Chapter 2: Markov Chains

L. Breuer
University of Kent, UK

October 23, 2013

Stationary Distributions

Let $\mathcal{X}$ denote a Markov chain with state space E.

Stationary Distributions

Let $\mathcal{X}$ denote a Markov chain with state space E. Let π denote a probability measure on E.

Stationary Distributions

Let $\mathcal{X}$ denote a Markov chain with state space E. Let π denote a probability measure on E. If $\mathbb{P}(X_0 = i) = \pi_i$ implies $\mathbb{P}(X_n = i) = \pi_i$ for all $n \in \mathbb{N}$ and $i \in E$,

Stationary Distributions

Let $\mathcal{X}$ denote a Markov chain with state space E. Let π denote a probability measure on E. If $\mathbb{P}(X_0 = i) = \pi_i$ implies $\mathbb{P}(X_n = i) = \pi_i$ for all $n \in \mathbb{N}$ and $i \in E$, then π is called a **stationary distribution** for $\mathcal{X}$.

Stationary Distributions

Let $\mathcal{X}$ denote a Markov chain with state space E. Let π denote a probability measure on E. If $\mathbb{P}(X_0 = i) = \pi_i$ implies $\mathbb{P}(X_n = i) = \pi_i$ for all $n \in \mathbb{N}$ and $i \in E$, then π is called a **stationary distribution** for $\mathcal{X}$. If π is a stationary distribution, then $c \cdot \pi$ for any $c \geq 0$ is called a **stationary measure**.

Theorem 2.18

Let $\mathcal{X}$ denote a Markov chain with state space E and transition matrix P.

Theorem 2.18

Let $\mathcal{X}$ denote a Markov chain with state space E and transition matrix P. Further, let π denote a probability distribution on E with $\pi P = \pi$,

Theorem 2.18

Let $\mathcal{X}$ denote a Markov chain with state space E and transition matrix P. Further, let π denote a probability distribution on E with $\pi P = \pi$, i.e.

$$\pi_j = \sum_{i \in E} \pi_i p_{ij} \qquad \text{and} \qquad \sum_{j \in E} \pi_j = 1$$

for all $j \in E$.

Theorem 2.18

Let $\mathcal{X}$ denote a Markov chain with state space E and transition matrix P. Further, let π denote a probability distribution on E with $\pi P = \pi$, i.e.

$$\pi_j = \sum_{i \in E} \pi_i p_{ij} \qquad \text{and} \qquad \sum_{j \in E} \pi_j = 1$$

for all $j \in E$. Then π is a stationary distribution for X.

Theorem 2.18

Let $\mathcal{X}$ denote a Markov chain with state space E and transition matrix P. Further, let π denote a probability distribution on E with $\pi P = \pi$, i.e.

$$\pi_j = \sum_{i \in E} \pi_i p_{ij} \qquad \text{and} \qquad \sum_{j \in E} \pi_j = 1$$

for all $j \in E$. Then π is a stationary distribution for X. If π is a stationary distribution for $\mathcal{X}$, then $\pi P = \pi$ holds.

Proof of theorem 2.18

Let $\mathbb{P}(X_0 = i) = \pi_i$ for all $i \in E$.

Proof of theorem 2.18

Let $\mathbb{P}(X_0 = i) = \pi_i$ for all $i \in E$. Then $\mathbb{P}(X_n = i) = \mathbb{P}(X_0 = i)$ for all $n \in \mathbb{N}$ and $i \in E$ follows by induction on n.

Proof of theorem 2.18

Let $\mathbb{P}(X_0 = i) = \pi_i$ for all $i \in E$. Then $\mathbb{P}(X_n = i) = \mathbb{P}(X_0 = i)$ for all $n \in \mathbb{N}$ and $i \in E$ follows by induction on n. The case $n = 1$ holds by assumption,

Proof of theorem 2.18

Let $\mathbb{P}(X_0 = i) = \pi_i$ for all $i \in E$. Then $\mathbb{P}(X_n = i) = \mathbb{P}(X_0 = i)$ for all $n \in \mathbb{N}$ and $i \in E$ follows by induction on n. The case $n = 1$ holds by assumption, and the induction step follows by induction hypothesis and the Markov property.

Proof of theorem 2.18

Let $\mathbb{P}(X_0 = i) = \pi_i$ for all $i \in E$. Then $\mathbb{P}(X_n = i) = \mathbb{P}(X_0 = i)$ for all $n \in \mathbb{N}$ and $i \in E$ follows by induction on n. The case $n = 1$ holds by assumption, and the induction step follows by induction hypothesis and the Markov property. The last statement is obvious.

Example 2.19

Let the transition matrix of a Markov chain $\mathcal{X}$ be given by

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

Example 2.19

Let the transition matrix of a Markov chain $\mathcal{X}$ be given by

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

Then $\pi = (0.5, 0.5, 0, 0)$,

Example 2.19

Let the transition matrix of a Markov chain $\mathcal{X}$ be given by

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

Then $\pi = (0.5, 0.5, 0, 0)$, $\pi' = (0, 0, 0.5, 0.5)$

Example 2.19

Let the transition matrix of a Markov chain $\mathcal{X}$ be given by

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

Then $\pi = (0.5, 0.5, 0, 0)$, $\pi' = (0, 0, 0.5, 0.5)$ as well as any linear combination of them are stationary distributions for $\mathcal{X}$.

Example 2.19

Let the transition matrix of a Markov chain $\mathcal{X}$ be given by

$$P = \begin{pmatrix} 0.8 & 0.2 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

Then $\pi = (0.5, 0.5, 0, 0)$, $\pi' = (0, 0, 0.5, 0.5)$ as well as any linear combination of them are stationary distributions for $\mathcal{X}$. This shows that a stationary distribution does not need to be unique.

Example 2.20: Bernoulli process

The transition matrix of a Bernoulli process has the structure

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & \dots \\ 0 & 1-p & p & 0 & \ddots \\ 0 & 0 & 1-p & p & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Example 2.20: Bernoulli process

The transition matrix of a Bernoulli process has the structure

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & \dots \\ 0 & 1-p & p & 0 & \ddots \\ 0 & 0 & 1-p & p & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Hence $\pi P = \pi$ implies first

$$\pi_0 \cdot (1-p) = \pi_0 \quad \Rightarrow \quad \pi_0 = 0$$

since $0 < p < 1$.

Example 2.20: Bernoulli process

The transition matrix of a Bernoulli process has the structure

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & \dots \\ 0 & 1-p & p & 0 & \ddots \\ 0 & 0 & 1-p & p & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Hence $\pi P = \pi$ implies first

$$\pi_0 \cdot (1-p) = \pi_0 \quad \Rightarrow \quad \pi_0 = 0$$

since $0 < p < 1$. Assume that $\pi_n = 0$ for any $n \in \mathbb{N}_0$.

