} - e^{u^-\cdot(b-l)}) \\
&\times \Big((u^+ e^{-u^+\cdot(b-l)} + u^- e^{u^-\cdot(b-l)})(e^{-u^+\cdot(u-b)} - e^{u^-\cdot(u-b)}) \\
&\quad + (u^- e^{-u^+\cdot(u-b)} + u^+ e^{u^-\cdot(u-b)})(e^{-u^+\cdot(b-l)} - e^{u^-\cdot(b-l)})\Big)^{-1} \\
&= (u^+ + u^-)(e^{-u^+\cdot(b-l)} - e^{u^-\cdot(b-l)}) \\
&\quad \times \left(u^+ e^{-u^+\cdot(u-l)} - u^- e^{u^-\cdot(u-l)} + u^- e^{-u^+\cdot(u-l)} - u^+ e^{u^-\cdot(u-l)}\right)^{-1} \\
&= \frac{e^{-u^+\cdot(b-l)} - e^{u^-\cdot(b-l)}}{e^{-u^+\cdot(u-l)} - e^{u^-\cdot(u-l)}}
\end{aligned}$$

which is the classical result, cf. equation (2.17) in [9].

4.2. Occupation times before a first passage

Choose some $b < u \in \mathbb{R}$. Define the column vectors $\zeta_k(u) := (\zeta_{kj}(u) : j \in E)$ for $k \in \{1,2\}$ via their entries

$$\zeta_{1,j}(u) := \int_0^{\tau(u)} \mathbb{I}_{\{X_t<b, J_t=j\}}\, dt \quad \text{and} \quad \zeta_{2,j}(u) := \int_0^{\tau(u)} \mathbb{I}_{\{X_t>b, J_t=j\}}\, dt$$

for $j \in E$. Further choose any exit rate vectors $\mathbf{r}_k = (r_{kj} : j \in E)$ for $k \in \{1,2\}$. We shall derive an expression for

$$E^+(u|a) := \mathbb{E}\left(e^{-\mathbf{r}_1\zeta_1(u)} e^{-\mathbf{r}_2\zeta_2(u)} | X_0 = a\right)$$

where $a < u$, thus providing the joint Laplace transform of the occupation times $\zeta_{kj}(u)$. There are three cases: If $a < u < b$, then $\sum_{j\in E} \zeta_{1,j}(u) = \tau(u)$ and thus

$$E^+(u|a) = \mathbb{E}\left(e^{-\int_0^{\tau(u)} \mathbf{r}_1 e_{J_s}\, ds} | X_0 = a\right)$$

which has been determined in (2). If $a < b < u$, then

$$E^+(u|a) = \mathbb{E}\left(e^{-\int_0^{\tau(b)} \mathbf{r}_1 e_{J_s}\, ds} | X_0 = a\right) E^+(u|b)$$

where again the first factor is known via (2). Finally, if $b < a < u$, then

$$E^+(u|a) = \Psi^+_{\mathbf{r}_2}(b, u|a) + \Psi^-_{\mathbf{r}_2}(b, u|a) E^+(u|b)$$

where the terms $\Psi^+_{\mathbf{r}_2}$ and $\Psi^-_{\mathbf{r}_2}$ are given in (7) and (8). Thus it suffices to determine $E^+(u|b)$ for $b < u$. Clearly $E^+_{(.,d)}(u|b) = \mathbf{0}$, since u cannot be passed from below in a descending phase. We further find the relation

$$E^+_{(d,a)}(u|b) = \mathbb{E}_{(d,a)}\left(e^{-\mathbf{r}_1\zeta_1(b)} | X_0 = b\right) E^+_{(a,a)}(u|b)$$

such that it remains to determine $E^+_{(a,a)}(u|b)$. Conditioning on the number n of possible returns to the level b in a strictly ascending phase $i \in E_s$ before passing the level u, we observe that

$$\begin{aligned}
E^+_{(s,a)}(u|b) &= \sum_{n=0}^{\infty} \left(\Psi^-_{\mathbf{r}_2} C^+(\mathbf{r}_1)\right)^n_{(s,s)} \left(\left(\Psi^+_{\mathbf{r}_2}\right)_{(s,a)} + \left(\Psi^-_{\mathbf{r}_2} C^+(\mathbf{r}_1)\right)_{(s,\sigma)} E^+_{(\sigma,a)}(u|b)\right) \\
&= \left(I_s - \left(\Psi^-_{\mathbf{r}_2} C^+(\mathbf{r}_1)\right)_{(s,s)}\right)^{-1} \left(\left(\Psi^+_{\mathbf{r}_2}\right)_{(s,a)} + \left(\Psi^-_{\mathbf{r}_2} C^+(\mathbf{r}_1)\right)_{(s,\sigma)} E^+_{(\sigma,a)}(u|b)\right)
\end{aligned}$$

