Applied Probability Trust (March 8, 2012)

THE RESOLVENT AND EXPECTED LOCAL TIMES FOR MARKOV-MODULATED BROWNIAN MOTION WITH PHASE DEPENDENT TERMINATION RATES

LOTHAR BREUER,* *University of Kent*

Abstract

We consider a Markov-modulated Brownian motion (MMBM) with phase-dependent termination rates, i.e. while in a phase i the process terminates with a constant hazard rate $r_i \geq 0$. For such a process we determine the matrix of expected local times (at zero) before termination and hence the resolvent. The results are applied to some recent questions arising in the framework of insurance risk. We further provide expressions for the resolvent and the local times at zero of an MMBM reflected at its infimum.

Keywords: Markov-modulated Brownian motion; local time; resolvent; Markov-additive process

2000 Mathematics Subject Classification: Primary 60J25

Secondary 60G51;60J55

1. Introduction

Markov-modulated Brownian motion (MMBM, see the definition below) has proved to be a powerful tool in stochastic modelling, with applications in queueing theory, insurance and finance. This is even more appparent after one considers the fact that exit problems for MAPs with phase-type jumps can be reduced to an analysis of MMBMs by standard transformation techniques (see e.g. [12, 6]).

Some results for MMBMs go back to the 1990s, with [13] investigating Wiener-Hopf factorisation and stationary distributions for the case that $\sigma_i = \varepsilon$ is independent of the phase process. Around the same time, [2] determined hitting probabilities and based on these expressions for the stationary distributions. More recent results are [10, 9], which analyse MMBMs with two reflecting barriers.

Let $\mathcal{J} = (J_t : t \geq 0)$ denote an irreducible Markov process with a finite state space $E = \{1, \ldots, m\}$ and infinitesimal generator matrix $Q = (q_{ij})_{i,j \in E}$. We call J_t the phase at

* Postal address: Institute of Mathematics and Statistics, University of Kent, Canterbury CT2 7NF, UK

time t and $\mathcal{J}$ the phase process. Choosing parameters $\mu_i \in \mathbb{R}$ and $\sigma_i \geq 0$ for all $i \in E$, we define the level process $\mathcal{X} = (X_t : t \geq 0)$ by

$$X_t = X_0 + \int_0^t \mu_{J_s}\, ds + \int_0^t \sigma_{J_s} dW_s$$

for all $t \geq 0$, where $\mathcal{W} = (W_t : t \geq 0)$ denotes a standard Wiener process that is independent of $\mathcal{J}$. Then $(\mathcal{X}, \mathcal{J})$ is called a Markov-modulated Brownian motion (MMBM). An MMBM is a Markov-additive process (MAP, see [3], chapter XI) without jumps.

Now add an absorbing phase, say Δ, to the phase space E to obtain $E' = E \cup \{\Delta\}$ and assume that this is entered from a phase i with a constant hazard rate $r_i \geq 0$. Denote the resulting process by $(\mathcal{X}, \mathcal{J})$ again. Define the exit rate vector by $\mathbf{r} := (r_i : i \in E)$. We shall assume throughout that Q is irreducible and $\|\mathbf{r}\| := \sum_{i \in E} r_i > 0$. Then the absorption time $\tau_\Delta := \min\{t \geq 0 : J_t = \Delta\}$ has a phase-type distribution $PH(\alpha, Q - \Delta_\mathbf{r})$ with $\alpha_i := \mathbb{P}(J_0 = i)$ for $i \in E$ and Δ_v denoting the diagonal matrix with entries taken from the vector v. We shall say that the MMBM $(\mathcal{X}, \mathcal{J})$ terminates at time τ_Δ and disregard any further evolution after this. The values r_i may be interpreted as state-dependent killing rates, see section 3 in [12].

The present paper aims to determine the matrix of expected local times (at zero) for a terminating MMBM. Based on this, the resolvent is given as a corollary. A particular application of the resolvent is the determination of the transition probabilities over phase-type distributed time distances.

The next section contains some preliminary results, while the appropriate notions of a resolvent and the expected local times for terminating MMBMs are introduced in section 3. Section 4 contains the main result and section 5 an application to insurance risk.

2. Preliminaries: First passage times

Define the first passage times $\tau^+(x) := \inf\{t \geq 0 : X_t > x\}$ for all $x \geq 0$ and assume that $X_0 = 0$. Consider an E-dimensional row vector $\mathbf{r} = (r_i : i \in E)$ with non-negative entries $r_i \geq 0$ for all $i \in E$. Define

$$\mathbb{E}_{ij}\left(e^{-\int_0^{\tau^+(x)} r_{J_s} ds}\right) := \mathbb{E}\left(e^{-\int_0^{\tau^+(x)} r_{J_s} ds}; J_{\tau^+(x)} = j | J_0 = i, X_0 = 0\right) \qquad (1)$$

for $i, j \in E$ and $\mathbb{E}\left(e^{-\int_0^{\tau^+(x)} r_{J_s} ds}\right)$ as the $E \times E$-matrix with these entries.

In order to simplify notations, we shall from now on exclude the case of a phase $i \in E$ with $\mu_i = \sigma_i = 0$. We distinguish the phases by the subspaces $E_p := \{i \in E : \sigma_i = 0, \mu_i > 0\}$ as well as $E_n := \{i \in E : \sigma_i = 0, \mu_i < 0\}$ and $E_\sigma := \{i \in E : \sigma_i > 0\}$. The same arguments as in [5], section 3, yield

$$\mathbb{E}\left(e^{-\int_0^{\tau^+(x)} r_{J_s} ds}\right) = \begin{pmatrix} I_a \\ A(\mathbf{r}) \end{pmatrix} \begin{pmatrix} e^{U(\mathbf{r})x} & \mathbf{0} \end{pmatrix} \tag{2}$$

where I_a denotes the identity matrix on $E_p \cup E_\sigma$ and $\mathbf{0}$ the zero matrix on $(E_p \cup E_\sigma) \times E_n$. The matrices $A = A(\mathbf{r})$ and $U = U(\mathbf{r})$ can be computed as follows. For arguments $\beta \geq 0$ define the functions $\phi_i(\beta) := \beta/\mu_i$ for $i \in E_p$ as well as

$$\phi_i(\beta) = \frac{1}{\sigma_i}\sqrt{2\beta + \frac{\mu_i^2}{\sigma_i^2}} - \frac{\mu_i}{\sigma_i^2} \qquad \text{and} \qquad \phi_i^*(\beta) = \frac{1}{\sigma_i}\sqrt{2\beta + \frac{\mu_i^2}{\sigma_i^2}} + \frac{\mu_i}{\sigma_i^2}$$

for $i \in E_\sigma$. The iteration to determine $A(\mathbf{r})$ and $U(\mathbf{r})$ is slightly changed from [6], section 2.2, to the following form: We obtain $(A(\mathbf{r}), U(\mathbf{r})) = \lim_{n\to\infty}(A_n, U_n)$ for initial values $A_0 := \mathbf{0}$, $U_0 := -diag(\phi_i(q_i + r_i))_{i \in E_p \cup E_\sigma}$ and iterations

$$e_i' U_{n+1} = -\frac{q_i + r_i}{\mu_i} e_i' + \frac{1}{\mu_i} \sum_{j \in E, j \neq i} q_{ij}\ e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} \tag{3}$$

for $i \in E_p$,

$$e_i' U_{n+1} = -\phi_i(q_i + r_i) e_i' + \frac{2}{\sigma_i^2} \sum_{j \in E, j \neq i} q_{ij}\ e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} (\phi_i^*(q_i + r_i) I - U_n)^{-1} \tag{4}$$

for $i \in E_\sigma$, and

$$e_i' A_{n+1} = \sum_{j \in E, j \neq i} q_{ij} e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} ((q_i + r_i) I + \mu_i U_n)^{-1} \tag{5}$$

for $i \in E_n$. Here e_i' denotes the ith canonical row base vector and $q_i := -q_{ii}$ for all $i \in E$. The case $\mathbf{r} = \mathbf{0}$ has been analysed earlier in [2].