Example 2.20: Bernoulli process

The transition matrix of a Bernoulli process has the structure

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & \dots \\ 0 & 1-p & p & 0 & \ddots \\ 0 & 0 & 1-p & p & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Hence $\pi P = \pi$ implies first

$$\pi_0 \cdot (1-p) = \pi_0 \quad \Rightarrow \quad \pi_0 = 0$$

since $0 < p < 1$. Assume that $\pi_n = 0$ for any $n \in \mathbb{N}_0$. This and the condition $\pi P = \pi$ further imply for π_{n+1}

$$\pi_n \cdot p + \pi_{n+1} \cdot (1-p) = \pi_{n+1} \quad \Rightarrow \quad \pi_{n+1} = 0$$

Example 2.20: Bernoulli process

The transition matrix of a Bernoulli process has the structure

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & \dots \\ 0 & 1-p & p & 0 & \ddots \\ 0 & 0 & 1-p & p & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Hence $\pi P = \pi$ implies first

$$\pi_0 \cdot (1-p) = \pi_0 \quad \Rightarrow \quad \pi_0 = 0$$

since $0 < p < 1$. Assume that $\pi_n = 0$ for any $n \in \mathbb{N}_0$. This and the condition $\pi P = \pi$ further imply for π_{n+1}

$$\pi_n \cdot p + \pi_{n+1} \cdot (1-p) = \pi_{n+1} \quad \Rightarrow \quad \pi_{n+1} = 0$$

which completes an induction argument proving $\pi_n = 0$ for all $n \in \mathbb{N}_0$.

Example 2.20: Bernoulli process

The transition matrix of a Bernoulli process has the structure

$$P = \begin{pmatrix} 1-p & p & 0 & 0 & \dots \\ 0 & 1-p & p & 0 & \ddots \\ 0 & 0 & 1-p & p & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Hence $\pi P = \pi$ implies first

$$\pi_0 \cdot (1-p) = \pi_0 \quad \Rightarrow \quad \pi_0 = 0$$

since $0 < p < 1$. Assume that $\pi_n = 0$ for any $n \in \mathbb{N}_0$. This and the condition $\pi P = \pi$ further imply for π_{n+1}

$$\pi_n \cdot p + \pi_{n+1} \cdot (1-p) = \pi_{n+1} \quad \Rightarrow \quad \pi_{n+1} = 0$$

which completes an induction argument proving $\pi_n = 0$ for all $n \in \mathbb{N}_0$. Hence the Bernoulli process does not have a stationary distribution.

Example 2.21

The solution of $\pi P = \pi$ and $\sum_{j \in E} \pi_j = 1$ is unique for

$$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

with $0 < p < 1$.

Example 2.21

The solution of $\pi P = \pi$ and $\sum_{j \in E} \pi_j = 1$ is unique for

$$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

with $0 < p < 1$. Thus there are transition matrices which have exactly one stationary distribution.

Theorem 2.22

A **transient Markov chain** (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof:

Theorem 2.22

A **transient Markov chain** (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof: Assume that $\pi P = \pi$ holds for some distribution π.

Theorem 2.22

A **transient Markov chain** (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof: Assume that $\pi P = \pi$ holds for some distribution π. Further let $E = \mathbb{N}$ without loss of generality.

Theorem 2.22

A **transient Markov chain** (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof: Assume that $\pi P = \pi$ holds for some distribution π. Further let $E = \mathbb{N}$ without loss of generality. Choose any state $m \in \mathbb{N}$ with $\pi_m > 0$.

Theorem 2.22

A **transient Markov chain** (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof: Assume that $\pi P = \pi$ holds for some distribution π. Further let $E = \mathbb{N}$ without loss of generality. Choose any state $m \in \mathbb{N}$ with $\pi_m > 0$. Since $\sum_{n=1}^{\infty} \pi_n = 1$ is bounded, there is an index $M > m$ such that $\sum_{n=M}^{\infty} \pi_n < \pi_m$.

Theorem 2.22

A **transient Markov chain** (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof: Assume that $\pi P = \pi$ holds for some distribution π. Further let $E = \mathbb{N}$ without loss of generality. Choose any state $m \in \mathbb{N}$ with $\pi_m > 0$. Since $\sum_{n=1}^{\infty} \pi_n = 1$ is bounded, there is an index $M > m$ such that $\sum_{n=M}^{\infty} \pi_n < \pi_m$. Set $\varepsilon := \pi_m - \sum_{n=M}^{\infty} \pi_n$.

Theorem 2.22

A **transient Markov chain** (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof: Assume that $\pi P = \pi$ holds for some distribution π. Further let $E = \mathbb{N}$ without loss of generality. Choose any state $m \in \mathbb{N}$ with $\pi_m > 0$. Since $\sum_{n=1}^{\infty} \pi_n = 1$ is bounded, there is an index $M > m$ such that $\sum_{n=M}^{\infty} \pi_n < \pi_m$. Set $\varepsilon := \pi_m - \sum_{n=M}^{\infty} \pi_n$. By theorem 2.17, there is an index $N \in \mathbb{N}$ such that $P^N(i, m) < \varepsilon$ for all $i \leq M$.

Theorem 2.22

A **transient Markov chain** (i.e. a Markov chain with transient states only) has no stationary distribution.

Proof: Assume that $\pi P = \pi$ holds for some distribution π. Further let $E = \mathbb{N}$ without loss of generality. Choose any state $m \in \mathbb{N}$ with $\pi_m > 0$. Since $\sum_{n=1}^{\infty} \pi_n = 1$ is bounded, there is an index $M > m$ such that $\sum_{n=M}^{\infty} \pi_n < \pi_m$. Set $\varepsilon := \pi_m - \sum_{n=M}^{\infty} \pi_n$. By theorem 2.17, there is an index $N \in \mathbb{N}$ such that $P^N(i,m) < \varepsilon$ for all $i \leq M$. Then the stationarity of π implies

$$\pi_m = \sum_{i=1}^{\infty} \pi_i P^N(i,m) = \sum_{i=1}^{M-1} \pi_i P^N(i,m) + \sum_{i=M}^{\infty} \pi_i P^N(i,m)$$

$$< \varepsilon + \sum_{i=M}^{\infty} \pi_i = \pi_m$$

which is a contradiction.

Positive / Null Recurrence

Define

$$N_i(n) := \sum_{k=0}^{n} \mathbb{I}_{\{X_k = i\}}$$

as the number of visits to state i until time n.

Positive / Null Recurrence

Define

$$N_i(n) := \sum_{k=0}^{n} \mathbb{I}_{\{X_k = i\}}$$

as the number of visits to state i until time n. Further define for a recurrent state $i \in E$ the mean time of return

$$m_i := \mathbb{E}(\tau_i | X_0 = i)$$

Positive / Null Recurrence

Define

$$N_i(n) := \sum_{k=0}^{n} \mathbb{I}_{\{X_k = i\}}$$

as the number of visits to state i until time n. Further define for a recurrent state $i \in E$ the mean time of return

$$m_i := \mathbb{E}(\tau_i | X_0 = i)$$

By definition $m_i > 0$ for all $i \in E$.

Positive / Null Recurrence

Define

$$N_i(n) := \sum_{k=0}^{n} \mathbb{I}_{\{X_k=i\}}$$

as the number of visits to state i until time n. Further define for a recurrent state $i \in E$ the mean time of return

$$m_i := \mathbb{E}(\tau_i | X_0 = i)$$

By definition $m_i > 0$ for all $i \in E$. A recurrent state $i \in E$ with $m_i < \infty$ will be called **positive recurrent**, Define

$$N_i(n) := \sum_{k=0}^{n} \mathbb{I}_{\{X_k = i\}}$$

as the number of visits to state i until time n. Further define for a recurrent state $i \in E$ the mean time of return

$$m_i := \mathbb{E}(\tau_i | X_0 = i)$$

By definition $m_i > 0$ for all $i \in E$. A recurrent state $i \in E$ with $m_i < \infty$ will be called **positive recurrent**, otherwise i is called **null recurrent**.

