A generalised Gerber-Shiu measure for Markov-additive risk processes with phase-type claims and capital injections

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September 4, 2011

Abstract

In this paper we consider a risk reserve process where the arrivals (either claims or capital injections) occur according to a Markovian point process. Both claim and capital injection sizes are phase-type distributed and the model allows for possible correlations between these and the inter-claim times. The premium income is modelled by a Markov-modulated Brownian motion which may depend on the underlying phases of the point arrival process. For this risk reserve model we derive a generalized Gerber-Shiu measure that is the joint distribution of the time to ruin, the surplus immediately before ruin, the deficit at ruin, the minimal risk reserve before ruin, and the time until this minimum is attained. Numerical examples illustrate the influence of the parameters on selected marginal distributions.

1 Introduction

Gerber and Shiu [15] derived the joint distribution of the time to ruin, the surplus immediately before ruin, and the deficit at ruin. Their analysis covered the classical Poisson risk model

$$R_t = u + ct - \sum_{j=1}^{N_t} X_j$$

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where $u \geq 0$ is the initial surplus, $c > 0$ is the rate of premium income, $(N_t : t \geq 0)$ is a Poisson process, and X_j, $j \in \mathbb{N}$, are iid positive random variables modelling the claim sizes. In this notation the time of ruin is given by

$$T = \inf\{t \geq 0 : R_t < 0\}$$

while the surplus immediately before ruin and the deficit at ruin are R_{T-} and $|R_T|$, respectively. Given a discount rate $\delta \geq 0$ and a non-negative function $w(x, y)$ on $x, y \geq 0$, Gerber and Shiu [16] investigated the function

$$\phi(u) = \mathbb{E}[w(R_{T-}, |R_T|) \cdot e^{-\delta T} \cdot \mathbb{I}_{\{T<\infty\}} | R_0 = u] \tag{1}$$

where $\mathbb{I}_A$ denotes the indicator function of some set A. This function ϕ has found much attention since then and was given the name Gerber-Shiu (GS) function or discounted penalty function. Many authors have contributed to its analysis, where the underlying risk reserve process has been generalised in several directions. The perturbed compound Poisson model has been considered in [14, 12, 20], while Markov-modulated (or regime switching) versions are analysed in [3, 23]. There are further related papers on the GS function for the Lévy risk process [13], the fluid flow model [7, 1], the Sparre Andersen model with Erlang inter-claim times [17] and its perturbed version [18]. [8, 9] extended the Gerber-Shiu function to a generalized discounted penalty function (GDPF) considering the last minimum of the surplus before ruin $\underline{R}_T$ in the analysis. The new defined GDPF can be represented as

$$\phi_G(u) = \mathbb{E}[w_G(R_{T-}, |R_T|, \underline{R}_T) \cdot e^{-\delta T} \cdot \mathbb{I}_{\{T<\infty\}} | R_0 = u], \tag{2}$$

where w_G is a bounded measurable function on $\mathbb{R}_+^3$.

One does not need to add more references to show that the GS functions enjoy great popularity among the research community. The almost universal approach of analysis is the derivation of some (defective) renewal equations, coming from a set of integro-differential equations which are obtained via Itô's formula or the infinitesimal generator of the risk reserve process (see discussion to [19]).

The present paper deals with the analysis of a generalized Gerber-Shiu type measure (to be introduced in Section 2) for Markov-additive risk processes. Combining the features of perturbation and Markov-modulation we render some of the aforementioned risk processes as special cases. The only restriction required for the analysis in this paper is that both claim and capital injection sizes have a phase-type representation.

Rather than employing the mainstream approach of defective renewal equations, we shall use a recent result presented in [11], where the joint distribution of the space-time positions of overshoots and undershoots has been derived for Markov-additive processes with phase-type jumps. Note that at times we will alternate the use of words overshoot/undershoot with deficit/surplus, in order to be consistent with the terminology encountered in the research areas where they are frequently used.

Our present paper aims to apply this result to the class of Markov–additive risk processes described in the second section. Extensive numerical examples shall illustrate the effect that different market conditions can have on the GDPF and its particular cases.

In the following section we shall present the model of Markov-additive risk processes and establish the relation to first passage times as well as overshoots and undershoots for Markov-additive processes (MAPs). In section 3 we collect all the necessary preliminary results for MAPs that we will need later on. In particular we simplify the results from [10] for the special kind of MAPs that we employ in this paper. Section 4 contains the main result with some corollaries. The final section acts as a utilization manual for the results obtained in Section 4 presented on two numerical examples.

2 The insurance risk model

We begin with a Markovian random environment. Let $\tilde{\mathcal{J}} = (\tilde{J}_t : t \geq 0)$ be an irreducible Markov (jump) process with finite state space $\tilde{E}$ and infinitesimal generator matrix $\tilde{Q} = (\tilde{q}_{ij})_{i,j \in \tilde{E}}$. We call $\tilde{J}_t$ the phase at time $t \geq 0$. Each phase $i \in \tilde{E}$ signifies a certain state of market conditions which may affect the intensity and severity of claims and capital injections as well as the rate and volatility of the premium income.

Based on the phase process $\tilde{\mathcal{J}}$, we define the risk reserve process $\mathcal{R} = (R_t : t \geq 0)$ as follows. Denote the initial risk reserve by $R_0 := u \geq 0$. We assume that the premium income between claims can be modelled by a Brownian motion, where the parameters c_i (drift) and $\tilde{\sigma}_i$ (variation) at time t may depend on the current phase $\tilde{J}_t = i$. For insurance risk we typically have $c_i > 0$ for all $i \in \tilde{E}$, which we shall assume from now on. We shall allow $\tilde{\sigma}_i = 0$ for some (or possibly all) phases, under which condition the Brownian motion becomes a linear drift. Thus the process of premium income is a Markov–modulated Brownian motion which we denote by $(\mathcal{B}, \tilde{\mathcal{J}}) = ((B_t, \tilde{J}_t) : t \geq 0)$. We assume that $B_0 = 0$.

Claims may occur in two ways. First, during $\{t \geq 0 : \tilde{J}_t = i\}$, i.e. when $\tilde{\mathcal{J}}$ is in phase i, claims occur at a constant (hazard) rate $\lambda_i^+ \geq 0$. The size of such a claim shall have a phase-type (PH) distribution with parameters $\alpha^{(ii)+}$ and $T^{(ii)+}$. Second, at time instances of phase changes from i to $j \neq i$ a claim may occur with probability p_{ij}^+. The size of such a claim shall have a PH distribution with parameters $\alpha^{(ij)+}$ and $T^{(ij)+}$. As usual, we write $\eta^{(ij)+} := -T^{(ij)+}\mathbf{1}$ for the so-called exit rate vector of the $PH(\alpha^{(ij)+}, T^{(ij)+})$ distribution, where $\mathbf{1}$ denotes a column vector of appropriate dimension with all entries being 1. Denote the claim arrival process by $(\mathcal{N}^+, \tilde{\mathcal{J}}) = ((N_t^+, \tilde{J}_t) : t \geq 0)$, i.e. let N_t^+ denote the number of claims that have occurred until time t. Denote the nth claim size by C_n, $n \in \mathbb{N}$.