Now define the downward first passage times $\tau^-(x) := \inf\{t \geq 0 : X_t < x\}$ for all $x \leq 0$ and assume that $X_0 = 0$. Let $(\mathcal{X}^+, \mathcal{J})$ denote the original MMBM and define the process $(\mathcal{X}^-, \mathcal{J}) =_d (-\mathcal{X}^+, \mathcal{J})$, where $=_d$ denotes equality in distribution. The two processes have the same generator matrix Q for $\mathcal{J}$, but the drift parameters are different. Denoting the variation and drift parameters for $\mathcal{X}^{\pm}$ by $\sigma_i^{\pm}$ and $\mu_i^{\pm}$, respectively, this means $\sigma_i^- = \sigma_i^+$ and $\mu_i^- = -\mu_i^+$ for all $i \in E$.

The generalised Laplace transforms for $\tau^-(x)$ can of course be obtained by considering the upward first passage times for the process $(\mathcal{X}^-, \mathcal{J})$. Let $A^{\pm}(\mathbf{r})$ and $U^{\pm}(\mathbf{r})$ denote the matrices that determine the first passage times of $\mathcal{X}^{\pm}$ in (2). Then

$$\mathbb{E}\left(e^{-\int_0^{\tau^-(x)} r_{J_s}\,ds}\right) = \begin{pmatrix} A^-(\mathbf{r}) \\ I_d \end{pmatrix} \begin{pmatrix} \mathbf{0} & e^{-U^-(\mathbf{r})x} \end{pmatrix} \tag{6}$$

for all $x \le 0$, where I_d denotes the identity matrix on $E_\sigma \cup E_n$ and $\mathbf{0}$ the zero matrix on $(E_\sigma \cup E_n) \times E_p$. We shall write $A^{\pm} = A^{\pm}(\mathbf{r})$ and $U^{\pm} = U^{\pm}(\mathbf{r})$ if we do not wish to underline the dependence on $\mathbf{r}$.

Remark 1. The generalised Laplace transforms of the first passage times $\tau^{\pm}(x)$ can be seen as transition probabilities among the transient phases $i, j \in E$ for the phase process $\mathcal{J}$, i.e.

$$\mathbb{E}_{ij}\left(e^{-\int_0^{\tau^{\pm}(x)} r_{J_s}\,ds}\right) = \mathbb{P}\left(\tau^{\pm}(x) < \tau_\Delta, J_{\tau^{\pm}(x)} = j | J_0 = i, X_0 = 0\right)$$

for $i, j \in E$. Likewise, $A_{ij}^{\pm} = \mathbb{P}\left(\tau^{\pm}(0) < \tau_\Delta, J_{\tau^{\pm}(0)} = j | J_0 = i, X_0 = 0\right)$ for $i \in E_n$ and $j \in E_p \cup E_\sigma$ resp. $i \in E_p$ and $j \in E_\sigma \cup E_n$.

3. The resolvent and expected local times for terminating MMBMs

Define the $\mathbf{r}$-resolvent of $(\mathcal{X}, \mathcal{J})$ as the matrix-valued function $S^{(\mathbf{r})}(x)$ with entries

$$\begin{aligned} S_{ij}^{(\mathbf{r})}(x) &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \mathbb{E}\left(\int_0^{\tau_\Delta} \mathbb{I}_{\{|X_t - x| < \varepsilon, J_t = j\}}\,dt \middle| X_0 = 0, J_0 = i\right) \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^\infty \mathbb{P}(t < \tau_\Delta, X_t \in]x - \varepsilon, x + \varepsilon[, J_t = j | X_0 = 0, J_0 = i)\,dt \end{aligned}$$

for $i, j \in E$ and $x \in \mathbb{R}$, where $\mathbb{I}_A$ denotes the indicator function of a set A. We shall also write more briefly $S^{(\mathbf{r})}(x)\,dx = \int_0^\infty \mathbb{P}(X_t \in dx)\,dt$. Note that the exponential time devaluation, which is the usual notion in the definition of a resolvent for Lévy processes, is now replaced by the phase-dependent termination rates contained in the vector $\mathbf{r}$.

Remark 2. Since $\mathcal{X}$ has continuous paths and the termination rates do not depend on the level process, we can use the resolvent to determine the transition probabilities over a phase-type time distance. More precisely,

$$\mathbb{P}\left(X_{\tau_\Delta} \in dy | X_0 = x\right) = \alpha S^{(\mathbf{r})}(y - x)\mathbf{r}\,dy$$

for all $x, y \in \mathbb{R}$, where $\mathbf{r}$ is seen as a column vector and $\alpha_i = \mathbb{P}(J_0 = i)$ for all $i \in E$. This property can be made more amenable to applications in the following way: Given a phase-type distributed time distance $Z \sim PH(\alpha, T)$ of order m and a MMBM $(\mathcal{X}, \mathcal{J})$ with generator matrix Q for $\mathcal{J}$ and phase space E, we construct a phase space $E' := E \times \{1, \ldots, m\}$ and a generator matrix $Q' := Q \oplus T + \Delta_{\mathbf{1}_E \otimes \eta}$, where $\oplus$ and $\otimes$ denote the Kronecker sum and product, respectively, $\eta = -T\mathbf{1}_m$, and $\mathbf{1}_m$ and $\mathbf{1}_E$ denote the column vectors on $\{1, \ldots, m\}$ and E, respectively, with all entries being 1. Set further $\mu_{(i,j)} := \mu_i$ and $\sigma_{(i,j)} := \sigma_i$ for all $i \in E$ and $j \in \{1, \ldots, m\}$ and denote the MMBM defined therewith by $(\mathcal{X}', \mathcal{J}')$. Then

$$\mathbb{P}\left(X_Z \in dy, J_Z = j | X_0 = x, J_0 = i\right) = \left(e_i' \otimes \alpha\right) S'(y - x) \left(e_j \otimes \eta\right)\, dy$$

for all $i, j \in E$ and $x, y \in \mathbb{R}$, where e_i denotes the ith canonical base column vector, e_i' its transpose, and S' denotes the $(\mathbf{1}_E \otimes \eta)$-resolvent of $(\mathcal{X}', \mathcal{J}')$.