Elementary renewal theorem

The elementary renewal theorem (which will be proven in chapter 4) states that

$$\lim_{n\to\infty} \frac{\mathbb{E}(N_i(n)|X_0 = j)}{n} = \frac{1}{m_i}$$

for all recurrent $i \in E$

Elementary renewal theorem

The elementary renewal theorem (which will be proven in chapter 4) states that

$$\lim_{n\to\infty} \frac{\mathbb{E}(N_i(n)|X_0 = j)}{n} = \frac{1}{m_i}$$

for all recurrent $i \in E$ and independently of $j \in E$ provided $j \leftrightarrow i$,

Elementary renewal theorem

The elementary renewal theorem (which will be proven in chapter 4) states that

$$\lim_{n\to\infty} \frac{\mathbb{E}(N_i(n)|X_0 = j)}{n} = \frac{1}{m_i}$$

for all recurrent $i \in E$ and independently of $j \in E$ provided $j \leftrightarrow i$, with the convention of $1/\infty := 0$.

Elementary renewal theorem

The elementary renewal theorem (which will be proven in chapter 4) states that

$$\lim_{n\to\infty} \frac{\mathbb{E}(N_i(n)|X_0 = j)}{n} = \frac{1}{m_i}$$

for all recurrent $i \in E$ and independently of $j \in E$ provided $j \leftrightarrow i$, with the convention of $1/\infty := 0$. Thus the asymptotic rate of visits to a recurrent state is determined by the mean recurrence time of this state.

Theorem 2.23

Positive recurrence and null recurrence are class properties with respect to the relation of communication between states.

Theorem 2.23

Positive recurrence and null recurrence are class properties with respect to the relation of communication between states.

Proof:
Assume that $i \leftrightarrow j$ for two states $i, j \in E$

Theorem 2.23

Positive recurrence and null recurrence are class properties with respect to the relation of communication between states.

Proof:
Assume that $i \leftrightarrow j$ for two states $i, j \in E$ and i is null recurrent.

Theorem 2.23

Positive recurrence and null recurrence are class properties with respect to the relation of communication between states.

Proof:
Assume that $i \leftrightarrow j$ for two states $i, j \in E$ and i is null recurrent. Thus there are numbers $m, n \in \mathbb{N}$ with $P^n(i, j) > 0$ and $P^m(j, i) > 0$.

Theorem 2.23

Positive recurrence and null recurrence are class properties with respect to the relation of communication between states.

Proof:
Assume that $i \leftrightarrow j$ for two states $i, j \in E$ and i is null recurrent. Thus there are numbers $m, n \in \mathbb{N}$ with $P^n(i,j) > 0$ and $P^m(j,i) > 0$. Because of the representation $\mathbb{E}(N_i(k)|X_0 = i) = \sum_{l=0}^{k} P^l(i,i)$, we obtain

Proof of theorem 2.23 (contd.)

$$0 = \lim_{k\to\infty} \frac{\sum_{l=0}^{k} P^l(i,i)}{k}$$

Proof of theorem 2.23 (contd.)

$$0 = \lim_{k\to\infty} \frac{\sum_{l=0}^{k} P^l(i,i)}{k}$$

$$\geq \lim_{k\to\infty} \frac{\sum_{l=0}^{k-m-n} P^l(j,j)}{k} \cdot P^n(i,j)P^m(j,i)$$

Proof of theorem 2.23 (contd.)

$$0 = \lim_{k\to\infty} \frac{\sum_{l=0}^{k} P^l(i,i)}{k}$$

$$\geq \lim_{k\to\infty} \frac{\sum_{l=0}^{k-m-n} P^l(j,j)}{k} \cdot P^n(i,j)P^m(j,i)$$

$$= \lim_{k\to\infty} \frac{k-m-n}{k} \cdot \frac{\sum_{l=0}^{k-m-n} P^l(j,j)}{k-m-n} \cdot P^n(i,j)P^m(j,i)$$

Proof of theorem 2.23 (contd.)

$$
\begin{aligned}
0 &= \lim_{k\to\infty} \frac{\sum_{l=0}^{k} P^l(i,i)}{k} \\
&\geq \lim_{k\to\infty} \frac{\sum_{l=0}^{k-m-n} P^l(j,j)}{k} \cdot P^n(i,j)P^m(j,i) \\
&= \lim_{k\to\infty} \frac{k-m-n}{k} \cdot \frac{\sum_{l=0}^{k-m-n} P^l(j,j)}{k-m-n} \cdot P^n(i,j)P^m(j,i) \\
&= \lim_{k\to\infty} \frac{\sum_{l=0}^{k} P^l(j,j)}{k} \cdot P^n(i,j)P^m(j,i)
\end{aligned}
$$

Proof of theorem 2.23 (contd.)

$$
\begin{aligned}
0 &= \lim_{k\to\infty} \frac{\sum_{l=0}^{k} P^l(i,i)}{k} \\
&\geq \lim_{k\to\infty} \frac{\sum_{l=0}^{k-m-n} P^l(j,j)}{k} \cdot P^n(i,j)P^m(j,i) \\
&= \lim_{k\to\infty} \frac{k-m-n}{k} \cdot \frac{\sum_{l=0}^{k-m-n} P^l(j,j)}{k-m-n} \cdot P^n(i,j)P^m(j,i) \\
&= \lim_{k\to\infty} \frac{\sum_{l=0}^{k} P^l(j,j)}{k} \cdot P^n(i,j)P^m(j,i) \\
&= \frac{P^n(i,j)P^m(j,i)}{m_j}
\end{aligned}
$$

Proof of theorem 2.23 (contd.)

$$
\begin{aligned}
0 &= \lim_{k\to\infty} \frac{\sum_{l=0}^{k} P^l(i,i)}{k} \\
&\geq \lim_{k\to\infty} \frac{\sum_{l=0}^{k-m-n} P^l(j,j)}{k} \cdot P^n(i,j)P^m(j,i) \\
&= \lim_{k\to\infty} \frac{k-m-n}{k} \cdot \frac{\sum_{l=0}^{k-m-n} P^l(j,j)}{k-m-n} \cdot P^n(i,j)P^m(j,i) \\
&= \lim_{k\to\infty} \frac{\sum_{l=0}^{k} P^l(j,j)}{k} \cdot P^n(i,j)P^m(j,i) \\
&= \frac{P^n(i,j)P^m(j,i)}{m_j}
\end{aligned}
$$

and thus $m_j = \infty$, which signifies the null recurrence of j.

Theorem 2.24

Let $i \in E$ be positive recurrent and define the mean first visit time $m_i := \mathbb{E}(\tau_i | X_0 = i)$.

Theorem 2.24

Let $i \in E$ be positive recurrent and define the mean first visit time $m_i := \mathbb{E}(\tau_i | X_0 = i)$. Then a stationary distribution π is given by

$$\pi_j := m_i^{-1} \cdot \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

for all $j \in E$.

Theorem 2.24

Let $i \in E$ be positive recurrent and define the mean first visit time $m_i := \mathbb{E}(\tau_i | X_0 = i)$. Then a stationary distribution π is given by

$$\pi_j := m_i^{-1} \cdot \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

for all $j \in E$. In particular, $\pi_i = m_i^{-1}$

Theorem 2.24

Let $i \in E$ be positive recurrent and define the mean first visit time $m_i := \mathbb{E}(\tau_i | X_0 = i)$. Then a stationary distribution π is given by

$$\pi_j := m_i^{-1} \cdot \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

for all $j \in E$. In particular, $\pi_i = m_i^{-1}$ and $\pi_k = 0$ for all states k outside of the communication class belonging to i.