where I_s indicates the identity matrix on E_s, $\Psi^\pm_{\mathbf{r}_2} = \Psi^\pm_{\mathbf{r}_2}(b, u|b)$, and $C^+(\mathbf{r}_1)$ is given in (6). We have thus reduced the problem to the determination of $E^+_{(\sigma,a)}(u|b)$. This can be obtained as the limit $E^+_{(\sigma,a)}(u|b) = \lim_{l\to-\infty} E^+_{(\sigma,a)}(l, u|b)$.

Corollary 1. Write $U_k^\pm := U^\pm(\mathbf{r}_k)$ for $k \in \{1,2\}$ and assume that $\|\mathbf{r}_1\| > 0$. Then

$$\begin{aligned}
E^+_{(\sigma,a)}(u|b) &= \mathbb{E}_{(\sigma,a)}\left(e^{-\mathbf{r}_1\zeta_1(u)}e^{-\mathbf{r}_2\zeta_2(u)}|X_0 = b\right) \\
&= 2\left(\left(D_2 C_1^+\right)_{(\sigma,\sigma)} + \left(2U_1^+\right)_{(\sigma,\sigma)}\right)^{-1} \\
&\quad \times \begin{pmatrix}\mathbf{0}_{(\sigma,s)} & I_\sigma\end{pmatrix}\left(U_2^+ + C_2^- U_2^- C_2^+\right)\left(e^{-U_2^+\cdot(u-b)} - C_2^- e^{U_2^-\cdot(u-b)}C_2^+\right)^{-1}
\end{aligned}$$

for $b < u$, where the matrix D_2 is given in lemma 2.

Proof: Looking at the formula in theorem 1 we find that only D_1 depends on l. For $l \to -\infty$ we obtain from lemma 1

$$\begin{aligned}
\lim_{l\to-\infty} D_1 &= \lim_{l\to-\infty} 2\begin{pmatrix}\mathbf{0}_{(\sigma,s)} & I_\sigma\end{pmatrix}\left(U_1^+ e^{-U_1^+\cdot(b-l)} + C_1^- U_1^- e^{U_1^-\cdot(b-l)}C_1^+\right) \\
&\quad \times \left(e^{-U_1^+\cdot(b-l)} - C_1^- e^{U_1^-\cdot(b-l)}C_1^+\right)^{-1} \\
&= \lim_{l\to-\infty} 2\begin{pmatrix}\mathbf{0}_{(\sigma,s)} & I_\sigma\end{pmatrix}\left(U_1^+ + C_1^- U_1^- e^{U_1^-\cdot(b-l)}C_1^+ e^{U_1^+\cdot(b-l)}\right) \\
&\quad \times \left(I_a - C_1^- e^{U_1^-\cdot(b-l)}C_1^+ e^{U_1^+\cdot(b-l)}\right)^{-1} \\
&= 2\begin{pmatrix}\mathbf{0}_{(\sigma,s)} & I_\sigma\end{pmatrix}U_1^+
\end{aligned}$$

since $\|\mathbf{r}_1\| > 0$ implies that U_1^+ is a strict sub-generator matrix and $\lim_{l\to-\infty} e^{U_1^+\cdot(b-l)} = \mathbf{0}$. □

Example 6. For the compound Poisson risk model with exponential claims the phase space is $E = E_s \cup E_d$, i.e. $E_\sigma = \emptyset$. Thus $E^+_{(a,a)}(u|b) = E^+_{(s,s)}(u|b)$ for which the formula before the corollary yields

$$E^+_{(s,s)}(u|b) = \left(1 - e_1'\Psi^-_{\mathbf{r}_2}A^+(\mathbf{r}_1)\right)^{-1}\ e_1'\Psi^+_{\mathbf{r}_2}$$