Define the cumulant functions $\kappa_i(\beta) := \beta^2 \sigma_i^2 / 2 + \beta \mu_i$ for all $i \in E$ and the cumulant matrix $K(\beta) := \Delta_{\kappa(\beta)} + Q$, where $\Delta_{\kappa(\beta)}$ denotes the diagonal matrix on E with entries $\kappa_i(\beta)$. A first observation is

$$\begin{aligned} \int_{\mathbb{R}} e^{\beta x} S^{(\mathbf{r})}(x)\, dx = \int_0^\infty \int_{\mathbb{R}} e^{\beta x} \mathbb{P}(X_t \in dx)\, dt = \int_0^\infty e^{(K(\beta) - \Delta_{\mathbf{r}})t}\, dt \\ = \left(\Delta_{\mathbf{r}} - K(\beta)\right)^{-1} \end{aligned} \tag{7}$$

for suitable values β, see proposition 2.2 in [3] for the second equality or theorem 7.11 in [11] for the whole statement.

The local time $L^{(\mathbf{r})}$ at zero of the terminating MMBM $(\mathcal{X}, \mathcal{J})$ may then be defined as the resolvent at zero, i.e. $L^{(\mathbf{r})} := S^{(\mathbf{r})}(0)$. The spatial homogeneity of the level process $\mathcal{X}$, conditional on the phase process $\mathcal{J}$, further leads to the obvious relation

$$S^{(\mathbf{r})}(x) = \begin{cases} \begin{pmatrix} I_a \\ A^+ \end{pmatrix} \begin{pmatrix} e^{U^+ x} & \mathbf{0} \end{pmatrix} L^{(\mathbf{r})}, & x > 0 \\ \begin{pmatrix} A^- \\ I_d \end{pmatrix} \begin{pmatrix} \mathbf{0} & e^{-U^- x} \end{pmatrix} L^{(\mathbf{r})}, & x < 0 \end{cases} \tag{8}$$

between the resolvent and the local time, see remark 1 and equations (2 - 6). Thus the resolvent is continuous in $x \neq 0$, therefore uniquely determined by its transform equation (7) on the one hand and by the local time on the other hand.

Note first that upon crossing zero in a pure drift phase $j \in E_p \cup E_n$, the phase will almost surely remain constant within a neighbourhoood of zero. This implies that we can determine the local time for phases $j \in E_p \cup E_n$ by simply counting the number of crossings, i.e.

$$L_{ij}^{(\mathbf{r})} = \frac{1}{|\mu_j|} \mathbb{E}\left(C\left(\{t < \tau_\Delta : X_t = 0, J_t = j\}\right) | X_0 = 0, J_0 = i\right) \tag{9}$$

where $C(A)$ denotes the number of elements in a set A. Note that we have excluded phases i with $\mu_i = \sigma_i = 0$.

4. Main result

We shall first provide (rather heuristic) probabilistic arguments as to how the block entries of $L^{(\mathbf{r})}$ should look like. Then we verify the result more rigorously in theorem 1. Write

$$L^{(\mathbf{r})} = \begin{pmatrix} L_{pp}^{(\mathbf{r})} & L_{p\sigma}^{(\mathbf{r})} & L_{pn}^{(\mathbf{r})} \\ L_{\sigma p}^{(\mathbf{r})} & L_{\sigma\sigma}^{(\mathbf{r})} & L_{\sigma n}^{(\mathbf{r})} \\ L_{np}^{(\mathbf{r})} & L_{n\sigma}^{(\mathbf{r})} & L_{nn}^{(\mathbf{r})} \end{pmatrix}$$

in obvious block notation and use the same notation for $A^\pm = A^\pm(\mathbf{r})$ and $U^\pm = U^\pm(\mathbf{r})$.

We begin with $L_{\sigma\sigma}^{(\mathbf{r})}$. Write Δ_{2/σ^2} for the diagonal matrix on E_σ with entries $2/\sigma_i^2$ for all phases $i \in E_\sigma$. In the absence of pure drift phases, i.e. if $E_p = E_n = \emptyset$, it has been shown in remark 2 of [8] that the expected local time before τ_Δ is given by the expression $-\left(U_{\sigma\sigma}^+ + U_{\sigma\sigma}^-\right)^{-1} \Delta_{2/\sigma^2}$. In the presence of positive drift phases (i.e. if $E_p \neq \emptyset$ but $E_n = \emptyset$), this should be extended to $-\left(U_{\sigma\sigma}^+ + U_{\sigma\sigma}^- + U_{\sigma p}^+ A_{p\sigma}^-\right)^{-1} \Delta_{2/\sigma^2}$, cf. the interpretation of $A_{p\sigma}^-$ in remark 1. If positive and negative drift phases are present, then the extension should be

$$\begin{aligned} L_{\sigma\sigma}^{(\mathbf{r})} = -\Big[& U_{\sigma p}^+ \left(I_p - A_{pn}^- A_{np}^+\right)^{-1} \left(A_{pn}^- A_{n\sigma}^+ + A_{p\sigma}^-\right) + \left(U_{\sigma\sigma}^+ + U_{\sigma\sigma}^-\right) \\ & + U_{\sigma n}^- \left(I_n - A_{np}^+ A_{pn}^-\right)^{-1} \left(A_{np}^+ A_{p\sigma}^- + A_{n\sigma}^+\right)\Big]^{-1} \Delta_{2/\sigma^2} \end{aligned} \tag{10}$$

The same point of view yields

$$L_{p\sigma}^{(\mathbf{r})} = \left(I_p - A_{pn}^- A_{np}^+\right)^{-1} \left(A_{pn}^- A_{n\sigma}^+ + A_{p\sigma}^-\right) L_{\sigma\sigma}^{(\mathbf{r})} \tag{11}$$

and

$$L_{n\sigma}^{(\mathbf{r})} = \left(I_n - A_{np}^+ A_{pn}^-\right)^{-1} \left(A_{np}^+ A_{p\sigma}^- + A_{n\sigma}^+\right) L_{\sigma\sigma}^{(\mathbf{r})} \tag{12}$$

Note that

$$L_{n\sigma}^{(\mathbf{r})} = A_{np}^+ L_{p\sigma}^{(\mathbf{r})} + A_{n\sigma}^+ L_{\sigma\sigma}^{(\mathbf{r})} \qquad \text{and} \qquad L_{p\sigma}^{(\mathbf{r})} = A_{pn}^- L_{n\sigma}^{(\mathbf{r})} + A_{p\sigma}^- L_{\sigma\sigma}^{(\mathbf{r})} \tag{13}$$