Proof of theorem 2.24

First of all, π is a probability measure

Proof of theorem 2.24

First of all, π is a probability measure since

$$\sum_{j \in E} \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i) = \sum_{n=0}^{\infty} \sum_{j \in E} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

Proof of theorem 2.24

First of all, π is a probability measure since

$$\sum_{j \in E} \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i) = \sum_{n=0}^{\infty} \sum_{j \in E} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(\tau_i > n | X_0 = i) = m_i$$

Proof of theorem 2.24

First of all, π is a probability measure since

$$\sum_{j \in E} \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i) = \sum_{n=0}^{\infty} \sum_{j \in E} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(\tau_i > n | X_0 = i) = m_i$$

The particular statements in the theorem are obvious from the definition of π

Proof of theorem 2.24

First of all, π is a probability measure since

$$\sum_{j \in E} \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i) = \sum_{n=0}^{\infty} \sum_{j \in E} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(\tau_i > n | X_0 = i) = m_i$$

The particular statements in the theorem are obvious from the definition of π and the fact that a recurrent communication class is closed.

Proof of theorem 2.24 (contd.)

The stationarity of π is shown as follows.

Proof of theorem 2.24 (contd.)

The stationarity of π is shown as follows. First we obtain

$$\pi_j = m_i^{-1} \cdot \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

Proof of theorem 2.24 (contd.)

The stationarity of π is shown as follows. First we obtain

$$\pi_j = m_i^{-1} \cdot \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

$$= m_i^{-1} \cdot \sum_{n=1}^{\infty} \mathbb{P}(X_n = j, \tau_i \geq n | X_0 = i)$$

Proof of theorem 2.24 (contd.)

The stationarity of π is shown as follows. First we obtain

$$\pi_j = m_i^{-1} \cdot \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

$$= m_i^{-1} \cdot \sum_{n=1}^{\infty} \mathbb{P}(X_n = j, \tau_i \geq n | X_0 = i)$$

$$= m_i^{-1} \cdot \sum_{n=1}^{\infty} \mathbb{P}(X_n = j, \tau_i > n - 1 | X_0 = i)$$

since $X_0 = X_{\tau_i} = i$ in the conditioning set $\{X_0 = i\}$.

Proof of theorem 2.24 (contd.)

The stationarity of π is shown as follows. First we obtain

$$\pi_j = m_i^{-1} \cdot \sum_{n=0}^{\infty} \mathbb{P}(X_n = j, \tau_i > n | X_0 = i)$$

$$= m_i^{-1} \cdot \sum_{n=1}^{\infty} \mathbb{P}(X_n = j, \tau_i \geq n | X_0 = i)$$

$$= m_i^{-1} \cdot \sum_{n=1}^{\infty} \mathbb{P}(X_n = j, \tau_i > n-1 | X_0 = i)$$

since $X_0 = X_{\tau_i} = i$ in the conditioning set $\{X_0 = i\}$. Further,

Proof of theorem 2.24 (contd.)

$$\mathbb{P}(X_n = j, \tau_i > n-1 | X_0 = i) = \frac{\mathbb{P}(X_n = j, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_0 = i)}$$

Proof of theorem 2.24 (contd.)

$$\mathbb{P}(X_n = j, \tau_i > n-1 | X_0 = i) = \frac{\mathbb{P}(X_n = j, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_0 = i)}$$

$$= \sum_{k \in E} \frac{\mathbb{P}(X_n = j, X_{n-1} = k, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_0 = i)}$$

Proof of theorem 2.24 (contd.)

$$\mathbb{P}(X_n = j, \tau_i > n-1 | X_0 = i) = \frac{\mathbb{P}(X_n = j, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_0 = i)}$$

$$= \sum_{k \in E} \frac{\mathbb{P}(X_n = j, X_{n-1} = k, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_0 = i)}$$

$$= \sum_{k \neq i} \frac{\mathbb{P}(X_n = j, X_{n-1} = k, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_{n-1} = k, \tau_i > n-1, X_0 = i)}$$

$$\times \frac{\mathbb{P}(X_{n-1} = k, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_0 = i)}$$

Proof of theorem 2.24 (contd.)

$$\mathbb{P}(X_n = j, \tau_i > n-1 | X_0 = i) = \frac{\mathbb{P}(X_n = j, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_0 = i)}$$

$$= \sum_{k \in E} \frac{\mathbb{P}(X_n = j, X_{n-1} = k, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_0 = i)}$$

$$= \sum_{k \neq i} \frac{\mathbb{P}(X_n = j, X_{n-1} = k, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_{n-1} = k, \tau_i > n-1, X_0 = i)}$$

$$\times \frac{\mathbb{P}(X_{n-1} = k, \tau_i > n-1, X_0 = i)}{\mathbb{P}(X_0 = i)}$$

$$= \sum_{k \in E} p_{kj} \mathbb{P}(X_{n-1} = k, \tau_i > n-1 | X_0 = i)$$

Proof of theorem 2.24 (contd.)

Hence we obtain

$$\pi_j = m_i^{-1} \cdot \sum_{n=1}^{\infty} \sum_{k \in E} p_{kj} \mathbb{P}(X_{n-1} = k, \tau_i > n-1 | X_0 = i)$$

Proof of theorem 2.24 (contd.)

Hence we obtain

$$\pi_j = m_i^{-1} \cdot \sum_{n=1}^{\infty} \sum_{k \in E} p_{kj} \mathbb{P}(X_{n-1} = k, \tau_i > n-1 | X_0 = i)$$

$$= \sum_{k \in E} p_{kj} \cdot m_i^{-1} \sum_{n=0}^{\infty} \mathbb{P}(X_n = k, \tau_i > n | X_0 = i)$$

Proof of theorem 2.24 (contd.)

Hence we obtain

$$\pi_j = m_i^{-1} \cdot \sum_{n=1}^{\infty} \sum_{k \in E} p_{kj} \mathbb{P}(X_{n-1} = k, \tau_i > n-1 | X_0 = i)$$

$$= \sum_{k \in E} p_{kj} \cdot m_i^{-1} \sum_{n=0}^{\infty} \mathbb{P}(X_n = k, \tau_i > n | X_0 = i)$$

$$= \sum_{k \in E} \pi_k p_{kj}$$

which completes the proof.

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain.

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Proof:
Existence has been shown in theorem 2.24.

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Proof:
Existence has been shown in theorem 2.24. Uniqueness of the stationary distribution can be seen as follows.

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Proof:
Existence has been shown in theorem 2.24. Uniqueness of the stationary distribution can be seen as follows. Let π denote the stationary distribution as constructed in theorem 2.24

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Proof:
Existence has been shown in theorem 2.24. Uniqueness of the stationary distribution can be seen as follows. Let π denote the stationary distribution as constructed in theorem 2.24 and i the positive recurrent state that served as recurrence point for π.

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Proof:
Existence has been shown in theorem 2.24. Uniqueness of the stationary distribution can be seen as follows. Let π denote the stationary distribution as constructed in theorem 2.24 and i the positive recurrent state that served as recurrence point for π. Further, let ν denote any stationary distribution for $\mathcal{X}$.

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Proof:
Existence has been shown in theorem 2.24. Uniqueness of the stationary distribution can be seen as follows. Let π denote the stationary distribution as constructed in theorem 2.24 and i the positive recurrent state that served as recurrence point for π. Further, let ν denote any stationary distribution for $\mathcal{X}$. Then there is a state $j \in E$ with $\nu_j > 0$

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Proof:
Existence has been shown in theorem 2.24. Uniqueness of the stationary distribution can be seen as follows. Let π denote the stationary distribution as constructed in theorem 2.24 and i the positive recurrent state that served as recurrence point for π. Further, let ν denote any stationary distribution for $\mathcal{X}$. Then there is a state $j \in E$ with $\nu_j > 0$ and a number $m \in \mathbb{N}$ with $P^m(j, i) > 0$,

Theorem 2.25

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then $\mathcal{X}$ has a unique stationary distribution.