as $C^+(\mathbf{r}_1) = A^+(\mathbf{r}_1)$ and $\Psi^\pm_{\mathbf{r}_2}$ are simply column vectors on E. Setting $\mathbf{r}_1 = (0,\gamma)$, $\mathbf{r}_2 = \mathbf{0}$ and $b = 0$ we obtain the Laplace transform (with argument γ) of the time spent above the initial risk reserve u before ruin. The values for $\mathbf{r}_2$ yield $A_2^+ = \lambda/(c\beta)$ and $U_2^+ = \lambda/c - \beta$ as well as $A_2^- = 1$ and $U_2^- = 0$. Hence

$$e_1'\Psi^-_{\mathbf{r}_2} = \frac{1 - e^{-(\beta-\lambda/c)\cdot(u-b)}}{1 - \frac{\lambda}{c\beta}e^{-(\beta-\lambda/c)\cdot(u-b)}} \quad \text{and} \quad e_1'\Psi^+_{\mathbf{r}_2} = \frac{1 - \frac{\lambda}{c\beta}}{1 - \frac{\lambda}{c\beta}e^{-(\beta-\lambda/c)\cdot(u-b)}}e^{-(\beta-\lambda/c)\cdot(u-b)}$$

while the values for $\mathbf{r}_1$ yield

$$A^+(\mathbf{r}_1) = \frac{1}{2c\beta}\left(\lambda + \gamma + c\cdot\beta - \sqrt{(\lambda+\gamma+c\cdot\beta)^2 - 4\lambda c\beta}\right)$$

5. Occupation times in more than two intervals

We now consider a a finite number of thresholds $b_1 < \ldots < b_N$. We can determine occupation times of $(\mathcal{X}, \mathcal{J})$ before a first passage or an exit from an interval in the following way. Fix the respective rate exit vector $\mathbf{r}_k$ for the (open) intervals I_k, $k \in \{1, \ldots, N+1\}$, resulting from $b_1 < \ldots < b_N$. Define $\tau^+(x) := \inf\{t \geq 0 : X_t > x\}$ and the column vectors $\zeta_k^+(u) := (\zeta_{kj}^+(u) : j \in E)$, $k \in \{1, \ldots, N+1\}$, with entries

$$\zeta_{kj}^+(u) := \int_0^{\tau^+(u)} \mathbb{I}_{\{X_t \in I_k, J_t = j\}} \, dt$$

for $j \in E$. As before, we shall abbreviate

$$E^+(u|a) := \mathbb{E}\left(e^{-\sum_{k=1}^{N+1} \mathbf{r}_k \zeta_k^+(u)} | X_0 = a\right)$$

for $a < u$. Similarly, define $\tau^-(x) := \inf\{t \geq 0 : X_t < x\}$ as well as the column vectors $\zeta_k^-(u) := (\zeta_{kj}^-(u) : j \in E)$, $k \in \{1, \ldots, N+1\}$, with entries

$$\zeta_{kj}^-(u) := \int_0^{\tau^-(u)} \mathbb{I}_{\{X_t \in I_k, J_t = j\}} \, dt$$

for $j \in E$. We shall write

$$E^-(l|a) := \mathbb{E}\left(e^{-\sum_{k=1}^{N+1} \mathbf{r}_k \zeta_k^-(u)} | X_0 = a\right)$$

for $l < a$. Define the column vectors $\zeta_k(l, u) := (\zeta_{kj}(l, u) : j \in E)$, $k \in \{1, \ldots, N+1\}$, with entries

$$\zeta_{kj}(l, u) := \int_0^{\tau(l,u)} \mathbb{I}_{\{X_t \in I_k, J_t = j\}} \, dt$$

for $j \in E$. We write

$$E^+(l, u|a) := \mathbb{E}\left(e^{-\sum_{k=1}^{N+1} \mathbf{r}_k \zeta_k(l,u)}; X_{\tau(l,u)} = u | X_0 = a\right)$$

for $l < a < u$, as well as

$$E^-(l, u|a) := \mathbb{E}\left(e^{-\sum_{k=1}^{N+1} \mathbf{r}_k \zeta_k(l,u)}; X_{\tau(l,u)} = l | X_0 = a\right)$$

The matrices $E^+(u|a)$ and $E^+(l, u|a)$ have been determined in sections 4.2 and 4.1, respectively. The matrices $E^-(l|a)$ and $E^-(l, u|a)$ are determined in the same way after reflection at the initial level a, i.e. interchanging A^+ and U^+ with A^- and U^-, cf. the relation between (7) and (8).