From a probabilistic point of view, this is obvious by the definition of $L^{(\mathbf{r})}$ and remark 1 for the matrices $A^{\pm}$. To verify this in an algebraic manner, one first observes that

$$\begin{aligned} A_{np}^{+} L_{p\sigma}^{(\mathbf{r})} &= A_{np}^{+} \sum_{k=0}^{\infty} \left(A_{pn}^{-} A_{np}^{+} \right)^{k} \left(A_{pn}^{-} A_{n\sigma}^{+} + A_{p\sigma}^{-} \right) L_{\sigma\sigma}^{(\mathbf{r})} \\ &= \left(I_n - A_{np}^{+} A_{pn}^{-} \right)^{-1} A_{np}^{+} \left(A_{pn}^{-} A_{n\sigma}^{+} + A_{p\sigma}^{-} \right) L_{\sigma\sigma}^{(\mathbf{r})} \end{aligned}$$

and then computes

$$\begin{aligned} L_{n\sigma}^{(\mathbf{r})} - A_{np}^{+} L_{p\sigma}^{(\mathbf{r})} &= \left(I_n - A_{np}^{+} A_{pn}^{-} \right)^{-1} \left(A_{np}^{+} A_{p\sigma}^{-} + A_{n\sigma}^{+} - A_{np}^{+} \left(A_{pn}^{-} A_{n\sigma}^{+} + A_{p\sigma}^{-} \right) \right) L_{\sigma\sigma}^{(\mathbf{r})} \\ &= \left(I_n - A_{np}^{+} A_{pn}^{-} \right)^{-1} \left(I_n - A_{np}^{+} A_{pn}^{-} \right) A_{n\sigma}^{+} L_{\sigma\sigma}^{(\mathbf{r})} \\ &= A_{n\sigma}^{+} L_{\sigma\sigma}^{(\mathbf{r})} \end{aligned}$$

which confirms the first equation. The second equation is obtained analogously. Now we turn to $L_{pp}^{(\mathbf{r})}$. Write $\Delta_{1/\mu}^{p}$ for the diagonal matrix on E_p with entries $1/\mu_i$ for all phases $i \in E_p$. In view of (9) it suffices to determine the expected number of crossings of the level 0 in a phase from E_p and divide this by the drift. In the absence of diffusion phases, i.e. if $E_\sigma = \emptyset$, this is simply $\left(I_p - A_{pn}^{-} A_{np}^{+} \right)^{-1} \Delta_{1/\mu}^{p}$. If $E_\sigma \neq \emptyset$, this should be extended to

$$\begin{aligned} L_{pp}^{(\mathbf{r})} = & \left[I_p - A_{pn}^{-} A_{np}^{+} + \left(A_{p\sigma}^{-} + A_{pn}^{-} A_{n\sigma}^{+} \right) \left(U_{\sigma\sigma}^{+} + U_{\sigma\sigma}^{-} + U_{\sigma n}^{-} A_{n\sigma}^{+} \right)^{-1} \left(U_{\sigma p}^{+} + U_{\sigma n}^{-} A_{np}^{+} \right) \right]^{-1} \\ & \times \Delta_{1/\mu}^{p} \end{aligned} \quad (14)$$

The main part of this extension is

$$L_{\sigma p}^{(\mathbf{r})} = - \left(U_{\sigma\sigma}^{+} + U_{\sigma\sigma}^{-} + U_{\sigma n}^{-} A_{n\sigma}^{+} \right)^{-1} \left(U_{\sigma p}^{+} + U_{\sigma n}^{-} A_{np}^{+} \right) L_{pp}^{(\mathbf{r})} \quad (15)$$

while remark 1 on A^{+} yields

$$L_{np}^{(\mathbf{r})} = A_{np}^{+} L_{pp}^{(\mathbf{r})} + A_{n\sigma}^{+} L_{\sigma p}^{(\mathbf{r})} \quad (16)$$

The expected local times for negative drift phases should of course have expressions that are symmetric to the ones for positive drift phases. Write $\Delta_{-1/\mu}^{n}$ for the diagonal matrix on E_n with entries $-1/\mu_i$ for all phases $i \in E_n$. Then

$$\begin{aligned} L_{nn}^{(\mathbf{r})} = & \left[I_n - A_{np}^{+} A_{pn}^{-} + \left(A_{n\sigma}^{+} + A_{np}^{+} A_{p\sigma}^{-} \right) \left(U_{\sigma\sigma}^{+} + U_{\sigma\sigma}^{-} + U_{\sigma p}^{+} A_{p\sigma}^{-} \right)^{-1} \left(U_{\sigma n}^{-} + U_{\sigma p}^{+} A_{pn}^{-} \right) \right]^{-1} \\ & \times \Delta_{-1/\mu}^{nn} \end{aligned} \quad (17)$$

as well as

$$L_{\sigma n}^{(\mathbf{r})} = -\left(U_{\sigma\sigma}^{+} + U_{\sigma\sigma}^{-} + U_{\sigma p}^{+} A_{p\sigma}^{-}\right)^{-1} \left(U_{\sigma n}^{-} + U_{\sigma p}^{+} A_{pn}^{-}\right) L_{nn}^{(\mathbf{r})} \tag{18}$$

and

$$L_{pn}^{(\mathbf{r})} = A_{pn}^{-} L_{nn}^{(\mathbf{r})} + A_{p\sigma}^{-} L_{\sigma n}^{(\mathbf{r})} \tag{19}$$

Theorem 1. *The entries of the matrix $L^{(\mathbf{r})}$ of expected local times are given by the equations (10 - 12) and (14 - 19).*

Proof: We verify that the resolvent as proposed solves the transform equation

$$\int_{\mathbb{R}} e^{\beta x} S^{(\mathbf{r})}(x) dx = (\Delta_{\mathbf{r}} - K(\beta))^{-1}$$

The same arguments as for equation (7) in [5] yield

$$\Delta_{\mathbf{r}} \begin{pmatrix} I_a \\ A^{+}(\mathbf{r}) \end{pmatrix} = \Delta_{\sigma^2/2} \begin{pmatrix} I_a \\ A^{+}(\mathbf{r}) \end{pmatrix} U^{+}(\mathbf{r})^2 - \Delta_{\mu} \begin{pmatrix} I_a \\ A^{+}(\mathbf{r}) \end{pmatrix} U^{+}(\mathbf{r}) + Q$$

where I_a denotes the identity matrix on $E_p \cup E_\sigma$ (use the function in (1) instead of $f_{ij}(x)$ as defined in (4) of [5]). Note that I_a is denoted by I_σ in [5] and there is a typo in equation (8) of [5], where it should state $-\Delta_\mu$ instead of $+\Delta_\mu$. The cumulant matrix can be written as $K(\beta) = \Delta_{\sigma^2/2}\beta^2 + \Delta_\mu \beta + Q$, whence we obtain

$$(K(\beta) - \Delta_{\mathbf{r}}) \begin{pmatrix} I_a \\ A^{+} \end{pmatrix} = \left(\Delta_{\sigma^2/2} \begin{pmatrix} I_a \\ A^{+} \end{pmatrix} (\beta I_a - U^{+}) + \Delta_{\mu} \begin{pmatrix} I_a \\ A^{+} \end{pmatrix} \right) (\beta I_a + U^{+}) \tag{20}$$