Proof:
Existence has been shown in theorem 2.24. Uniqueness of the stationary distribution can be seen as follows. Let π denote the stationary distribution as constructed in theorem 2.24 and i the positive recurrent state that served as recurrence point for π. Further, let ν denote any stationary distribution for $\mathcal{X}$. Then there is a state $j \in E$ with $\nu_j > 0$ and a number $m \in \mathbb{N}$ with $P^m(j,i) > 0$, since $\mathcal{X}$ is irreducible.

Proof of theorem 2.25 (contd.)

Consequently we obtain

$$\nu_i = \sum_{k \in E} \nu_k P^m(k,i) \geq \nu_j P^m(j,i) > 0$$

Proof of theorem 2.25 (contd.)

Consequently we obtain

$$\nu_i = \sum_{k \in E} \nu_k P^m(k,i) \geq \nu_j P^m(j,i) > 0$$

Hence we can multiply ν by a factor $c > 0$ such that $c \cdot \nu_i = \pi_i = 1/m_i$.

Proof of theorem 2.25 (contd.)

Consequently we obtain

$$\nu_i = \sum_{k \in E} \nu_k P^m(k,i) \geq \nu_j P^m(j,i) > 0$$

Hence we can multiply ν by a factor $c > 0$ such that $c \cdot \nu_i = \pi_i = 1/m_i$. Denote $\tilde{\nu} := c \cdot \nu$,

Proof of theorem 2.25 (contd.)

Consequently we obtain

$$\nu_i = \sum_{k \in E} \nu_k P^m(k,i) \geq \nu_j P^m(j,i) > 0$$

Hence we can multiply ν by a factor $c > 0$ such that $c \cdot \nu_i = \pi_i = 1/m_i$. Denote $\tilde{\nu} := c \cdot \nu$, i.e. $\tilde{\nu}_k := c \cdot \nu_k$ for all $k \in E$.

Proof of theorem 2.25 (contd.)

Consequently we obtain

$$\nu_i = \sum_{k \in E} \nu_k P^m(k,i) \geq \nu_j P^m(j,i) > 0$$

Hence we can multiply ν by a factor $c > 0$ such that $c \cdot \nu_i = \pi_i = 1/m_i$. Denote $\tilde{\nu} := c \cdot \nu$, i.e. $\tilde{\nu}_k := c \cdot \nu_k$ for all $k \in E$.
Let $\tilde{P}$ denote the transition matrix P without the ith column, i.e. $\tilde{P} = (\tilde{p}_{hk})_{h,k \in E}$ with

$$\tilde{p}_{hk} = \begin{cases} p_{hk}, & k \neq i \\ 0, & k = i \end{cases}$$

Proof of theorem 2.25 (contd.)

Consequently we obtain

$$\nu_i = \sum_{k \in E} \nu_k P^m(k,i) \geq \nu_j P^m(j,i) > 0$$

Hence we can multiply ν by a factor $c > 0$ such that $c \cdot \nu_i = \pi_i = 1/m_i$. Denote $\tilde{\nu} := c \cdot \nu$, i.e. $\tilde{\nu}_k := c \cdot \nu_k$ for all $k \in E$. Let $\tilde{P}$ denote the transition matrix P without the ith column, i.e. $\tilde{P} = (\tilde{p}_{hk})_{h,k \in E}$ with

$$\tilde{p}_{hk} = \begin{cases} p_{hk}, & k \neq i \\ 0, & k = i \end{cases}$$

Denote further the Dirac measure on i by δ^i, i.e.

$$\delta^i_k = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}$$

Proof of theorem 2.25 (contd.)

Then the stationary distribution π can be represented by

$$\pi = m_i^{-1} \cdot \delta^i \sum_{n=0}^{\infty} \tilde{P}^n$$

Proof of theorem 2.25 (contd.)

Then the stationary distribution π can be represented by

$$\pi = m_i^{-1} \cdot \delta^i \sum_{n=0}^{\infty} \tilde{P}^n$$

We first claim that

$$m_i \tilde{\nu} = \delta^i + m_i \tilde{\nu} \tilde{P}$$

Proof of theorem 2.25 (contd.)

Then the stationary distribution π can be represented by

$$\pi = m_i^{-1} \cdot \delta^i \sum_{n=0}^{\infty} \tilde{P}^n$$

We first claim that

$$m_i \tilde{\nu} = \delta^i + m_i \tilde{\nu} \tilde{P}$$

This is clear for the entry $\tilde{\nu}_i$

Proof of theorem 2.25 (contd.)

Then the stationary distribution π can be represented by

$$\pi = m_i^{-1} \cdot \delta^i \sum_{n=0}^{\infty} \tilde{P}^n$$

We first claim that

$$m_i \tilde{\nu} = \delta^i + m_i \tilde{\nu} \tilde{P}$$

This is clear for the entry $\tilde{\nu}_i$ and easily seen for $\tilde{\nu}_k$ with $k \neq i$

Proof of theorem 2.25 (contd.)

Then the stationary distribution π can be represented by

$$\pi = m_i^{-1} \cdot \delta^i \sum_{n=0}^{\infty} \tilde{P}^n$$

We first claim that

$$m_i \tilde{\nu} = \delta^i + m_i \tilde{\nu} \tilde{P}$$

This is clear for the entry $\tilde{\nu}_i$ and easily seen for $\tilde{\nu}_k$ with $k \neq i$ because in this case

$$(\tilde{\nu}\tilde{P})_k = c \cdot (\nu P)_k = c \cdot \nu_k = \tilde{\nu}_k$$

Proof of theorem 2.25 (contd.)

Now we can proceed with the same argument to see that

$$m_i\tilde{\nu} = \delta^i + (\delta^i + m_i\tilde{\nu}\tilde{P})\tilde{P} = \delta^i + \delta^i\tilde{P} + m_i\tilde{\nu}\tilde{P}^2 = \dots$$

Now we can proceed with the same argument to see that

$$m_i\tilde{\nu} = \delta^i + (\delta^i + m_i\tilde{\nu}\tilde{P})\tilde{P} = \delta^i + \delta^i\tilde{P} + m_i\tilde{\nu}\tilde{P}^2 = \dots$$

$$= \delta^i \sum_{n=0}^{\infty} \tilde{P}^n = m_i\pi$$

Proof of theorem 2.25 (contd.)

Now we can proceed with the same argument to see that

$$m_i\tilde{\nu} = \delta^i + (\delta^i + m_i\tilde{\nu}\tilde{P})\tilde{P} = \delta^i + \delta^i\tilde{P} + m_i\tilde{\nu}\tilde{P}^2 = \ldots$$

$$= \delta^i \sum_{n=0}^{\infty} \tilde{P}^n = m_i\pi$$

Hence $\tilde{\nu}$ already is a probability measure

Proof of theorem 2.25 (contd.)

Now we can proceed with the same argument to see that

$$m_i\tilde{\nu} = \delta^i + (\delta^i + m_i\tilde{\nu}\tilde{P})\tilde{P} = \delta^i + \delta^i\tilde{P} + m_i\tilde{\nu}\tilde{P}^2 = \dots$$

$$= \delta^i \sum_{n=0}^{\infty} \tilde{P}^n = m_i\pi$$

Hence $\tilde{\nu}$ already is a probability measure and thus $c = 1$.

Proof of theorem 2.25 (contd.)