5.1. Occupation times before a first passage

We seek a computational scheme for $E^+(u|a)$ where $a < u$. If $u \leq b_2$, then the solution is given by the results in section 4.2 with $b = b_1$. For $u > b_2$ let $k := \max\{n \geq 2 : b_n < u\}$. Path continuity yields for $a \leq b_k < u$

$$E^+(u|a) = E^+(b_k|a)\ E^+(u|b_k)$$

where

$$E^+(u|b_k) = E^+(b_{k-1}, u|b_k) + E^-(b_{k-1}, u|b_k)E^+(b_k|b_{k-1})E^+(u|b_k)$$

which implies

$$E^+(u|b_k) = \left(I - E^-(b_{k-1}, u|b_k)E^+(b_k|b_{k-1})\right)^{-1} E^+(b_{k-1}, u|b_k)$$

In the case $b_k < a < u$ we obtain

$$E^+(u|a) = \Psi^+_{k+1}(b_k, u|a) + \Psi^-_{k+1}(b_k, u|a)E^+(u|b_k)$$

where Ψ^+_{k+1} and Ψ^-_{k+1} denote the two-sided exit matrices as defined in (7) and (8) with parameters taken from the $k + 1$st regime. Since the matrices $E^+(b_{k-1}, u|b_k)$ and $E^-(b_{k-1}, u|b_k)$ have been determined in section 4.1, this provides a recursion scheme for $E^+(u|a)$.

5.2. Occupation times before an exit from an interval

We shall determine $E^+(l, u|a)$ with $l < a < u$. First note that the problem can be reduced to the results obtained in section 5.1 by exploiting the probabilistic interpretation at the beginning of section 3. This yields

$$E^+(l, u|a) = \left(E^+(u|a) - E^-(l|a)E^+(u|l)\right)\left(I - E^-(l|u)E^+(u|l)\right)^{-1}$$

We further wish to provide a recursion that involves only matrices of the form $E^\pm(x, y|z)$. For $h := \min\{n \geq 1 : b_n > l\}$, the matrix $E^+(l, b_{h+1}|b_h)$ has been determined in section 4.1. Define $k := \max\{n \geq 1 : b_n < u\}$. If $k = h$, then $E^+(l, u|a)$ is given by the results in section 4.1. Thus assume that $k > h \geq 1$. We obtain by path continuity

$$E^+(l, u|a) = E^+(l, b_k|a)E^+(l, u|b_k)$$

where

$$E^+(l, u|b_k) = E^+(b_{k-1}, u|b_k) + E^-(b_{k-1}, u|b_k)E^+(l, b_k|b_{k-1})E^+(l, u|b_k)$$

This yields

$$E^+(l,u|b_k) = \left(I - E^-(b_{k-1},u|b_k)E^+(l,b_k|b_{k-1})\right)^{-1} E^+(b_{k-1},u|b_k)$$

Since the matrices $E^+(b_{k-1},u|b_k)$ and $E^-(b_{k-1},u|b_k)$ have been determined in section 4.1, this provides a recursion scheme for $E^+(l,u|a)$.

Appendix

In this appendix, the lemmata that have been used in the proof of theorem 1 are collected. Recall the abbreviations $\Psi_1^+(\varepsilon)$ for the $(E_\sigma \cup E_d) \times E_a$-block of $\Psi_{\mathbf{r}_1}^+(l, b+\varepsilon|b-\varepsilon)$, $\Psi_2^-(\varepsilon)$ for the $E_a \times (E_\sigma \cup E_d)$-block of $\Psi_{\mathbf{r}_2}^-(b-\varepsilon, u|b+\varepsilon)$, and $\Psi_2^+(\varepsilon)$ for the $E_a \times E_a$-block of $\Psi_{\mathbf{r}_2}^+(b-\varepsilon, u|b+\varepsilon)$. Further let $\mathbf{0}_{(\sigma,d)}$ and $\mathbf{0}_{(\sigma,s)}$ denote the zero matrices on $E_\sigma \times E_d$ and on $E_\sigma \times E_s$, respectively.