For the negative process $(\mathcal{X}^{-}, \mathcal{J})$ we obtain in the same way

$$\Delta_{\mathbf{r}} \begin{pmatrix} A^{-}(\mathbf{r}) \\ I_d \end{pmatrix} = \Delta_{\sigma^2/2} \begin{pmatrix} A^{-}(\mathbf{r}) \\ I_d \end{pmatrix} U^{-}(\mathbf{r})^2 + \Delta_{\mu} \begin{pmatrix} A^{-}(\mathbf{r}) \\ I_d \end{pmatrix} U^{-}(\mathbf{r}) + Q$$

where I_d denotes the identity matrix on $E_\sigma \cup E_n$, and hence

$$(K(\beta) - \Delta_{\mathbf{r}}) \begin{pmatrix} A^{-} \\ I_d \end{pmatrix} = \left(\Delta_{\sigma^2/2} \begin{pmatrix} A^{-} \\ I_d \end{pmatrix} (\beta I_d + U^{-}) + \Delta_{\mu} \begin{pmatrix} A^{-} \\ I_d \end{pmatrix} \right) (\beta I_d - U^{-}) \tag{21}$$

Equation (8) yields (for $|\beta|$ small enough)

$$\begin{aligned}\int_{\mathbb{R}} e^{\beta x} U^{(\mathbf{r})}(x)dx &= \int_0^\infty e^{\beta x} \begin{pmatrix} I_a \\ A^+ \end{pmatrix} \begin{pmatrix} e^{U^+ x} & \mathbf{0} \end{pmatrix} dx L^{(\mathbf{r})} \\ &\quad + \int_0^\infty e^{-\beta x} \begin{pmatrix} A^- \\ I_d \end{pmatrix} \begin{pmatrix} \mathbf{0} & e^{U^- x} \end{pmatrix} dx L^{(\mathbf{r})} \\ &= \begin{pmatrix} I_a \\ A^+ \end{pmatrix} \begin{pmatrix} -\left(\beta I_a + U^+\right)^{-1} & \mathbf{0} \end{pmatrix} L^{(\mathbf{r})} \\ &\quad + \begin{pmatrix} A^- \\ I_d \end{pmatrix} \begin{pmatrix} \mathbf{0} & \left(\beta I_d - U^-\right)^{-1} \end{pmatrix} L^{(\mathbf{r})}\end{aligned}$$

Hence we obtain

$$\begin{aligned}(\Delta_{\mathbf{r}} - K(\beta)) \int_{\mathbb{R}} e^{\beta x} U^{(\mathbf{r})}(x)dx &= \begin{pmatrix} \Delta_{\sigma^2/2} \begin{pmatrix} I_a \\ A^+ \end{pmatrix} (\beta I_a - U^+) + \Delta_\mu \begin{pmatrix} I_a \\ A^+ \end{pmatrix} & \mathbf{0} \end{pmatrix} L^{(\mathbf{r})} \\ &\quad - \begin{pmatrix} \mathbf{0} & \Delta_{\sigma^2/2} \begin{pmatrix} A^- \\ I_d \end{pmatrix} (\beta I_d + U^-) + \Delta_\mu \begin{pmatrix} A^- \\ I_d \end{pmatrix} \end{pmatrix} L^{(\mathbf{r})} \\ &= \begin{pmatrix} \Delta_\mu^{pp} & -\Delta_\mu^{pp} A_{p\sigma}^- & -\Delta_\mu^{pp} A_{pn}^- \\ -\Delta_{\sigma^2/2} U_{\sigma p}^+ & -\Delta_{\sigma^2/2} \left(U_{\sigma\sigma}^+ + U_{\sigma\sigma}^-\right) & -\Delta_{\sigma^2/2} U_{\sigma n}^- \\ \Delta_\mu^{nn} A_{np}^+ & \Delta_\mu^{nn} A_{n\sigma}^+ & -\Delta_\mu^{nn} \end{pmatrix} L^{(\mathbf{r})}\end{aligned} \tag{22}$$

It remains to verify that $L^{(\mathbf{r})}$ as given in (14 - 19) yields the identity matrix as the result. Indeed, equations (14 - 16) yield

$$\begin{aligned}\Delta_\mu^{pp} L_{pp}^{(\mathbf{r})} - \Delta_\mu^{pp} A_{p\sigma}^- L_{\sigma p}^{(\mathbf{r})} - \Delta_\mu^{pp} A_{pn}^- L_{np}^{(\mathbf{r})} &= \Delta_\mu^{pp} \left(I_p - A_{pn}^- A_{np}^+\right) L_{pp}^{(\mathbf{r})} \\ &\quad - \Delta_\mu^{pp} \left(A_{p\sigma}^- + A_{pn}^- A_{n\sigma}^+\right) L_{\sigma p}^{(\mathbf{r})} \\ &= I_p\end{aligned}$$

as well as

$$\begin{aligned}&-\Delta_{\sigma^2/2} U_{\sigma p}^+ L_{pp}^{(\mathbf{r})} - \Delta_{\sigma^2/2} \left(U_{\sigma\sigma}^+ + U_{\sigma\sigma}^-\right) L_{\sigma p}^{(\mathbf{r})} - \Delta_{\sigma^2/2} U_{\sigma n}^- L_{np}^{(\mathbf{r})} \\ &\quad = -\Delta_{\sigma^2/2} \left(U_{\sigma p}^+ + U_{\sigma n}^- A_{np}^+\right) L_{pp}^{(\mathbf{r})} - \Delta_{\sigma^2/2} \left(U_{\sigma\sigma}^+ + U_{\sigma\sigma}^- + U_{\sigma n}^- A_{n\sigma}^+\right) L_{\sigma p}^{(\mathbf{r})} \\ &\quad = \mathbf{0}\end{aligned}$$

while $\Delta_\mu^{nn} A_{np}^+ L_{pp}^{(\mathbf{r})} + \Delta_\mu^{nn} A_{n\sigma}^+ L_{\sigma p}^{(\mathbf{r})} - \Delta_\mu^{nn} L_{np}^{(\mathbf{r})} = \mathbf{0}$ is trivial from (16). Likewise, the equality $\Delta_\mu^{pp} L_{p\sigma}^{(\mathbf{r})} - \Delta_\mu^{pp} A_{p\sigma}^- L_{\sigma\sigma}^{(\mathbf{r})} - \Delta_\mu^{pp} A_{pn}^- L_{n\sigma}^{(\mathbf{r})} = \mathbf{0}$ is trivial from the second equation in (13). Equations (10 - 12) yield further

$$-\Delta_{\sigma^2/2} U_{\sigma p}^+ L_{p\sigma}^{(\mathbf{r})} - \Delta_{\sigma^2/2} \left(U_{\sigma\sigma}^+ + U_{\sigma\sigma}^-\right) L_{\sigma\sigma}^{(\mathbf{r})} - \Delta_{\sigma^2/2} U_{\sigma n}^- L_{n\sigma}^{(\mathbf{r})} = I_\sigma$$

while $\Delta_\mu^{nn} A_{np}^+ L_{p\sigma}^{(\mathbf{r})} + \Delta_\mu^{nn} A_{n\sigma}^+ L_{\sigma\sigma}^{(\mathbf{r})} - \Delta_\mu^{nn} L_{n\sigma}^{(\mathbf{r})} = \mathbf{0}$ is trivial from the first equation in (13). Finally, $\Delta_\mu^{pp} L_{pn}^{(\mathbf{r})} - \Delta_\mu^{pp} A_{p\sigma}^- L_{\sigma n}^{(\mathbf{r})} - \Delta_\mu^{pp} A_{pn}^- L_{nn}^{(\mathbf{r})} = \mathbf{0}$ follows from (19), while