Now we can proceed with the same argument to see that

$$m_i\tilde{\nu} = \delta^i + (\delta^i + m_i\tilde{\nu}\tilde{P})\tilde{P} = \delta^i + \delta^i\tilde{P} + m_i\tilde{\nu}\tilde{P}^2 = \dots$$

$$= \delta^i \sum_{n=0}^{\infty} \tilde{P}^n = m_i\pi$$

Hence $\tilde{\nu}$ already is a probability measure and thus $c = 1$. This yields $\nu = \tilde{\nu} = \pi$ and thus the statement.

Theorem 2.27

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain.

Theorem 2.27

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then the stationary distribution π of $\mathcal{X}$ is given by

$$\pi_j = m_j^{-1} = \frac{1}{\mathbb{E}(\tau_j | X_0 = j)}$$

for all $j \in E$.

Theorem 2.27

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then the stationary distribution π of $\mathcal{X}$ is given by

$$\pi_j = m_j^{-1} = \frac{1}{\mathbb{E}(\tau_j | X_0 = j)}$$

for all $j \in E$.

Proof:
Since all states in E are positive recurrent,

Theorem 2.27

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then the stationary distribution π of $\mathcal{X}$ is given by

$$\pi_j = m_j^{-1} = \frac{1}{\mathbb{E}(\tau_j | X_0 = j)}$$

for all $j \in E$.

Proof:
Since all states in E are positive recurrent, the construction in theorem 2.24 can be pursued for any inital state j.

Theorem 2.27

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then the stationary distribution π of $\mathcal{X}$ is given by

$$\pi_j = m_j^{-1} = \frac{1}{\mathbb{E}(\tau_j | X_0 = j)}$$

for all $j \in E$.

Proof:
Since all states in E are positive recurrent, the construction in theorem 2.24 can be pursued for any inital state j. This yields $\pi_j = m_j^{-1}$ for all $j \in E$.

Theorem 2.27

Let $\mathcal{X}$ denote an irreducible, positive recurrent Markov chain. Then the stationary distribution π of $\mathcal{X}$ is given by

$$\pi_j = m_j^{-1} = \frac{1}{\mathbb{E}(\tau_j | X_0 = j)}$$

for all $j \in E$.

Proof:
Since all states in E are positive recurrent, the construction in theorem 2.24 can be pursued for any inital state j. This yields $\pi_j = m_j^{-1}$ for all $j \in E$. The statement now follows from the uniqueness of the stationary distribution.

Theorem 2.28

For an irreducible, positive recurrent Markov chain,

Theorem 2.28

For an irreducible, positive recurrent Markov chain, the stationary probability π_j of a state j coincides with its asymptotic rate of recurrence,

Theorem 2.28

For an irreducible, positive recurrent Markov chain, the stationary probability π_j of a state j coincides with its asymptotic rate of recurrence, i.e.

$$\lim_{n\to\infty} \frac{\mathbb{E}(N_j(n)|X_0 = i)}{n} = \pi_j$$

for all $j \in E$

Theorem 2.28

For an irreducible, positive recurrent Markov chain, the stationary probability π_j of a state j coincides with its asymptotic rate of recurrence, i.e.

$$\lim_{n\to\infty} \frac{\mathbb{E}(N_j(n)|X_0 = i)}{n} = \pi_j$$

for all $j \in E$ and independently of $i \in E$.

Theorem 2.28

For an irreducible, positive recurrent Markov chain, the stationary probability π_j of a state j coincides with its asymptotic rate of recurrence, i.e.

$$\lim_{n\to\infty} \frac{\mathbb{E}(N_j(n)|X_0 = i)}{n} = \pi_j$$

for all $j \in E$ and independently of $i \in E$. Further, if an asymptotic distribution $p_j = \lim_{n\to\infty} \mathbb{P}(X_n = j)$ for all $j \in E$ does exist,

Theorem 2.28

For an irreducible, positive recurrent Markov chain, the stationary probability π_j of a state j coincides with its asymptotic rate of recurrence, i.e.

$$\lim_{n\to\infty} \frac{\mathbb{E}(N_j(n)|X_0 = i)}{n} = \pi_j$$

for all $j \in E$ and independently of $i \in E$. Further, if an asymptotic distribution $p_j = \lim_{n\to\infty} \mathbb{P}(X_n = j)$ for all $j \in E$ does exist, then it coincides with the stationary distribution.

Theorem 2.28

For an irreducible, positive recurrent Markov chain, the stationary probability π_j of a state j coincides with its asymptotic rate of recurrence, i.e.

$$\lim_{n\to\infty} \frac{\mathbb{E}(N_j(n)|X_0 = i)}{n} = \pi_j$$

for all $j \in E$ and independently of $i \in E$. Further, if an asymptotic distribution $p_j = \lim_{n\to\infty} \mathbb{P}(X_n = j)$ for all $j \in E$ does exist, then it coincides with the stationary distribution. In particular, it is independent of the initial distribution of $\mathcal{X}$.

Proof of theorem 2.28

The first statement immediately follows from the elementary renewal theorem.

Proof of theorem 2.28

The first statement immediately follows from the elementary renewal theorem. For the second statement, it suffices to employ $\mathbb{E}(N_j(n)|X_0 = i) = \sum_{l=0}^{n} P^l(i,j)$.

Proof of theorem 2.28

The first statement immediately follows from the elementary renewal theorem. For the second statement, it suffices to employ $\mathbb{E}(N_j(n)|X_0 = i) = \sum_{l=0}^{n} P^l(i,j)$. If an asymptotic distribution does exist, The first statement immediately follows from the elementary renewal theorem. For the second statement, it suffices to employ $\mathbb{E}(N_j(n)|X_0 = i) = \sum_{l=0}^n P^l(i,j)$. If an asymptotic distribution does exist, then for any initial distribution ν we obtain

$$p_j = \lim_{n\to\infty}(\nu P^n)_j = \sum_{i\in E} \nu_i \lim_{n\to\infty} P^n(i,j)$$

The first statement immediately follows from the elementary renewal theorem. For the second statement, it suffices to employ $\mathbb{E}(N_j(n)|X_0 = i) = \sum_{l=0}^{n} P^l(i,j)$. If an asymptotic distribution does exist, then for any initial distribution ν we obtain

$$p_j = \lim_{n\to\infty} (\nu P^n)_j = \sum_{i\in E} \nu_i \lim_{n\to\infty} P^n(i,j)$$

$$= \sum_{i\in E} \nu_i \lim_{n\to\infty} \frac{\sum_{l=0}^{n} P^l(i,j)}{n} = \sum_{i\in E} \nu_i \pi_j$$

Proof of theorem 2.28

The first statement immediately follows from the elementary renewal theorem. For the second statement, it suffices to employ $\mathbb{E}(N_j(n)|X_0 = i) = \sum_{l=0}^{n} P^l(i,j)$. If an asymptotic distribution does exist, then for any initial distribution ν we obtain

$$p_j = \lim_{n\to\infty} (\nu P^n)_j = \sum_{i\in E} \nu_i \lim_{n\to\infty} P^n(i,j)$$

$$= \sum_{i\in E} \nu_i \lim_{n\to\infty} \frac{\sum_{l=0}^{n} P^l(i,j)}{n} = \sum_{i\in E} \nu_i \pi_j$$

$$= \pi_j$$

Example

Let $\mathcal{X}$ denote a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example

Let $\mathcal{X}$ denote a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\mathcal{X}$ has no asymptotic distribution,

Example

Let $\mathcal{X}$ denote a Markov chain with transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\mathcal{X}$ has no asymptotic distribution, but a stationary distribution, namely $\pi = (1/2, 1/2)$.