We further allow for capital injections, i.e. sudden increases of the risk reserve at particular time instances. Here we use the same model as for the claim arrival process. Thus we may see a capital injection during $\{t \geq 0 : \tilde{J}_t = i\}$ at a constant rate $\lambda_i^- \geq 0$. Its size shall have a PH distribution with parameters $\alpha^{(ii)-}$ and $T^{(ii)-}$. At time instances of phase changes from i to $j \neq i$ a capital injection may occur with probability p_{ij}^-. The size of such an injection shall have a PH distribution with parameters $\alpha^{(ij)-}$ and $T^{(ij)-}$. Again we define $\eta^{(ij)-} := -T^{(ij)-}\mathbf{1}$ for $i, j \in \tilde{E}$. We further denote the process generating capital injections by $(\mathcal{N}^-, \tilde{\mathcal{J}}) = ((N_t^-, \tilde{J}_t) : t \geq 0)$, i.e. let N_t^- denote the number of capital injections that have occurred until time t. Denote the size of the nth injection by K_n, $n \in \mathbb{N}$.

Thus the point arrival process is a Markovian point process (MPP)[1] with parameter matrices D_0 and D_1 on $\tilde{E}$ having entries

$$D_{0;ij} = \begin{cases} \tilde{q}_{ii} - \lambda_i^+ - \lambda_i^-, & j = i \\ \tilde{q}_{ij} \cdot (1 - p_{ij}^+ - p_{ij}^-), & j \neq i \end{cases} \qquad \text{and} \qquad D_{1;ij} = \begin{cases} \lambda_i^+ + \lambda_i^-, & j = i \\ \tilde{q}_{ij} \cdot (p_{ij}^+ + p_{ij}^-), & j \neq i \end{cases}$$

where we assume that $p_{ij}^+ + p_{ij}^- \leq 1$ for all $i, j \in \tilde{E}$.

As expected, we observe that the sum of the elements in each row of the matrix $D_0 + D_1$ equals 0. A common phase space enables us to model correlations between point arrivals, claim/capital injection sizes, and the premium income. It is shown in [5] that the class of MPPs is dense within the class of marked point processes. Further, it is shown in [24] that the class of phase–type distributions is dense within the class of all distributions on the positive real numbers. Thus we incur no serious restriction in generality.

With the definitions above, the risk reserve process $\mathcal{R} = (R_t : t \geq 0)$ is given by

$$R_t = u + B_t - \sum_{n=1}^{N_t^+} C_n + \sum_{n=1}^{N_t^-} K_n =: u + B_t - \mathfrak{C}_t + \mathfrak{K}_t,$$

for $t \geq 0$, where $\mathfrak{C}_t$ represents the aggregate claim process and $\mathfrak{K}_t$ the aggregate capital infusion process. The net profit condition for such an insurance risk model can be written as

$$\mathbb{E}_{\tilde{\pi}}[B_1] > \mathbb{E}_{\tilde{\pi}}[\mathfrak{C}_1] - \mathbb{E}_{\tilde{\pi}}[\mathfrak{K}_1]. \tag{3}$$

[1] see [22, 21, 5]. This has traditionally been called Markovian arrival process and abbreviated as MAP. Since we use the shortcut MAP for the more general class of Markov–additive processes already, we prefer to use the term Markovian point process and the abbreviation MPP instead. Some authors use the shortcut MArP.

where $\tilde{\pi}$ is the stationary distribution of $\tilde{\mathcal{J}}$, i.e. $\tilde{\pi}\tilde{Q} = \mathbf{0}$ and $\tilde{\pi}\mathbf{1} = 1$. In terms of the given parameters this translates into

$$\sum_{i\in\tilde{E}} \tilde{\pi}_i c_i > \sum_{i\in\tilde{E}} \tilde{\pi}_i \left(\lambda_i^- \alpha^{(ii)-} \left(T^{(ii)-}\right)^{-1} \mathbf{1} - \lambda_i^+ \alpha^{(ii)+} \left(T^{(ii)+}\right)^{-1} \mathbf{1}\right)$$
$$+ \sum_{i\in\tilde{E}} \tilde{\pi}_i \sum_{j\neq i} \tilde{q}_{ij} \left(p_{ij}^- \alpha^{(ij)-} \left(T^{(ij)-}\right)^{-1} \mathbf{1} - p_{ij}^+ \alpha^{(ij)+} \left(T^{(ij)+}\right)^{-1} \mathbf{1}\right)$$

In typical applications many of the parameters $p_{ij}^\pm$ and $\lambda_i^\pm$ turn out to be zero, which reduces the number of terms in the sums above. In all examples given in this paper the parameters $p_{ij}^\pm$ will vanish.

We denote the net loss process by $\tilde{\mathcal{X}} = (\tilde{X}_t : t \geq 0)$ where $\tilde{X}_t := u - R_t$ for all $t \geq 0$. Then the process $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ is a Markov-additive process (MAP) with phase–type jumps. In order to determine the GDPF for the risk reserve process described above, we shall make use of a recently published quintuple law for MAPs, see [11]. Thus it will help to translate the variables derived from the risk reserve process $\mathcal{R}$ into variables related to the net loss process $\tilde{\mathcal{X}}$.

Clearly, $\tilde{X}_0 = 0$. The time of ruin is given by

$$T = \inf\{t \geq 0 : R_t < 0\} = \inf\{t \geq 0 : \tilde{X}_t > u\} =: \tilde{\tau}(u)$$

where $\tilde{\tau}(u)$ is the first passage time to level u of the net loss process $\tilde{\mathcal{X}}$. The surplus immediately before ruin and the deficit at ruin are then given by

$$R_{T-} = u - \tilde{X}_{\tilde{\tau}(u)-} \qquad \text{and} \qquad |R_T| = \tilde{X}_{\tilde{\tau}(u)} - u$$

respectively. Thus it suffices to look at undershoots and overshoots for MAPs in order to determine the surplus prior to and the deficit at ruin under the risk reserve process. As mentioned before, in this paper we go a step further and derive additionally the minimal surplus before ruin, i.e.

$$\underline{R}_{\,T} := \min\{R_t : t < T\} = u - \max\{\tilde{X}_t : t < \tilde{\tau}(u)\} := u - \tilde{M}_{\tilde{\tau}(u)}$$

and the time until this minimum is reached, i.e.

$$S(\underline{R}_T) := \min\{t \geq 0 : R_t = \underline{R}_T\} = \max\{t \geq 0 : \tilde{X}_t = \tilde{M}_{\tilde{\tau}(u)}\} =: \tilde{G}_{\tilde{\tau}(u)} \qquad (4)$$

The last equality holds because we have excluded constant (null) movements and allowed only phase-type jumps. An actuarial motivation for the analysis of $\underline{R}_{\,T}$ and $S(\underline{R}_{\,T})$ is provided in [9], p.92: "The lower this minimum, the worse the financing conditions that can be negotiated by the company with capital providers. Similarly, the closer the last minimum was to the bankruptcy level, and the shorter the time elapsed since that minimum, the more urgent was the need to correct the course and steer away from dangerous waters."

The main result of this paper is an explicit formula for the generalized Gerber-Shiu measure defined as a density function

$$f_{\gamma,\gamma^*}(x,y,z)\;dx\;dy\;dz := \mathbb{E}\left(e^{-\gamma S(\underline{R}_T)-\gamma^*\cdot(T-S(\underline{R}_T))}; R_{T-}\in dx, |R_T|\in dy, \underline{R}_T\in dz\right) \tag{5}$$

where $\gamma,\gamma^*\geq 0$ are time discounting factors and $0<z<u$. Note that $\underline{R}_{\,T}\leq R_0=u$ such that $f_{\gamma,\gamma^*}(x,y,z)=0$ for $z>u$.