$$\begin{aligned}
-\Delta_{\sigma^2/2} U_{\sigma p}^+ L_{pn}^{(\mathbf{r})} &- \Delta_{\sigma^2/2} \left(U_{\sigma\sigma}^+ + U_{\sigma\sigma}^-\right) L_{\sigma n}^{(\mathbf{r})} - \Delta_{\sigma^2/2} U_{\sigma n}^- L_{nn}^{(\mathbf{r})} \\
&= -\Delta_{\sigma^2/2} \left[\left(U_{\sigma p}^+ A_{pn}^- + U_{\sigma n}^-\right) L_{nn}^{(\mathbf{r})} + \left(U_{\sigma p}^+ A_{p\sigma}^- + U_{\sigma\sigma}^+ + U_{\sigma\sigma}^-\right) L_{\sigma n}^{(\mathbf{r})} \right] \\
&= \mathbf{0}
\end{aligned}$$

is due to (18 - 19) and

$$\begin{aligned}
\Delta_\mu^{nn} A_{np}^+ L_{pn}^{(\mathbf{r})} &+ \Delta_\mu^{nn} A_{n\sigma}^+ L_{\sigma n}^{(\mathbf{r})} - \Delta_\mu^{nn} L_{nn}^{(\mathbf{r})} \\
&= -\Delta_\mu^{nn} \left[\left(I_n - A_{np}^+ A_{pn}^-\right) L_{nn}^{(\mathbf{r})} - \left(A_{np}^+ A_{p\sigma}^- + A_{n\sigma}^+\right) L_{\sigma n}^{(\mathbf{r})} \right] \\
&= I_n
\end{aligned}$$

because of (17 - 19).

□

Example 1. The case of a Brownian motion with variance σ^2 and drift μ is of course covered by exercise 2 of section VII in [4]. There it is stated that the resolvent density is given by

$$u^q(x) = \Phi'(q) e^{-\Phi(q)x}$$

for x > 0, where $\Phi(q)$ is the positive inverse of the cumulant function $\psi(\beta) := \beta^2\sigma^2/2 + \beta\mu$. Thus

$$\Phi(q) = -\frac{1}{\sigma^2}\left(\mu - \sqrt{\mu^2 + 2q\sigma^2}\right)$$

and because of

$$U^\pm(q) = \frac{1}{\sigma^2}\left(\pm\mu - \sqrt{\mu^2 + 2q\sigma^2}\right)$$

we obtain $\Phi(q) = -U^+(q)$. Since further $U^+(q) + U^-(q) = -2/\sigma^2\sqrt{\mu^2 + 2q\sigma^2}$ and hence

$$\Phi'(q) = \left(\sqrt{\mu^2 + 2q\sigma^2}\right)^{-1} = -\frac{2}{\sigma^2}\left(U^+(q) + U^-(q)\right)^{-1}$$

we obtain $L^q = \Phi'(q)$ and $S^q(x) = u^q(x)$ according to theorem 1 and (8), respectively.

Example 2. For the case $E_n = \emptyset$, the matrix $L^{(\mathbf{r})}$ plays a role in the investigation of the scale function as in section 7.5 of [11]. In particular, section 7.7 therein states that in this case $L^{(\mathbf{r})} = \Xi^{-1}$ where

$$\Xi = \Delta_\mu \left(\Pi^+ - \Pi^-\Pi_-^+\right) - \frac{1}{2}\Delta_\sigma^2 \left(\Pi^+ \left(\Lambda^+ - \alpha I\right) + \Pi^- \left(\Lambda^- + \alpha I\right) \Pi_-^+\right)$$

in the notation of [11]. This translates for the case $E_n = \emptyset$ as $\Pi^+ = I_a$, $\Pi^- = \begin{pmatrix} A^- \\ I_\sigma \end{pmatrix}$, $\Pi_-^+ = \begin{pmatrix} \mathbf{0} & I_\sigma \end{pmatrix}$ and $\Lambda^\pm = U^\pm$. Thus, in our notation

$$\begin{aligned}\Xi &= \Delta_\mu \begin{pmatrix} I_p & -A^- \\ \mathbf{0} & \mathbf{0} \end{pmatrix} - \Delta_{\sigma^2/2} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ U_{\sigma p}^+ & U_{\sigma\sigma}^+ + U^- \end{pmatrix} \\ &= \begin{pmatrix} \Delta_\mu^p & -\Delta_\mu^p A_{p\sigma}^- \\ -\Delta_{\sigma^2/2} U_{\sigma p}^+ & -\Delta_{\sigma^2/2}\left(U_{\sigma\sigma}^+ + U_{\sigma\sigma}^-\right) \end{pmatrix}\end{aligned}$$

since $E_n = \emptyset$ implies $A^- = A_{p\sigma}^-$ and $U^- = U_{\sigma\sigma}^-$. This is the upper left-hand part of the matrix in (22).

5. An application to insurance risk

Inspired by [1] we consider a risk reserve process observed at discrete times. There are several values of interest that we wish to address. First of all we would like to determine the transition probabilities between the discrete time observations. Then we might want to know how large the risk of ruin between observations is. Clearly there is a trade-off between longer inter-observation times and a higher risk of unobserved ruin. This leads to an optimisation problem.

The authors in [1] consider a compound Poisson model, which is observed at the times of a renewal process with Erlang distributed renewal intervals. We shalll consider a compound Poisson model with phase-type claim sizes and inter-observation times. The rate of the exponential inter-claim times is denoted by $\lambda > 0$. Let $T \sim PH(\alpha^{(o)}, T^{(o)})$ denote a generic observation interval and $C \sim PH(\alpha^{(c)}, T^{(c)})$ a generic claim size. denote the dimensions of $\alpha^{(o)}$ and $\alpha^{(c)}$ by $m^{(o)}$ and $m^{(c)}$, respectively. Further denote the exit rate vectors by $\eta^{(o)} := -T^{(o)}\mathbf{1}$ and $\eta^{(c)} := T^{(c)}\mathbf{1}$, where $\mathbf{1}$ denote column vectors of appropriate dimension with all entries being one.

In order to determine the values of interest, we employ the construction described in remark 2 and consider an MMBM with phase space $E = E_p \cup E_n$ where $E_p = \{1, \ldots, m^{(o)}\}$ and $E_n = \{(i,k) : i \in E_p, 1 \leq k \leq m^{(c)}\}$. The positive drift phases from E_p simply store the phase of the observation period. The second variable in a negative drift phase from E_n stores the phase of the claim size distribution, while the first variable remembers the phase of the observation period into which to jump back after the claim is paid out. The phase-dependent parameters are given as $\mu_i = c$ and $\sigma_i = 0$ for $i \in E_p$, where $c > 0$ denotes the rate of premium income. For $i \in E_n$ we set $\mu_i = -1$ and $\sigma_i = 0$. The generator matrix of the phase process is given as $Q = (q_{ij})_{i,j \in E}$ with

$$q_{ij} = \begin{cases} -\lambda + T^{(o)}_{ii} + \eta^{(o)}_i, & j = i \in E_p \\ T^{(o)}_{ij}, & j \neq i \in E_a \\ \lambda \cdot \alpha^{(c)}_k, & i \in E_p, j = (i,k) \in E_n \\ T^{(c)}_{kl}, & i = (h,k), j = (h,l) \in E_n, h \in E_p \\ \eta^{(c)}_l, & i = (j,l) \in E_n, j \in E_p \end{cases}$$

Remark 3. Since the claims are modelled as linear drifts of slope -1 in the MMBM model, the $PH(\alpha^{(o)}, T^{(o)})$ distribution of the inter-observation times translates to the somewhat more complicated $PH\left(\begin{pmatrix} \alpha^{(o)} \\ \mathbf{0} \end{pmatrix}, Q - \Delta_{(\eta^{(o)}, \mathbf{0})}\right)$ distribution.