Lemma 1. For $l < b$,

$$\begin{aligned} D_1 &:= \begin{pmatrix} I_\sigma & \mathbf{0}_{(\sigma,d)} \end{pmatrix} \frac{d}{d\varepsilon}\Psi_1^+(\varepsilon)\Big|_{\varepsilon=0} \\ &= 2\begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(U_1^+ e^{-U_1^+\cdot(b-l)} + C_1^- U_1^- e^{U_1^-\cdot(b-l)} C_1^+\right) \left(e^{-U_1^+\cdot(b-l)} - C_1^- e^{U_1^-\cdot(b-l)} C_1^+\right)^{-1} \end{aligned}$$

Proof: According to (7),

$$\begin{aligned} \begin{pmatrix} I_\sigma & \mathbf{0}_{(\sigma,d)} \end{pmatrix} \Psi_1^+(\varepsilon) &= \begin{pmatrix} I_\sigma & \mathbf{0}_{(\sigma,d)} \end{pmatrix} \left(C_1^+ e^{U_1^+\cdot(2\varepsilon)} - e^{U_1^-\cdot(b-l-\varepsilon)} C_1^+ e^{U_1^+\cdot(b-l+\varepsilon)}\right) \\ &\quad \times \left(I_a - C_1^- e^{U_1^-\cdot(b-l+\varepsilon)} C_1^+ e^{U_1^+\cdot(b-l+\varepsilon)}\right)^{-1} \\ &= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(e^{U_1^+\cdot(2\varepsilon)} - C_1^- e^{U_1^-\cdot(b-l-\varepsilon)} C_1^+ e^{U_1^+\cdot(b-l+\varepsilon)}\right) \\ &\quad \times \left(I_a - C_1^- e^{U_1^-\cdot(b-l+\varepsilon)} C_1^+ e^{U_1^+\cdot(b-l+\varepsilon)}\right)^{-1} \\ &= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(e^{-U_1^+\cdot(b-l-\varepsilon)} - C_1^- e^{U_1^-\cdot(b-l-\varepsilon)} C_1^+\right) \\ &\quad \times \left(e^{-U_1^+\cdot(b-l+\varepsilon)} - C_1^- e^{U_1^-\cdot(b-l+\varepsilon)} C_1^+\right)^{-1} \end{aligned}$$

After abbreviating

$$F(\varepsilon) := e^{-U_1^+\cdot(b-l-\varepsilon)} - C_1^- e^{U_1^-\cdot(b-l-\varepsilon)} C_1^+, \quad G(\varepsilon) := e^{-U_1^+\cdot(b-l+\varepsilon)} - C_1^- e^{U_1^-\cdot(b-l+\varepsilon)} C_1^+$$

we apply the formal rules of derivation for functions of a real variable (see [4], sections I.1.3-4) to obtain

$$D_1 = \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(F'(0)G(0)^{-1} - F(0)G(0)^{-1}G'(0)G(0)^{-1}\right)$$

where $F(0) = G(0) = e^{-U_1^+ \cdot (b-l)} - C_1^- e^{U_1^- \cdot (b-l)} C_1^+$ and

$$F'(0) = U_1^+ e^{-U_1^+ \cdot (b-l)} + C_1^- U_1^- e^{U_1^- \cdot (b-l)} C_1^+ = -G'(0)$$

This yields the statement.

□

Lemma 2. For $b < u$,

$$\begin{aligned} D_2 &:= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \frac{d}{d\varepsilon} \Psi_2^-(\varepsilon) \bigg|_{\varepsilon=0} \\ &= 2 \begin{pmatrix} I_\sigma & \mathbf{0}_{(\sigma,d)} \end{pmatrix} \left(U_2^- e^{-U_2^- \cdot (u-b)} + C_2^+ U_2^+ e^{U_2^+ \cdot (u-b)} C_2^- \right) \left(e^{-U_2^- \cdot (u-b)} - C_2^+ e^{U_2^+ \cdot (u-b)} C_2^- \right)^{-1} \end{aligned}$$

Proof: According to (8),

$$\begin{aligned} \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \Psi_2^-(\varepsilon) &= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(C_2^- e^{U_2^- 2\varepsilon} - e^{U_2^+ \cdot (u-b-\varepsilon)} C_2^- e^{U_2^- \cdot (u-b+\varepsilon)} \right) \\ &\quad \times \left(I - C_2^+ e^{U_2^+ \cdot (u-b+\varepsilon)} C_2^- e^{U_2^- \cdot (u-b+\varepsilon)} \right)^{-1} \\ &= \begin{pmatrix} I_\sigma & \mathbf{0}_{(\sigma,d)} \end{pmatrix} \left(e^{U_2^- 2\varepsilon} - C_2^+ e^{U_2^+ \cdot (u-b-\varepsilon)} C_2^- e^{U_2^- \cdot (u-b+\varepsilon)} \right) \\ &\quad \times \left(I - C_2^+ e^{U_2^+ \cdot (u-b+\varepsilon)} C_2^- e^{U_2^- \cdot (u-b+\varepsilon)} \right)^{-1} \\ &= \begin{pmatrix} I_\sigma & \mathbf{0}_{(\sigma,d)} \end{pmatrix} \left(e^{-U_2^- \cdot (u-b-\varepsilon)} - C_2^+ e^{U_2^+ \cdot (u-b-\varepsilon)} C_2^- \right) \\ &\quad \times \left(e^{-U_2^- \cdot (u-b+\varepsilon)} - C_2^+ e^{U_2^+ \cdot (u-b+\varepsilon)} C_2^- \right)^{-1} \end{aligned}$$