There are two singular points for $\underline{R}_{\,T}$. One is $\underline{R}_{\,T}=u$, which means that the risk reserve will never fall below the initial value before ruin occurs by a claim. In this case $S(\underline{R}_{\,T})=0$. The other singular point is $\underline{R}_{\,T}=0$, which means that ruin occurs by creeping, i.e. not by a claim but by the volatility of the premium income process. In this case $S(\underline{R}_{\,T})=T$ and $R_{T-}=|R_T|=0$ follow necessarily.

This result provides all the information that is usually contained in the GDPF. In addition, it yields the distributions of $S(\underline{R}_{\,T})$ and $\underline{R}_{\,T}$. Given a discount rate $\delta\geq 0$ and a non-negative penalty function $w_G(x,y,z)$ on $x,y,z\geq 0$, the original GDPF (2) can be determined by setting $\gamma=\gamma^*=\delta$ as follows

$$\begin{aligned}\phi_G(u)=\int_{x,y=0}^{\infty}\int_{z=0}^{u} & w_G(x,y,z)f_{\delta,\delta}(x,y,z)\;dz\;dx\;dy\\ & +\int_{x,y=0}^{\infty}w_G(x,y,u)\;\mathbb{E}\left[e^{-\delta T}; R_{T-}\in dx, |R_T|\in dy, \underline{R}_{\,T}=u\right]\\ & + w_G(0,0,0)\;\mathbb{E}[e^{-\delta T}; R_{T-}=|R_T|=\underline{R}_{\,T}=0].\end{aligned}$$

In particular, the term $\int_{z=0}^{u} f_{\delta,\delta}(x,y,z)\;dz+\mathbb{E}\left[e^{-\delta T}; R_{T-}\in dx, |R_T|\in dy, \underline{R}_{\,T}=u\right]$ may be seen as the original Gerber-Shiu density function.

3 Preliminaries

Looking at the problem from the angle described above and in order to obtain results pertinent to the insurance risk process R_t, we first need to collect some necessary preliminary results for MAPs from the existing literature. This shall be the purpose of the present section.

3.1 Markov-additive processes with phase-type jumps

The joint net loss and phase process $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ form a MAP with the following parameters. The phase space is $\tilde{E}$ and the infinitesimal generator matrix for $\tilde{\mathcal{J}}$ is $\tilde{Q}$. The real–valued level process $\tilde{\mathcal{X}} = (\tilde{X}_t : t \geq 0)$ evolves like a Lévy process $\tilde{\mathcal{X}}^{(i)}$ with parameters $\tilde{\mu}_i := -c_i$ (drift), $\tilde{\sigma}_i^2$ (variation), and Lévy measure

$$\tilde{\nu}_i(dx) = \lambda_i^+ \mathbb{I}_{\{x>0\}}\, \alpha^{(ii)+} \exp(T^{(ii)+}x)\eta^{(ii)+}dx \\ + \lambda_i^- \mathbb{I}_{\{x<0\}}\, \alpha^{(ii)-} \exp(-T^{(ii)-}x)\eta^{(ii)-}dx \quad (6)$$

during intervals when the phase equals $i \in \tilde{E}$. Whenever $\tilde{\mathcal{J}}$ jumps from a state $i \in \tilde{E}$ to another state $j \in \tilde{E}$, $j \neq i$, this may be accompanied (with probability p_{ij}^+) by an upward jump of $\tilde{\mathcal{X}}$ with distribution function $\tilde{F}_{ij}^+ = PH(\alpha^{(ij)+}, T^{(ij)+})$ or (with probability p_{ij}^-) by a downward jump with distribution function $\tilde{F}_{ij}^- = PH(\alpha^{(ij)-}, T^{(ij)-})$. Denote the order of $PH(\alpha^{(ij)\pm}, T^{(ij)\pm})$ by $m_{ij}^\pm$.

The main advantage of the PH restriction on the jump distributions is the possibility of transforming the jumps into a succession of linear pieces of exponential duration (each with slope 1 or -1) and retrieving the original process via a simple time change, see [4], section 8, or [6]. This will transform our original MAP process $\tilde{\mathcal{X}}$ into a new one, denoted by $\mathcal{X}$, in which there will be a time evolution during those periods when the claims and the capital injections are paid. This is done in the following way. Without the jumps, the Lévy process $\tilde{\mathcal{X}}^{(i)}$ during a phase $i \in \tilde{E}$ is either a linear drift (of slope $\tilde{\mu}_i$) or a Brownian motion (with parameters $\tilde{\sigma}_i$ and $\tilde{\mu}_i$). Considering this MAP (without the jumps) we can partition its phase space $\tilde{E}$ into the subspaces E_p (for positive drifts), E_σ (for Brownian motions), and E_n (for negative drifts). We thus define

$$E_p := \{i \in \tilde{E} : \tilde{\mu}_i > 0, \tilde{\sigma}_i = 0\}, E_n := \{i \in \tilde{E} : \tilde{\mu}_i < 0, \tilde{\sigma}_i = 0\}, E_\sigma := \{i \in \tilde{E} : \tilde{\sigma}_i > 0\}. \quad (7)$$

Note that $\tilde{E} = E_p \cup E_\sigma \cup E_n$, since we have excluded the case of $\tilde{\mu}_i = \tilde{\sigma}_i^2 = 0$ for any phase $i \in \tilde{E}$. Then we introduce two new phase spaces

$$E_\pm := \{(i, j, k, \pm) : i, j \in E_p \cup E_\sigma \cup E_n, 1 \leq k \leq m_{ij}^\pm\} \quad (8)$$

to model the jumps. Define now the enlarged phase space $E = E_+ \cup \tilde{E} \cup E_-$. We define the modified MAP $(\mathcal{X}, \mathcal{J})$ over the enlarged phase space E as follows. Set the parameters $(\mu_i, \sigma_i^2, \nu_i)$ for $i \in E$ as

$$(\mu_i, \sigma_i^2, \nu_i) := \begin{cases} (\pm 1, 0, \mathbf{0}), & i \in E_\pm \\ (\tilde{\mu}_i, \tilde{\sigma}_i^2, \mathbf{0}), & i \in \tilde{E} = E_p \cup E_\sigma \cup E_n \end{cases} \quad (9)$$

The modified phase process $\mathcal{J}$ is determined by its generator matrix $Q = (q_{ij})_{i,j\in E}$. For this the construction above yields

$$q_{ih} = \begin{cases} \tilde{q}_{ii} - \lambda_i^+ - \lambda_i^-, & h = i \in \tilde{E} \\ \tilde{q}_{ih} \cdot (1 - p_{ih}^+ - p_{ih}^-), & h \in \tilde{E}, h \neq i \\ \lambda_i^\pm \alpha_k^{(ii)\pm}, & h = (i,i,k,\pm) \\ \tilde{q}_{ij} \cdot p_{ij}^\pm \cdot \alpha_k^{(ij)\pm}, & h = (i,j,k,\pm) \end{cases} \tag{10}$$

for $i \in \tilde{E}$ as well as

$$q_{(i,j,k,\pm),(i,j,l,\pm)} = T_{kl}^{(ij)\pm} \qquad \text{and} \qquad q_{(i,j,k,\pm),j} = \eta_k^{(ij)\pm} \tag{11}$$

for $i,j \in \tilde{E}$ and $1 \leq k,l \leq m_{ij}$. For later use we define $q_i := -q_{ii}$ for all $i \in E$.