Theorem 2.31

An irreducible Markov chain with finite state space F is positive recurrent.

Theorem 2.31

An irreducible Markov chain with finite state space F is positive recurrent.

Proof:
For all $n \in \mathbb{N}$ and $i \in F$ we have

$$\sum_{j \in F} P^n(i,j) = 1$$

Theorem 2.31

An irreducible Markov chain with finite state space F is positive recurrent.

Proof:
For all $n \in \mathbb{N}$ and $i \in F$ we have

$$\sum_{j \in F} P^n(i,j) = 1$$

Hence it is not possible that $\lim_{n\to\infty} P^n(i,j) = 0$ for all $j \in F$.

Theorem 2.31

An irreducible Markov chain with finite state space F is positive recurrent.

Proof:
For all $n \in \mathbb{N}$ and $i \in F$ we have

$$\sum_{j \in F} P^n(i,j) = 1$$

Hence it is not possible that $\lim_{n\to\infty} P^n(i,j) = 0$ for all $j \in F$. Thus there is one state $h \in F$ such that

$$\sum_{n=0}^{\infty} P^n(i,h) = r_{ih} = f_{ih} r_{hh} = \infty$$

Theorem 2.31

An irreducible Markov chain with finite state space F is positive recurrent.

Proof:
For all $n \in \mathbb{N}$ and $i \in F$ we have

$$\sum_{j \in F} P^n(i,j) = 1$$

Hence it is not possible that $\lim_{n\to\infty} P^n(i,j) = 0$ for all $j \in F$.
Thus there is one state $h \in F$ such that

$$\sum_{n=0}^{\infty} P^n(i,h) = r_{ih} = f_{ih} r_{hh} = \infty$$

which means by corollary 2.15 that h is recurrent

Theorem 2.31

An irreducible Markov chain with finite state space F is positive recurrent.

Proof:
For all $n \in \mathbb{N}$ and $i \in F$ we have

$$\sum_{j \in F} P^n(i,j) = 1$$

Hence it is not possible that $\lim_{n\to\infty} P^n(i,j) = 0$ for all $j \in F$. Thus there is one state $h \in F$ such that

$$\sum_{n=0}^{\infty} P^n(i,h) = r_{ih} = f_{ih} r_{hh} = \infty$$

which means by corollary 2.15 that h is recurrent and by irreducibility that the chain is recurrent.

Theorem 2.31

An irreducible Markov chain with finite state space F is positive recurrent.

Proof:
For all $n \in \mathbb{N}$ and $i \in F$ we have

$$\sum_{j \in F} P^n(i,j) = 1$$

Hence it is not possible that $\lim_{n\to\infty} P^n(i,j) = 0$ for all $j \in F$. Thus there is one state $h \in F$ such that

$$\sum_{n=0}^{\infty} P^n(i,h) = r_{ih} = f_{ih} r_{hh} = \infty$$

which means by corollary 2.15 that h is recurrent and by irreducibility that the chain is recurrent.

Proof of theorem 2.31

If the chain were null recurrent,

If the chain were null recurrent, then according to the elementary renewal theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k(i,j) = 0$$

would hold for all $j \in F$,

Proof of theorem 2.31

If the chain were null recurrent, then according to the elementary renewal theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k(i,j) = 0$$

would hold for all $j \in F$, independently of i because of irreducibility.

Proof of theorem 2.31

If the chain were null recurrent, then according to the elementary renewal theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k(i,j) = 0$$

would hold for all $j \in F$, independently of i because of irreducibility. But this would imply that

$$\lim_{n\to\infty} P^n(i,j) = 0$$

for all $j \in F$,

Proof of theorem 2.31

If the chain were null recurrent, then according to the elementary renewal theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k(i,j) = 0$$

would hold for all $j \in F$, independently of i because of irreducibility. But this would imply that

$$\lim_{n\to\infty} P^n(i,j) = 0$$

for all $j \in F$, which contradicts our first observation in this proof.

Proof of theorem 2.31

If the chain were null recurrent, then according to the elementary renewal theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k(i,j) = 0$$

would hold for all $j \in F$, independently of i because of irreducibility. But this would imply that

$$\lim_{n\to\infty} P^n(i,j) = 0$$

for all $j \in F$, which contradicts our first observation in this proof. Hence the chain must be positive recurrent.

The Geo/Geo/1 queue in discrete time

The Geo/Geo/1 queue in discrete time

Choose any parameters $0 < p, q < 1$.

The Geo/Geo/1 queue in discrete time

Choose any parameters $0 < p, q < 1$. Let the arrival process be distributed as a Bernoulli process with parameter p

The Geo/Geo/1 queue in discrete time

Choose any parameters $0 < p, q < 1$. Let the arrival process be distributed as a Bernoulli process with parameter p and the service times $(S_n : n \in \mathbb{N}_0)$ be iid

The Geo/Geo/1 queue in discrete time

Choose any parameters $0 < p, q < 1$. Let the arrival process be distributed as a Bernoulli process with parameter p and the service times $(S_n : n \in \mathbb{N}_0)$ be iid according to the geometric distribution with parameter q.

Theorem 2.34 (memoryless property)

Let S be distributed geometrically with parameter q,

Theorem 2.34 (memoryless property)

Let S be distributed geometrically with parameter q, i.e. let $\mathbb{P}(S = k) = (1-q)^{k-1}q$ for all $k \in \mathbb{N}$.

Theorem 2.34 (memoryless property)

Let S be distributed geometrically with parameter q, i.e. let $\mathbb{P}(S = k) = (1-q)^{k-1}q$ for all $k \in \mathbb{N}$. Then $\mathbb{P}(S = k | S > k-1) = q$,

Theorem 2.34 (memoryless property)

Let S be distributed geometrically with parameter q, i.e. let $\mathbb{P}(S = k) = (1 - q)^{k-1}q$ for all $k \in \mathbb{N}$. Then $\mathbb{P}(S = k | S > k - 1) = q$, independently of k.

Theorem 2.34 (memoryless property)

Let S be distributed geometrically with parameter q, i.e. let $\mathbb{P}(S = k) = (1-q)^{k-1}q$ for all $k \in \mathbb{N}$. Then $\mathbb{P}(S = k|S > k-1) = q$, independently of k.

Proof:

$$\mathbb{P}(S = k|S > k-1) = \frac{\mathbb{P}(S = k, S > k-1)}{\mathbb{P}(S > k-1)}$$

Theorem 2.34 (memoryless property)

Let S be distributed geometrically with parameter q, i.e. let $\mathbb{P}(S = k) = (1-q)^{k-1}q$ for all $k \in \mathbb{N}$. Then $\mathbb{P}(S = k | S > k-1) = q$, independently of k.

Proof:

$$\mathbb{P}(S = k | S > k-1) = \frac{\mathbb{P}(S = k, S > k-1)}{\mathbb{P}(S > k-1)}$$

$$= \frac{\mathbb{P}(S = k)}{\mathbb{P}(S > k-1)}$$

Theorem 2.34 (memoryless property)

Let S be distributed geometrically with parameter q, i.e. let $\mathbb{P}(S = k) = (1-q)^{k-1}q$ for all $k \in \mathbb{N}$. Then $\mathbb{P}(S = k|S > k-1) = q$, independently of k.

Proof:

$$\mathbb{P}(S = k|S > k-1) = \frac{\mathbb{P}(S = k, S > k-1)}{\mathbb{P}(S > k-1)}$$

$$= \frac{\mathbb{P}(S = k)}{\mathbb{P}(S > k-1)}$$

$$= \frac{(1-q)^{k-1}q}{(1-q)^{k-1}} = q$$

The Geo/Geo/1 queue as a Markov chain

Let Q_n denote the number of users in the system at time $n \in \mathbb{N}_0$.