Let $p(x,y) := \mathbb{P}(X_T \in dy | X_0 = x)$ denote the transition densities between observation points. Setting $r_i := \eta^{(o)}_i$ for all $i \in E_p$ and $r_i := 0$ for all $i \in E_n$ and noting that we observe only phases in E_p (for which $1/\mu_i = c$), we can evaluate those as

$$\begin{aligned} p(x,y) &= \alpha^{(o)} S^{(\mathbf{r})}_{pp}(y-x) \eta^{(o)} \\ &= \begin{cases} c \cdot \alpha^{(o)} e^{U^+ \cdot (y-x)} \left(I_p - A^-_{pn} A^+_{np}\right)^{-1} \eta^{(o)}, & y \geq x \\ c \cdot \alpha^{(o)} A^-_{pn} e^{U^- \cdot (x-y)} A^+_{np} \left(I_p - A^-_{pn} A^+_{np}\right)^{-1} \eta^{(o)}, & y < x \end{cases} \end{aligned}$$

for all $x, y \in \mathbb{R}$, where we write $U^\pm = U^\pm(\mathbf{r})$ and $A^\pm = A^\pm(\mathbf{r})$.

In particular, if the claim sizes and the inter-observation times are exponential with parameters $\nu > 0$ and $\gamma > 0$, respectively, then we obtain $E = \{1, 2\}$ and $Q = \begin{pmatrix} -\lambda & \lambda \\ \nu & -\nu \end{pmatrix}$ as well as $\mathbf{r} = (\gamma, 0)$. Example 3 in [8] with $\beta = \nu$ yields

$$U^+(\mathbf{r}) = \frac{1}{2c}\left(c\nu - \gamma - \lambda - \sqrt{(c\nu - \gamma - \lambda)^2 + 4c\nu\gamma}\right), \qquad A^+(\mathbf{r}) = \frac{\nu}{\nu - U^+(\mathbf{r})}$$
$$U^-(\mathbf{r}) = \frac{1}{2c}\left(\gamma + \lambda - c\nu - \sqrt{(c\nu - \gamma - \lambda)^2 + 4c\nu\gamma}\right), \qquad A^-(\mathbf{r}) = \frac{\nu + U^-(\mathbf{r})}{\nu}$$

Since $\alpha^{(o)} = 1$ and $\eta^{(o)} = \gamma$, we obtain

$$p(0, y) = \begin{cases} c \cdot e^{U^+ x} \frac{\nu - U^+}{-U^+ - U^-}, & y > 0 \\ c \cdot e^{U^- y} \frac{\nu + U^-}{-U^+ - U^-}, & y < 0 \end{cases}$$

Setting $\delta = 0$ in [1], we see from (13) therein that the notation translates as $U^(\mathbf{r}) = -\rho_\gamma$ and $U^-(\mathbf{r}) = -R_\gamma$. Noting that in [1] the authors consider the net claim process, the increments of which are the negative increments of the risk reserve, we find that the result for the density function above coincides with $g_\delta(y)$ as given in section 3.1 of [1].

Coming back to the more general setting of phase-type claim sizes and inter-observation times, we now consider the expected number of ruin events between observation times. This may be considered as a risk measure in order to determine a suitable distribution for the inter-observation times. Given that the risk reserve starts with $x > 0$ at the beginning of an observation interval, denote this as $R(\alpha^{(o)}, T^{(o)}|x)$. Recalling (9) and $|\mu_i| = 1$ for all $i \in E_n$, we obtain

$$R(\alpha^{(o)}, T^{(o)}|x) = \alpha^{(o)} A_{pn}^- e^{U^- x} \left(I_p - A_{np}^+ A_{pn}^-\right)^{-1}$$

for all $x > 0$. Atttaching a proportional cost factor to an expected ruin event and the rate of observations, we then would need to minimise the function

$$C(\alpha^{(o)}, T^{(o)}|x) = \frac{R(\alpha^{(o)}, T^{(o)}|x)}{-\alpha^{(o)}(T^{(o)})^{-1}\mathbf{1}}$$

over $(\alpha^{(o)}, T^{(o)})$ for each $x > 0$. Thus the optimal values $(\alpha^{(o)}, T^{(o)})$ will in general be functions of x.

6. The resolvent and the local times at zero for reflected MMBMs

Now we consider a MMBM that is reflected upwards at zero. Define the infimum process $\mathcal{I} = (I_t : t \geq 0)$ by $I_t := \inf_{s \leq t} X_s \wedge 0$ for all $t \geq 0$, and the reflected process by $\mathcal{Y} := \mathcal{X} - \mathcal{I}$.

Again we assume throughout that $X_0 = 0$. Define the $\mathbf{r}$-resolvent of $(\mathcal{Y}, \mathcal{J})$ as the matrix-valued function $R^{(\mathbf{r})}(x)$ with entries

$$R_{ij}^{(\mathbf{r})}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \mathbb{E}\left(\int_0^{\tau_\Delta} \mathbb{I}_{\{|Y_t - x| < \varepsilon, J_t = j\}} \, dt \middle| Y_0 = 0, J_0 = i \right)$$

for $x > 0$ and $i, j \in E$. Note that the local time at zero, say $L^{(\mathbf{r})}(0)$, equals the absolute infimum before τ_Δ, i.e. $L^{(\mathbf{r})}(0) = -I_{\tau_\Delta}$.

In order to state the result, we first introduce some abbreviations to simplify notations. Define the matrices

$$C^+ := C^+(\mathbf{r}) := \begin{pmatrix} \mathbf{0} \quad I_\sigma \\ A^+(\mathbf{r}) \end{pmatrix} \qquad \text{and} \qquad C^- := C^-(\mathbf{r}) := \begin{pmatrix} A^-(\mathbf{r}) \\ I_\sigma \quad \mathbf{0} \end{pmatrix} \tag{23}$$

of dimensions $E_d \times E_a$ and $E_a \times E_d$, respectively. Further define

$$W^+ := W^+(\mathbf{r}) := \begin{pmatrix} I_a \\ A^+(\mathbf{r}) \end{pmatrix} \qquad \text{and} \qquad W^- := W^-(\mathbf{r}) := \begin{pmatrix} A^-(\mathbf{r}) \\ I_d \end{pmatrix}$$

which are matrices of dimensions $E \times E_a$ and $E \times E_d$.