We denote the MAP constructed in (9), (10), and (11) by $(\mathcal{X}, \mathcal{J})$. The original level process $\tilde{\mathcal{X}}$ is retrieved via the time change

$$c(t) := \int_0^t 1_{J_s \in \tilde{E}}\, ds \quad \text{and} \quad \tilde{X}_{c(t)} = X_t \tag{12}$$

for all $t \geq 0$. The inverses of the cumulant functions ψ_i for the so–called ascending phases $i \in E_a := E_+ \cup E_p \cup E_\sigma$ can be given explicitly as

$$\phi_i(\beta) = \begin{cases} \frac{\beta}{\mu_i}, & i \in E_+ \cup E_p \\ \frac{1}{\sigma_i}\sqrt{2\beta + \frac{\mu_i^2}{\sigma_i^2}} - \frac{\mu_i}{\sigma_i^2}, & i \in E_\sigma \end{cases} \tag{13}$$

Example 1 We consider the classical compound Poisson model. Inter–claim times and claim sizes are iid exponential with parameter $\lambda > 0$ and $\beta > 0$, respectively. The rate of premium income is $c > 0$. The net profit condition is then $\lambda/(c\beta) < 1$. This model has been examined in [16]. Under our notation the net loss at time $t \geq 0$ is given by

$$\tilde{X}_t = \sum_{n=0}^{N_t} C_n - ct \tag{14}$$

where $(N_t : t \geq 0)$ is a Poisson process with intensity λ and the C_n, $n \in \mathbb{N}$, are iid random variables with exponential distribution of parameter β. The net loss process can be analysed as a MAP with exponential (and hence phase–type) positive jumps with parameter β. For this, we would need only one phase, i.e. $\tilde{E} = \{1\}$. This phase governs a Lévy process with parameters $\tilde{\sigma} = 0$, $\tilde{\mu} = -c$, and $\tilde{\nu}(dx) = \lambda e^{-\beta x}\beta dx$ for all $x > 0$.

We obtain the modified MAP $(\mathcal{X}, \mathcal{J})$ as follows. The new phase space is given as $E = \{(1,1), 1\}$, where $E_+ = \{(1,1)\}$, $E_n = \{1\}$, and $E_\sigma = E_p = E_- = \emptyset$. We set $\lambda_1 = \lambda$ and $m_{11} = 1$ since the positive jumps have an exponential distribution. The parameters are given by $\sigma_{(1,1)} = \sigma_1 = 0$, $\mu_{(1,1)} = 1$, $\mu_1 = -c$, $\nu_{(1,1)} = \nu_1 = \mathbf{0}$, according to (9). The generator matrix for the phase process $\mathcal{J}$ is given as

$$Q = \begin{pmatrix} -\beta & \beta \\ \lambda & -\lambda \end{pmatrix}$$

according to (10) and (11).

Example 2 To illustrate the use of environment phases we resort to an example first presented in [6]. There is a predominant normal state, A and a “rare" state B to represent periods of contagion. The system switches from A to B at rate α_A and from B to A at rate α_B. Environment A features standard claim rates and claim sizes, while environmental B features a supplemental stream of claims due to a highly infectious disease.

Claims occur in two ways. There is at all times a Poisson process with parameter δ_1 of small exponential claims, their mean being $\frac{1}{\mu_1}$. While in environmental B, there is in addition a second process, with parameter δ_2 of exponential claims; their expected value is $\frac{1}{\mu_2}$.

The following five states are identified:

1. environment A, normal claim payment in progress;
2. environment B, normal claim payment in progress;
3. environment B, contagion claim payment in progress.
4. environment A, during an interval between claims;
5. environment B, during an interval between claims;

Thus $E_+ = \{1, 2, 3\}$ and $E_n = \{4, 5\}$. The generator for $\mathcal{J}$ is

$$Q = \begin{pmatrix} -\mu_1 & 0 & 0 & \mu_1 & 0 \\ 0 & -\mu_1 & 0 & 0 & \mu_1 \\ 0 & 0 & -\mu_2 & 0 & \mu_2 \\ \delta_1 & 0 & 0 & -\alpha_A - \delta_1 & \alpha_A \\ 0 & \delta_1 & \delta_2 & \alpha_B & -\alpha_B - \delta_1 - \delta_2 \end{pmatrix}. \tag{15}$$

The parameters for $\mathcal{X}$ are

$$(\mu_i, \sigma_i, \nu_i) = \begin{cases} (0, 1, \mathbf{0}), & i \in \{1, 2, 3\} \\ (0, -c_A, \mathbf{0}), & i = 4 \\ (0, -c_B, \mathbf{0}), & i = 5 \end{cases}$$

where c_A and c_B are the rates of premium collection in environments A and B, respectively.

Example 3 The joint density function of the surplus prior to ruin and the deficit at ruin has been derived in [7] for the fluid flow case from an insurance perspective. The fluid queue $\{(L_t, J_t) : t \geq 0\}$ as defined on p. 434 therein is a MAP with phase space $S = S_1 \cup S_2$ and parameters

$$(\sigma_i, \mu_i, \nu_i) = \begin{cases} (0, 1, \mathbf{0}), & i \in S_1 \\ (0, -1, \mathbf{0}), & i \in S_2 \end{cases}$$

as well as generator matrix

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

Phases in S_1 are considered as premium income phases, while phases in S_2 pertain to claims. Not counting the time during claim phases (in S_2) and setting $L_0 := u$, which shall denote the initial risk reserve, $\{(L_t, J_t) : t \geq 0\}$ uniquely defines a risk reserve process $R_{c'(t)} := L_t$, $t \geq 0$, via the time change $c'(t) := \int_0^t 1_{J_s \in S_1} ds$. Thus the approach in [7] is very similar to the present paper, only the parameters are more restricted (they do not allow perturbations by diffusion).

It is therefore quite simple to compare results between [7] and the present paper. The net claim amount at time $t \geq 0$ is $\tilde{X}_t := u - R_t$. For the modified MAP $(\mathcal{X}, \mathcal{J})$, we obtain $E_p = E_\sigma = E_- = \emptyset$ and $E_+ = S_2$, $E_n = S_1$. Comparing the notations for the generator matrix of the phase process $\mathcal{J}$, we get the block partition

$$Q = \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix} = \begin{pmatrix} T_{22} & T_{21} \\ T_{12} & T_{11} \end{pmatrix}$$

The assumption $c = 1$ therein translates to $\mu_i = -1$ for all $i \in E_n$ in our notation.