The Geo/Geo/1 queue as a Markov chain

Let Q_n denote the number of users in the system at time $n \in \mathbb{N}_0$. Then the state space is $E = \mathbb{N}_0$.

The Geo/Geo/1 queue as a Markov chain

Let Q_n denote the number of users in the system at time $n \in \mathbb{N}_0$.
Then the state space is $E = \mathbb{N}_0$.
The transition probabilities are $p_{01} := p$, $p_{00} := 1 - p$,

The Geo/Geo/1 queue as a Markov chain

Let Q_n denote the number of users in the system at time $n \in \mathbb{N}_0$.
Then the state space is $E = \mathbb{N}_0$.
The transition probabilities are $p_{01} := p$, $p_{00} := 1 - p$, and

$$p_{ij} := \begin{cases} p(1-q), & j = i+1 \\ pq + (1-p)(1-q), & j = i \\ q(1-p), & j = i-1 \end{cases}$$

for $i \geq 1$.

Transition matrix

Thus the transition matrix is triagonal,

Transition matrix

Thus the transition matrix is triagonal, i.e.

$$P = \begin{pmatrix} 1-p & p & 0 & \dots \\ q(1-p) & pq+(1-p)(1-q) & p(1-q) & \ddots \\ 0 & q(1-p) & pq+(1-p)(1-q) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Transition matrix

Thus the transition matrix is triagonal, i.e.

$$P = \begin{pmatrix} 1-p & p & 0 & \dots \\ q(1-p) & pq+(1-p)(1-q) & p(1-q) & \ddots \\ 0 & q(1-p) & pq+(1-p)(1-q) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Abbreviate $p' := p(1-q)$ and $q' := q(1-p)$.

Stationarity condition for the Geo/Geo/1 queue

Then the condition $\pi P = \pi$

Stationarity condition for the Geo/Geo/1 queue

Then the condition $\pi P = \pi$ means

$$\pi_0 = \pi_0(1-p) + \pi_1 q'$$

Stationarity condition for the Geo/Geo/1 queue

Then the condition $\pi P = \pi$ means

$$\pi_0 = \pi_0(1 - p) + \pi_1 q'$$

$$\pi_1 = \pi_0 p + \pi_1(1 - p - q') + \pi_2 q'$$

Stationarity condition for the Geo/Geo/1 queue

Then the condition $\pi P = \pi$ means

$$\pi_0 = \pi_0(1 - p) + \pi_1 q'$$

$$\pi_1 = \pi_0 p + \pi_1(1 - p - q') + \pi_2 q'$$

and

$$\pi_n = \pi_{n-1} p' + \pi_n(1 - (p' + q')) + \pi_{n+1} q'$$

for all $n \geq 2$.

Stationary distribution - 1

We try the geometric form

$$\pi_{n+1} = \pi_n \cdot r$$

for all $n \geq 1$, with $0 < r < 1$.

Stationary distribution - 1

We try the geometric form

$$\pi_{n+1} = \pi_n \cdot r$$

for all $n \geq 1$, with $0 < r < 1$. Then stationarity yields

$$0 = \pi_n p' - \pi_n r(p' + q') + \pi_n r^2 q'$$

Stationary distribution - 1

We try the geometric form

$$\pi_{n+1} = \pi_n \cdot r$$

for all $n \geq 1$, with $0 < r < 1$. Then stationarity yields

$$0 = \pi_n p' - \pi_n r(p' + q') + \pi_n r^2 q'$$

$$= \pi_n \left(p' - r(p' + q') + r^2 q'\right)$$

Stationary distribution - 1

We try the geometric form

$$\pi_{n+1} = \pi_n \cdot r$$

for all $n \geq 1$, with $0 < r < 1$. Then stationarity yields

$$0 = \pi_n p' - \pi_n r(p' + q') + \pi_n r^2 q'$$

$$= \pi_n \left(p' - r(p' + q') + r^2 q' \right)$$

and hence $r = p'/q'$

Stationary distribution - 1

We try the geometric form

$$\pi_{n+1} = \pi_n \cdot r$$

for all $n \geq 1$, with $0 < r < 1$. Then stationarity yields

$$0 = \pi_n p' - \pi_n r(p' + q') + \pi_n r^2 q'$$

$$= \pi_n \left(p' - r(p' + q') + r^2 q'\right)$$

and hence $r = p'/q' < 1 \iff p < q$.

Stationary distribution - 1

We try the geometric form

$$\pi_{n+1} = \pi_n \cdot r$$

for all $n \geq 1$, with $0 < r < 1$. Then stationarity yields

$$0 = \pi_n p' - \pi_n r(p' + q') + \pi_n r^2 q'$$

$$= \pi_n \left(p' - r(p' + q') + r^2 q'\right)$$

and hence $r = p'/q' < 1 \iff p < q$. Further,

$$\pi_1 = \pi_0 \frac{p}{q'} = \pi_0 \frac{\rho}{1-p}$$

with $\rho := p/q$,

Stationary distribution - 2

and

$$\pi_2 = \frac{1}{q'} \left(\pi_1 (p' + q') - \pi_0 p \right)$$

Stationary distribution - 2

and

$$\pi_2 = \frac{1}{q'} \left(\pi_1 (p' + q') - \pi_0 p \right)$$

$$= \frac{1}{q'} \left(\frac{p}{q'} (p' + q') - p \right) \pi_0$$

Stationary distribution - 2

and

$$\pi_2 = \frac{1}{q'}\left(\pi_1(p'+q') - \pi_0 p\right)$$

$$= \frac{1}{q'}\left(\frac{p}{q'}(p'+q') - p\right)\pi_0$$

$$= \pi_0 \frac{p}{q'}\left(\frac{p'+q'}{q'} - 1\right)$$

and

$$
\begin{aligned}
\pi_2 &= \frac{1}{q'}\left(\pi_1(p'+q') - \pi_0 p\right) \\
&= \frac{1}{q'}\left(\frac{p}{q'}(p'+q') - p\right)\pi_0 \\
&= \pi_0 \frac{p}{q'}\left(\frac{p'+q'}{q'} - 1\right) \\
&= \pi_1 \frac{p'}{q'}
\end{aligned}
$$

Stationary distribution - 3

Normalisation of π yields

$$1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 \left(1 + \frac{p}{q'} \sum_{n=1}^{\infty} \left(\frac{p'}{q'} \right)^{n-1} \right)$$

Stationary distribution - 3

Normalisation of π yields

$$1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 \left(1 + \frac{p}{q'} \sum_{n=1}^{\infty} \left(\frac{p'}{q'} \right)^{n-1} \right)$$

and hence

$$\pi_0 = \left(1 + \frac{p}{q'} \sum_{n=1}^{\infty} \left(\frac{p'}{q'} \right)^{n-1} \right)^{-1} = 1 - \rho$$

Stationary distribution - 3

Normalisation of π yields

$$1 = \sum_{n=0}^{\infty} \pi_n = \pi_0 \left(1 + \frac{p}{q'} \sum_{n=1}^{\infty} \left(\frac{p'}{q'} \right)^{n-1} \right)$$

and hence

$$\pi_0 = \left(1 + \frac{p}{q'} \sum_{n=1}^{\infty} \left(\frac{p'}{q'} \right)^{n-1} \right)^{-1} = 1 - \rho$$

Verify this as an exercise!