We abbreviate

$$F(\varepsilon) := e^{-U_2^- \cdot (u-b-\varepsilon)} - C_2^+ e^{U_2^+ \cdot (u-b-\varepsilon)} C_2^-, \quad G(\varepsilon) := e^{-U_2^- \cdot (u-b+\varepsilon)} - C_2^+ e^{U_2^+ \cdot (u-b+\varepsilon)} C_2^-$$

where $F(0) = G(0) = e^{-U_2^- \cdot (u-b)} - C_2^+ e^{U_2^+ \cdot (u-b)} C_2^-$ and

$$F'(0) = U_2^- e^{-U_2^- \cdot (u-b)} + C_2^+ U_2^+ e^{U_2^+ \cdot (u-b)} C_2^- = -G'(0)$$

Hence

$$\begin{aligned} D_2 &= \begin{pmatrix} I_\sigma & \mathbf{0}_{(\sigma,d)} \end{pmatrix} \left(F'(0) G(0)^{-1} - F(0) G(0)^{-1} G'(0) G(0)^{-1} \right) \\ &= 2 \begin{pmatrix} I_\sigma & \mathbf{0}_{(\sigma,d)} \end{pmatrix} F'(0) G(0)^{-1} \end{aligned}$$

which is the statement.

□

Lemma 3. For $b < u$,

$$
\begin{aligned}
D_3 &:= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \frac{d}{d\varepsilon} \Psi_2^+(\varepsilon) \Big|_{\varepsilon=0} \\
&= -2 \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(U_2^+ + C_2^- U_2^- C_2^+ \right) \left(e^{-U_2^+ \cdot (u-b)} - C_2^- e^{U_2^- \cdot (u-b)} C_2^+ \right)^{-1}
\end{aligned}
$$

Proof: According to (7),

$$
\begin{aligned}
\begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \Psi_2^+(\varepsilon) &= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(e^{U_2^+ \cdot (u-b-\varepsilon)} - C_2^- e^{U_2^- 2\varepsilon} C_2^+ e^{U_2^+ \cdot (u-b+\varepsilon)} \right) \\
&\quad \times \left(I_a - C_2^- e^{U_2^- \cdot (u-b+\varepsilon)} C_2^+ e^{U_2^+ \cdot (u-b+\varepsilon)} \right)^{-1} \\
&= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(e^{-U_2^+ \cdot 2\varepsilon} - C_2^- e^{U_2^- 2\varepsilon} C_2^+ \right) \\
&\quad \times \left(e^{-U_2^+ \cdot (u-b+\varepsilon)} - C_2^- e^{U_2^- \cdot (u-b+\varepsilon)} C_2^+ \right)^{-1}
\end{aligned}
$$

We abbreviate

$$
F(\varepsilon) = e^{-2U_2^+ \varepsilon} - C_2^- e^{2U_2^- \varepsilon} C_2^+, \quad G(\varepsilon) = e^{-U_2^+ \cdot (u-b+\varepsilon)} - C_2^- e^{U_2^- \cdot (u-b+\varepsilon)} C_2^+
$$

to obtain $F'(0) = -2 \left(U_2^+ + C_2^- U_2^- C_2^+ \right)$ and further $\begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} F(0) = \mathbf{0}_{(\sigma,a)}$ as well as $G(0) = e^{-U_2^+ \cdot (u-b)} - C_2^- e^{U_2^- \cdot (u-b)} C_2^+$. Altogether this yields

$$
\begin{aligned}
D_3 &= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} \left(F'(0) G(0)^{-1} - F(0) G(0)^{-1} G'(0) G(0)^{-1} \right) \\
&= \begin{pmatrix} \mathbf{0}_{(\sigma,s)} & I_\sigma \end{pmatrix} F'(0) G(0)^{-1}
\end{aligned}
$$

which is the statement.

□

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