3.2 First passage times

Of central use in the present paper will be the matrices $A(\gamma)$ and $U(\gamma)$ that determine the Laplace transforms for the first passage times of MAPs with phase–type jumps as given in [10]. Define $\tilde{\tau}(x) := \inf\{t \geq 0 : \tilde{X}_t > x\}$ and $\tau(x) := \inf\{t \geq 0 : X_t > x\}$ for all $x \geq 0$

and assume that $X_0 = 0$. The time change in (12) yields $\tilde{\tau}(x) = c(\tau(x)) = \int_0^{\tau(x)} 1_{J_s \in \tilde{E}} ds$, i.e. we may compute expectations over $\tilde{\tau}(x)$ using the distribution of the modified MAP $(\mathcal{X}, \mathcal{J})$ only. For $\gamma \geq 0$ denote

$$\mathbb{E}_{ij}[e^{-\gamma\tilde{\tau}(x)}] := \mathbb{E}[e^{-\gamma\tilde{\tau}(x)}; J_{\tau(x)} = j | J_0 = i, X_0 = 0]$$

for all $i, j \in E$. Note that the phases i, j may be taken from the enlarged phase space E, thus we include phases $i, j \in E_+ \cup E_-$ that model the jumps. Let $\mathbb{E}[e^{-\gamma\tilde{\tau}(x)}]$ denote the matrix with these entries and write

$$\mathbb{E}[e^{-\gamma\tilde{\tau}(x)}] = \begin{pmatrix} \mathbb{E}_{(a,a)}[e^{-\gamma\tilde{\tau}(x)}] & \mathbb{E}_{(a,d)}[e^{-\gamma\tilde{\tau}(x)}] \\ \mathbb{E}_{(d,a)}[e^{-\gamma\tilde{\tau}(x)}] & \mathbb{E}_{(d,d)}[e^{-\gamma\tilde{\tau}(x)}] \end{pmatrix}$$

in obvious block notation with respect to the subspaces $E_a = E_+ \cup E_p \cup E_\sigma$ (ascending phases) and $E_d = E_n \cup E_-$ (descending phases).

Since a first passage to a level above cannot occur in a descending phase, we obtain first $\mathbb{P}(J_{\tau(x)} = j) = 0$ for all $j \in E_d$ and thus $\mathbb{E}_{(d,d)}[e^{-\gamma\tilde{\tau}(x)}] = \mathbb{E}_{(a,d)}[e^{-\gamma\tilde{\tau}(x)}] = \mathbf{0}$ where $\mathbf{0}$ denotes a zero matrix of suitable dimension. The exponential form

$$e^{-\gamma\tilde{\tau}(x)} = e^{-\gamma \int_0^{\tau(x)} 1_{J_s \in \tilde{E}} ds}$$

as well as path continuity of $\mathcal{X}$ and spatial homogeneity of $(\mathcal{X}, \mathcal{J})$ lead to the functional equation

$$\mathbb{E}_{ij}[e^{-\gamma\tilde{\tau}(x+y)}] = \sum_{k \in E_a} \mathbb{E}_{ik}[e^{-\gamma\tilde{\tau}(x)}] \mathbb{E}_{kj}[e^{-\gamma\tilde{\tau}(y)}]$$

for all $i \in E$ and $j \in E_a$. Hence we obtain

$$\mathbb{E}_{(d,a)}[e^{-\gamma\tilde{\tau}(x)}] = A(\gamma) e^{U(\gamma)x} \quad \text{and} \quad \mathbb{E}_{(a,a)}[e^{-\gamma\tilde{\tau}(x)}] = e^{U(\gamma)x} \tag{16}$$

for some sub–generator matrix $U(\gamma)$ of dimension $E_a \times E_a$ and a sub–transition matrix $A(\gamma)$ of dimension $E_d \times E_a$, cf. equation (6) in [10].

Write e_i' for the ith canonical row base vector, according to context either on E, on E_a, or on E_d. According to [11], the matrices $A(\gamma)$ and $U(\gamma)$ can be determined by successive approximation as the limit of the sequence $((A_n, U_n) : n \geq 0)$ with initial values $A_0 := \mathbf{0}$,

$U_0 := -diag(\phi_i(q_i+\gamma)1_{i\in E_\sigma\cup E_p} + \phi_i(q_i)1_{i\in E_+})_{i\in E_a}$ and the following iteration:

$$e'_h U_{n+1} = \sum_{l=1}^{m_{ij}^+} T_{kl}^{(ij)+} e'_{(i,j,l,+)} + \eta_k^{(ij)+} e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} \qquad \text{for } h=(i,j,k,+)\in E_+,$$

$$e'_i U_{n+1} = -\frac{q_i+\gamma}{\mu_i} e'_i + \frac{1}{\mu_i} \sum_{j\in E, j\neq i} q_{ij}\ e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} \qquad \text{for } i\in E_p,$$

$$e'_i A_{n+1} = \sum_{j\in E, j\neq i} q_{ij} e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} \left((q_i+\gamma)I + \mu_i U_n\right)^{-1} \qquad \text{for } i\in E_n,$$

$$e'_i A_{n+1} = \sum_{j\in E, j\neq i} q_{ij} e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} \left(q_i I - U_n\right)^{-1} \qquad \text{for } i\in E_-, \text{ and}$$

$$e'_i U_{n+1} = -\phi_i(q_i+\gamma)e'_i + \frac{2}{\sigma_i^2} \sum_{j\in E, j\neq i} q_{ij}\ e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} \left(\phi_i^*(q_i+\gamma)I - U_n\right)^{-1} \qquad (17)$$

for $i \in E_\sigma$. Note that I_a represents an identity matrix whose dimension is given by the number of ascending phases from E_a.

Example 4 Coming back to example 1, it is shown in [10], example 5, that the Laplace transform of the first passage time $\tilde{\tau}(x) := \inf\{t \geq 0 : \tilde{X}_t > x\}$ to a level $x > 0$ is given by

$$\mathbb{E}[e^{-\gamma\tilde{\tau}(x)}] = A(\gamma)e^{U(\gamma)x} \quad \text{where} \quad A(\gamma) = \frac{\beta - R}{\beta}, \quad U(\gamma) = -R$$

and

$$-R = \frac{1}{2c}\left(\lambda + \gamma - c\beta - \sqrt{(c\beta-\gamma-\lambda)^2 + 4c\beta\gamma}\right)$$

which coincides with equation (4.24) in [16], noting that γ is denoted as δ there.

3.3 Time–reversed MAPs

Denote the number of phases in E by $m := |E|$. Let $\pi = (\pi_1, \ldots, \pi_m)$ denote the stationary phase distribution, which can be computed by $\pi Q = \mathbf{0}$ and $\pi\mathbf{1} = \sum_{i=1}^m \pi_i = 1$, where $\mathbf{0}$ denotes the zero row vector and $\mathbf{1}$ the column vector with all entries being one. Define the matrix $Q^* = (q^*_{ij})_{i,j\in E}$ by $q^*_{ij} := \pi_j q_{ji}/\pi_i$ for all $i,j \in E$ or in shorter notation $Q^* := \Delta_\pi^{-1} Q' \Delta_\pi$, where $\Delta_\pi = diag(\pi_1, \ldots, \pi_m)$ is the diagonal matrix with entry π_i in its ith row and the superscript $'$ denotes transposition of a matrix. Then the Markov process with state space E and generator matrix Q^* is a time–reversed version of the original phase process $\mathcal{J}$. We denote it by $\mathcal{J}^* = (J^*_t : t \geq 0)$.