Theorem 2. *The $\mathbf{r}$-resolvent of the reflected MMBM $(\mathcal{Y}, \mathcal{J})$ is given by*

$$R^{(\mathbf{r})}(x) = \begin{pmatrix} \Psi_{\mathbf{r}}^+(0, x|0) - \Psi_{\mathbf{r}}^-(0, x|0) G_{\mathbf{r}}(x)^{-1} H_{\mathbf{r}}(x) \\ -G_{\mathbf{r}}(x)^{-1} H_{\mathbf{r}}(x) \end{pmatrix} \left(I_a + C^- e^{U^- x} G_{\mathbf{r}}(x)^{-1} H_{\mathbf{r}}(x) \right)^{-1}$$
$$\times L_{(a,.)}^{(\mathbf{r})} \left(I - W^- e^{U^- x} C^+ e^{U^+ x} \right)$$

for all $x > 0$ and $\|r\| > 0$, where

$$G_{\mathbf{r}}(x) = \left(U^- e^{-U^- x} + C^+ e^{U^+ x} U^+ C^- \right) \left(e^{-U^- x} - C^+ e^{U^+ x} C^- \right)^{-1}$$

and

$$H_{\mathbf{r}}(x) = \left(C^+ U^+ + U^- C^+ \right) \left(C^- e^{U^- x} C^+ - e^{-U^+ x} \right)^{-1}$$

The distribution of the local times at zero is given by

$$\mathbb{P}(L^{(\mathbf{r})}(0) > y) = W^- e^{U^- y}$$

for all $y \geq 0$.

Proof: We first observe that, starting at $x > 0$, the local times of $(\mathcal{Y}, \mathcal{J})$ at x before hitting the zero level coincide with the local times of the free MMBM $(\mathcal{X}, \mathcal{J})$ before $\tau^-(0)$. Considering that the level x must have been reached from below, i.e. in an ascending phase, these are given by the term

$$L^{(\mathbf{r})}_{(a,.)}\left(I - W^- e^{U^- x} C^+ e^{U^+ x}\right)$$

cf. equation (7.4) in [11]. Using the results from theorem 1 in [7] (adapted to the present reflection at the infimum) we can determine the probabilities to go from level x (in an ascending phase) to level 0 and then back to x before $\tau_\triangle$ as $-C^- e^{U^- x} G_\mathbf{r}(x)^{-1} H_\mathbf{r}(x)$. Thus the expected number of such down and up crossings is given by $\left(I_a + C^- e^{U^- x} G_\mathbf{r}(x)^{-1} H_\mathbf{r}(x)\right)^{-1}$. Premultiplying by the probabilities of reaching level x from $X_0 = 0$ before $\tau_\triangle$ for the first time, namely by $-G_\mathbf{r}(x)^{-1} H_\mathbf{r}(x)$ if $J_0 \in E_d$ and by $\Psi^+_\mathbf{r}(0, x|0) - \Psi^-_\mathbf{r}(0, x|0) G_\mathbf{r}(x)^{-1} H_\mathbf{r}(x)$ if $J_0 \in E_p$, yields the stated formula for the resolvent. The statement for the local time at zero follows immediately from $L^{(\mathbf{r})}(0) = -I_{\tau_\triangle}$.

□

Example 3. We consider a Brownian motion with variation σ^2 and drift μ that is reflected at zero. In this case, $\mathbf{r}$ is a number, $U^\pm$ are given as in example 1, and

$$G_\mathbf{r}(x) = \frac{U^- + U^+ e^{(U^+ + U^-)x}}{1 - e^{(U^+ + U^-)x}} \qquad H_\mathbf{r}(x) = \frac{-(U^+ + U^-) e^{U^+ x}}{1 - e^{(U^- + U^+)x}}$$

Thus

$$-G_\mathbf{r}(x)^{-1} H_\mathbf{r}(x) = \frac{U^+ + U^-}{U^- + U^+ e^{(U^+ + U^-)x}} e^{U^+ x}$$

and

$$1 + e^{U^- x} G_\mathbf{r}(x)^{-1} H_\mathbf{r}(x) = U^- \frac{1 - e^{(U^- + U^+)x}}{U^- + U^+ e^{(U^- + U^+)x}}$$

Further $L^{(\mathbf{r})} = -U^+ U^- / (U^+ + U^-) \mathbf{r}^{-1}$ and hence

$$\begin{aligned} R^{(\mathbf{r})}(x) &= \frac{(U^+ + U^-) e^{U^+ x}}{U^- + U^+ e^{(U^+ + U^-)x}} \cdot \frac{U^- + U^+ e^{(U^- + U^+)x}}{U^-(1 - e^{(U^- + U^+)x})} \cdot \frac{U^+ U^- (1 - e^{(U^- + U^+)x})}{-\mathbf{r}(U^+ + U^-)} \\ &= \frac{-U^+ e^{U^+ x}}{\mathbf{r}} \end{aligned}$$

Thus a reflected Brownian motion observed at an exponential time has an exponential distribution with parameter $-U^+ = (\sqrt{\mu^2 + 2q\sigma^2} - \mu)/\sigma^2$. This result is part of the statement in [3], chapter IX, problem 3.3.

References

[1] ALBRECHER, H., CHEUNG, E. AND THONHAUSER, S. (2011). Randomized observation times for the compound Poisson risk model: Dividends. *ASTIN Bulletin* **41,** 645–672.

[2] ASMUSSEN, S. (1995). Stationary distributions for fluid flow models with or without Brownian motion. *Stochastic Models* **11,** 1–20.

[3] ASMUSSEN, S. (2003). *Applied Probability and Queues*. New York etc.: Springer.

[4] BERTOIN, J. (1996). *Lévy Processes*. Cambridge University Press, Cambridge etc.

[5] BREUER, L. (2008). First passage times for Markov–additive processes with positive jumps of phase–type. *J. Appl. Prob.* **45,** 779–799.

[6] BREUER, L. (2010). A quintuple law for Markov-additive processes with phase-type jumps. *J. Appl. Prob.* **47,** 441–458.

[7] BREUER, L. (2012). Exit problems for reflected Markov-modulated Brownian motion. *J. Appl. Prob.* **49,** to appear.

[8] BREUER, L. (2012). Occupation times for Markov-modulated Brownian motion. *J. Appl. Prob.* **49,** to appear.

[9] D'AURIA, B., IVANOVS, J., KELLA, O. AND MANDJES, M. (2012). Two-Sided Reflection of Markov-Modulated Brownian Motion. *Stochastic Models* **28,** to appear.

[10] IVANOVS, J. (2010). Markov-modulated Brownian motion with two reflecting barriers. *J. Appl. Prob.* **47,** 1034–1047.

[11] IVANOVS, J. (2011). One-sided Markov Additive Processes and Related Exit Problems. *PhD thesis*. Universiteit van Amsterdam.

[12] JIANG, Z. AND PISTORIUS, M. (2008). On perpetual American put valuation and first-passage in a regime-switching model with jumps. *Finance Stoch.* **12,** 331–355.

[13] ROGERS, L. (1994). Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *Annals of Applied Probability* **4,** 390–413.