Based on $\mathcal{J}^*$ we define a time-reversal $(\mathcal{X}^*, \mathcal{J}^*)$ of the original MAP $(\mathcal{X}, \mathcal{J})$ by the rule that $\mathcal{X}^*$ evolves like a Lévy process with parameters $-\mu_i$ (drift) and σ_i^2 (variation) during intervals when the time–reversed phase J_t^* equals $i \in E$. Note that the sign change of the μ_i leads to $E_\pm^* = E_\mp$, $E_p^* = E_n$, $E_n^* = E_p$, and $E_\sigma^* = E_\sigma$. We denote the first passage times for $(\mathcal{X}^*, \mathcal{J}^*)$ by $\tau^*(x) := \inf\{t \geq 0 : X_t^* > x\}$ for any level $x \geq 0$.

4 Main result

We necessarily have $0 \leq \underline{R}_{\,T} \leq u$, where $\underline{R}_{\,T} = u$ means that the risk reserve $\mathcal{R}$ does not fall below its initial value u before a claim causes ruin. The case $\underline{R}_{\,T} = 0$ means that passage occurs by creeping, i.e. ruin is not caused by a claim but by the volatility in premium income.

Our aim is to derive a computable expression for the measure defined in equation (5). Note that using the connections (developed in Section 2) between the risk process R_t and the MAP process $\tilde{X}_t$ one can write

$$\begin{aligned} & \mathbb{E}\left[e^{-\gamma S(\underline{R}_{\,T}) - \gamma^* \cdot (T - S(\underline{R}_{\,T}))}; R_{T-} \in dx, |R_T| \in dy, \underline{R}_{\,T} \in dz\right] \\ = \; & \mathbb{E}\left[e^{-\gamma \tilde{G}_{\tilde{\tau}(u)} - \gamma^*(\tilde{\tau}(u) - \tilde{G}_{\tilde{\tau}(u)})}; u - \tilde{X}_{\tilde{\tau}(u)-} \in dx, \tilde{X}_{\tilde{\tau}(u)} - u \in dy, u - \tilde{M}_{\tilde{\tau}(u)} \in dz,\right] \end{aligned} \tag{18}$$

where $\gamma, \gamma^* \geq 0$ are arguments for the double Laplace transform, $x, y \geq 0$, and $0 \leq z \leq u$. Note that necessarily $x \geq u - z$. It is then clear that on one side the knowledge of the time to ruin, the surplus prior to ruin, the deficit at ruin, the minimum prior to ruin and the time of the minimum in the risk process R_t correspond on the other side to finding the first passage time over a level u, the undershoot and the overshoot at this passage time, the maximum before this passage time and the time of this maximum for the MAP $\tilde{X}_t$. Thus we can make use of the quintuple law for MAPs as derived in [11]. The results shall be phrased in terms of the variables T, R_T, R_{T-}, $\underline{R}_T$, and $S(\underline{R}_T)$ as they are more immediate to insurance risk.

We shall use the parameters of the modified MAP $(\mathcal{X}, \mathcal{J})$ as constructed in section 3.1. Set $P = \Delta_q^{-1} Q + I$, where Q denotes the generator matrix of $\mathcal{J}$, see (10) and (11), and Δ_q is the diagonal matrix with entries $q_i = -q_{ii}$ for all $i \in E$. Define $p_{ij}^{(+,-)} := \delta_{ij}$ for $i \in E_\sigma$ and $p_{ij}^{(+,-)} := p_{ij}$ for $i \in E_+ \cup E_p$, $j \in E_n \cup E_-$. Further define

$$P^{(+,-)} := \left(p_{ij}^{(+,-)}\right)_{i \in E_a, j \in E_\sigma \cup E_d} \qquad \text{and} \qquad P^{(\sim,+)} := (p_{ij} 1_{i \in \tilde{E}})_{i \in E, j \in E_+} \tag{19}$$

The matrices $P^{(+,-)}$ and $P^{(\sim,+)}$ subsume the transition probabilities from ascending to descending phases and from continuous to positive jump phases, respectively. Write

$$\Delta_\phi := diag(\phi_i(q_i))_{i\in E_a} \quad \text{and} \quad \Delta_{\phi^*} := diag(\phi_i(q_i)1_{i\in E_p} + \phi^*(q_i)1_{i\in E_\sigma\cup E_n})_{i\in E} \tag{20}$$

and define the block diagonal matrix $T = diag\left(T^{(ij)}\right)_{(i,j)\in\tilde{E}\times\tilde{E}}$ and the block column vector $\eta = \left(\eta^{(ij)}\right)_{(i,j)\in\tilde{E}\times\tilde{E}}$. Finally, define the diagonal matrices

$$\Pi_a^* = diag(1/\pi_i)_{i\in E_\sigma\cup E_d} \qquad \text{and} \qquad \Pi_\sim = diag(\pi_j 1_{j\in\tilde{E}})_{j\in E} \tag{21}$$

Now we can state the main results. Note that M' denotes the transpose of a matrix M, I_a the identity matrix on E_a, and I_a^* the identity matrix on $E_\sigma \cup E_d$.

Theorem 1 *Let $(\mathcal{R}, \tilde{\mathcal{J}})$ denote a Markov-additive risk process with phase–type claims and possible capital injections. Let $\tilde{\alpha}$ denote its initial phase distribution, i.e. $\tilde{\alpha}_i := \mathbb{P}(\tilde{J}_0 = i)$ for all $i \in \tilde{E}$. Define the row vector $\alpha = (\alpha_i : i \in E)$ on the phase space E by $\alpha_i := \tilde{\alpha}_i$ for all $i \in \tilde{E}$ and $\alpha_i := 0$ for $i \in E_+ \cup E_-$. Then*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\gamma S(\underline{R}_T)-\gamma^*(T-S(\underline{R}_T))}; R_{T-}\in dx, |R_T|\in dy, u-\underline{R}_T\in dz,\right] \\
&\qquad = \alpha \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} e^{U(\gamma)z}\Delta_\phi P^{(+,-)} \\
&\qquad\qquad \left(\Pi_\sim \begin{pmatrix} A^*(\gamma^*) \\ I_a^* \end{pmatrix} e^{U^*(\gamma^*)\cdot(z-(u-x))}\Pi_a^*\right)' \Delta_{\phi^*} P^{(\sim,+)} e^{T\cdot(x+y)}\eta
\end{aligned}$$

for all $\gamma, \gamma^* \geq 0$, $0 < z < u$, $x > u - z$, and $y > 0$.

If the process starts in a phase $i \in \tilde{E}$ with $c_i > 0$ and $\tilde{\sigma}_i = 0$, then the singular case $\underline{R}_T = u$ is possible. This implies $S(\underline{R}_T) = 0$ and $R_{T-} > u$. The remaining quadruple law is given in the following corollary. For $\gamma^* = 0$ and $E_\sigma = \emptyset$ it yields equation (3.6) in [2] and theorem 1 in [23].

Corollary 1 *Let $\tilde{\alpha}$ be an initial phase distribution with support on E_n and define α as in theorem 1. Then*

$$\begin{aligned}
&\mathbb{E}\left[e^{-\gamma^* T}; R_{T-}\in dx, |R_T|\in dy, \underline{R}_T = u,\right] \\
&\qquad\qquad = \alpha\left(\Pi_\sim \begin{pmatrix} A^*(\gamma^*) \\ I_{E_\sigma\cup E_d} \end{pmatrix} e^{U^*(\gamma^*)\cdot(x-u)}\Pi_a^*\right)' \Delta_{\phi^*} P^{(\sim,+)} e^{T\cdot(x+y)}\eta
\end{aligned}$$

Another singular case that may arise is given in the following corollary. The reasoning is the same as for theorem 1. Note that in typical insurance applications there are no negative premiums and thus $E_p = \emptyset$ for the net loss process.

Corollary 2 *Ruin by diffusion*
The probability for ruin by diffusion is given as

$$\mathbb{E}\left[e^{-\gamma T}; R_{T-} = 0, |R_T| = 0, \underline{R}_T = 0\right] = \alpha \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} e^{U(\gamma)u} \mathbf{1}_{E_p \cup E_\sigma}$$

where $\mathbf{1}_{E_p \cup E_\sigma}$ is a column vector of dimension E_a with ith entry being 0 for $i \in E_+$ and 1 for $i \in E_p \cup E_\sigma$.

Corollary 3 *Deficit at ruin*
The unconditional deficit at the time of ruin has a phase-type distribution with representation $PH\left(\alpha \begin{pmatrix} I_a \\ A(0) \end{pmatrix} e^{U(0)u} I_{E_+}, T\right)$ where I_{E_+} denotes the diagonal matrix on E_a with entries 1 for rows $i \in E_+$ and 0 otherwise. The probability of a zero deficit at ruin is given as

$$\mathbb{P}(|R_T| = 0) = \alpha \begin{pmatrix} I_a \\ A(0) \end{pmatrix} e^{U(0)u} \mathbf{1}_{E_p \cup E_\sigma}$$

where $\mathbf{1}_{E_p \cup E_\sigma}$ is defined as in corollary 2.

Remark 1 For comparisons of the results in this paper to existing results in the literature we direct the interested reader to section 4 of [11].

5 Numerical illustrations

In the last section we consider two particular cases to illustrate some of our main results. Both cases shall serve as guidance to theorem 1, explaining where to find the relevant formulas needed to compute the ingredients for it.

5.1 Classical Poisson case with both positive and negative jumps

In the first case we consider a risk process with linear premiums at rate $c > 0$ which is superposed by two compound Poisson processes. One of them has positive jumps of exponential size with parameter $\beta^+ > 0$ and jump intensity $\lambda^+ > 0$. The other one has negative jumps of exponential size with parameter $\beta^- > 0$ and jump intensity $\lambda^- > 0$. Together this forms a Lévy process $\tilde{\mathcal{X}}$ with parameters $\mu = -c$ for the drift, $\sigma^2 = 0$ (i.e. there is no diffusion part), and Lévy measure

$$\nu(dx) = \lambda^+ \cdot \mathbb{I}_{\{x>0\}} \cdot e^{-\beta^+ x}\beta^+ \, dx + \lambda^- \cdot \mathbb{I}_{\{x<0\}} \cdot e^{-\beta^- x}\beta^- \, dx.$$

The process $\tilde{\mathcal{X}}$ can of course be represented as a MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ with trivial phase space $\tilde{E} = 1$ and the trivial generator matrix $\tilde{Q} = 0$ for $\tilde{\mathcal{J}}$. According to section 3.1 we construct the MAP $(\mathcal{X}, \mathcal{J})$ as follows. The enlarged phase space E consists of the subsets $E_n = \{1\}$, $E_+ = \{(1,1,1,+)\}$, and $E_- = \{(1,1,1,-)\}$, according to (7) and (8). The subsets E_σ and E_p are empty. For ease of notation, denote the jump phases by $1+ = (1,1,1,+)$ and $1- = (1,1,1,-)$. The parameters for each phase are set according to (9) as

$$(\mu_i, \sigma_i, \nu_i) = \begin{cases} (1, 0, \mathbf{0}), & i = 1+ \\ (-c, 0, \mathbf{0}), & i = 1 \\ (-1, 0, \mathbf{0}), & i = 1- \end{cases}$$

The generator matrix Q for $\mathcal{J}$ is given by (10 - 11) as

$$Q = \begin{pmatrix} -\beta^+ & \beta^+ & 0 \\ \lambda^+ & -\lambda & \lambda^- \\ 0 & \beta^- & -\beta^- \end{pmatrix}$$

where $\lambda = \lambda^+ + \lambda^-$. This completely defines the MAP $(\mathcal{X}, \mathcal{J})$. The matrices $A(\gamma)$ and $U(\gamma)$ have dimension 2×1 and 1×1, respectively. According to (17), they are determined as the limit $(U(\gamma), A(\gamma)) = \lim_{n\to\infty}(U_n, A_n)$, with initial values $A_0 = \mathbf{0}$ and $U_0 = -\beta^+$ and iteration

$$\begin{aligned} U_{n+1} &= -\beta^+ + \beta^+ e_1' A_n, \\ e_1' A_{n+1} &= \frac{1}{\lambda + \gamma - cU_n} \left(\lambda^+ + \lambda^- e_{1-}' A_n \right), \\ e_{1-}' A_{n+1} &= \frac{1}{\beta^- - U_n} \beta^- e_1' A_n. \end{aligned} \tag{22}$$

Here e_1' and e_{1-}' are the canonical row base vectors on the space $E_d = \{1, 1-\}$. Using $A(\gamma) = (a_1, a_2)'$ after trivial calculations we obtain

$$\begin{aligned} U &= -\beta^+ + \beta^+ a_1, \\ a_1 &= \frac{\lambda^+ + \lambda^- a_2}{\lambda + \gamma - cU}, \\ a_2 &= \frac{\beta^- a_1}{\beta^- - U}. \end{aligned} \tag{23}$$

In this case, the scalar Δ_ϕ given in (13) yields $\Delta_\phi = \beta^+$ according to (20). The transition matrix of phase changes is given by

$$P = \Delta_q^{-1} Q + I = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\lambda^+}{\lambda} & 0 & \frac{\lambda^-}{\lambda} \\ 0 & 1 & 0 \end{pmatrix}.$$

According to (19) we obtain the row vector $P^{(+,-)} = (1,0)$ and the column vector $P^{(\sim,+)} = (0, \frac{\lambda^+}{\lambda}, 0)'$. The matrices $\Pi_\sim$ and Π_a^* are determined by the stationary row vector π which satisfies $\pi Q = \mathbf{0}$ and $\pi \mathbf{1} = 1$. We thus obtain

$$\pi_{1+} = \left(1 + \frac{\beta^+}{\lambda^+} + \frac{\beta^+}{\lambda^+}\frac{\lambda^-}{\beta^-}\right)^{-1}, \qquad \pi_1 = \pi_{1+}\frac{\beta^+}{\lambda^+}, \qquad \pi_{1-} = \pi_{1+}\frac{\beta^+}{\lambda^+}\frac{\lambda^-}{\beta^-},$$

which determines $\Pi_\sim = diag(0, \pi_1, 0)$ and $\Pi_a^* = diag(1/\pi_1, 1/\pi_{1-})$ according to (21). In order to compute the matrices $A^*(\gamma^*)$ and $U^*(\gamma^*)$, we first need to determine the time-reversion $(\mathcal{X}^*, \mathcal{J}^*)$ of $(\mathcal{X}, \mathcal{J})$. This is described in section 3.3. The generator matrix Q^* of $\mathcal{J}^*$ is given by $Q^* = \Delta_\pi^{-1} Q' \Delta_\pi$. It turns out that $Q^* = Q$, i.e. $\mathcal{J}$ is reversible. The other parameters of $(\mathcal{X}^*, \mathcal{J}^*)$ are

$$(\mu_i^*, \sigma_i^*, \nu_i^*) = \begin{cases} (-1, 0, \mathbf{0}), & i = 1+ \\ (c, 0, \mathbf{0}), & i = 1 \\ (1, 0, \mathbf{0}), & i = 1- \end{cases}.$$

Thus $E_a^* = E_p \cup E_+^* = \{1, 1-\}$ and $E_d^* = E_-^* = 1+$ which means that $U^*(\gamma^*)$ is a matrix of dimensions 2×2 and $A^*(\gamma^*)$ is a row vector of dimension 1×2. According to (17), the pair $(U^*(\gamma^*), A^*(\gamma^*))$ is determined as the limit $\lim_{n\to\infty}(U_n^*, A_n^*)$ with initial values $A_0^* = \mathbf{0}$ and $U_0^* = diag(-(\lambda+\gamma)/c, -\beta^-)$ and iteration

$$\begin{aligned} e_{1-}' U_{n+1}^* &= -\beta^- e_{1-}' + \beta^- e_1' = (-\beta^-, \beta^-), \\ e_1' U_{n+1}^* &= -\frac{\lambda+\gamma}{c} e_1' + \frac{1}{c}\left(\lambda^- e_{1-}' + \lambda^+ A_n^*\right), \\ A_{n+1}^* &= \beta^+ e_1' \left(\beta^+ I - U_n^*\right)^{-1}, \end{aligned} \tag{24}$$

where e_i' denotes the ith canonical row vector on the space $E_a^* = \{1, 1-\}$. Writing $A^*(\gamma^*) = (a_1^*, a_2^*)$ and $U^* = U^*(\gamma^*)$, we obtain

$$U^* = \begin{pmatrix} -\frac{\lambda+\gamma}{c} + \frac{\lambda^+}{c} a_1^* & \frac{\lambda^+}{c} a_2^* + \frac{\lambda^-}{c} \\ \beta^- & -\beta^- \end{pmatrix},$$

and due to the third equation (24) we have to solve

$$(a_1^*, a_2^*)\begin{pmatrix} \beta^+ + \frac{\lambda+\gamma}{c} - \frac{\lambda^+}{c}a_1^* & -\frac{\lambda^+}{c}a_2^* - \frac{\lambda^-}{c} \\ -\beta^- & \beta^+ + \beta^- \end{pmatrix} = (\beta^+, 0).$$

This determines all ingredients we need from the time-reversed MAP $(\mathcal{X}^*, \mathcal{J}^*)$. According to (20), $\Delta_{\phi^*} = diag(0, \lambda/c, 0)$. The matrix $P^{(\sim,+)}$ is a column vector of dimension 3. We obtain $P^{(\sim,+)} = (0, \lambda^+/\lambda, 0)'$ according to (19). Regarding the positive jump part, we obtain finally the parameters $T = -\beta^+$ and $\eta = \beta^+$. This completes the derivations that we need for Theorem 1.

Example 5 Our numerical example that illustrates this case is taken from [26], Table 1. The authors consider the classical risk process with claims occurring at a rate $\lambda^+ = 1$ and claims sizes with mean $\beta^+ = 1$. The numerical ruin probabilities are calculated for several values of the initial surplus u and premium rates c. Note that for this we used Theorem 1 together with the Gaver-Stehfest Laplace transform numerical procedure (see [25]). Some values are illustrated in the following table.

TABLE 5.1 : Ruin probabilities for the classical case - "only claims"

t	u	$c = 0$	$c = 0.1$	$c = 0.9$	$c = 1$	$c = 1.1$
10	0	0.999999	0.999962	0.858597	0.822716	0.785428
	10	0.479780	0.401569	0.055613	0.042251	0.031984
100	0	0.999995	0.999989	0.979087	0.943611	0.889980
	10	0.999641	0.99991	0.671573	0.447682	0.260392
∞	0	1	1	1	1	0.909091
	10	1	1	1	1	0.366264

As one can see the values agree with those in [26]. As the loading condition is $c > 1$ the infinite time ruin probabilities are equal to 1 except for the case $c = 1.1$. We purposely chose values for $c \leq 1$ to analyze the ruin probabilities in the capital injection case, where the safety loading condition will be satisfied for a larger range of values for c.

We now compare these results to the more general case that includes capital injections. For this we assume the rate of capital injection occurrences to be $\lambda^- = 1$ and the mean injection amount $\beta^- = 1$. Under this scenario the positive safety loading condition (3) reduces to

$$c > \lambda^+ E[C] - \lambda^- E[K] = \lambda^+\beta^+ - \lambda^-\beta^- = 0.$$

The numerical values for the ruin probabilities are illustrated in the table below.

TABLE 5.2 : Ruin probabilities for the classical case - “claims and capital injections”

t	u	$c=0$	$c=0.1$	$c=0.9$	$c=1$	$c=1.1$
10	0	0.872239	0.846041	0.605712	0.578405	0.552538
	10	0.086230	0.067593	0.008527	0.006604	0.008142
100	0	0.960050	0.930334	0.616260	0.585779	0.557662
	10	0.581408	0.423523	0.013280	0.009308	0.006685
∞	0	1	0.950124	0.616264	0.587861	0.557663
	10	1	0.576998	0.013282	0.009307	0.006685

Comparing Tables 6.1 and 6.2., we remark that by adding capital injections the ruin probabilities will decrease. For the case $c = 0$, as the safety condition is not satisfied the infinite time ruin probabilities will be equal to 1. However, for values of $c > 0$, unlike the "only claim" case, the infinite time ruin probabilities will be strictly smaller than 1.

To illustrate some of the new quantities introduced in this paper, we first consider the density of the minimum surplus prior to ruin $\underline{R}_T$ for a choice of the premium rate $c = 0.1$. As before, we consider two cases the “only claims” case, as well as the both “claims and capital injections” case. The two densities are presented in the following graph with the dotted line representing the density in the “only claims” case, and the full line the density in the both “claims and capital injections” case. The initial surplus is setup at $u = 10$. For this choice of the premium rate we remind the reader that the infinite time ruin probability in the “only claims” case is 1 and this is in accordance with the fact that the total area under the dotted line is 1. However, when capital injections are considered, not all the sample paths will lead to ruin, the ruin probability calculated in the first part of the example being 0.576998. When integrating the area under the full line density we obtain 0.576954. Using Corollary 1 we obtain the value of the mass point at $u = 10$ being 0.000044. This very small value can be explained by the fact that in average a claim is of value 1, being much smaller when compared with the initial surplus $u = 10$. Furthermore, adding these two numbers will give us the corresponding value of the infinite time ruin probability. Another observation for this example is the difference in scale between the two densities. One can easily remark that the pdf of the minimum surplus prior to ruin is roughly 2 times less likely in the “claims and capital injections” case than in the “only claims” case.

A last numerical illustration related to this example is designated to the comparison between the time to ruin and the time until the minimum surplus prior to ruin. For that we consider $c = 0.1$ in the case when we deal with both claims and capital injections. When one considers the time of the minimum surplus before ruin, one has to take into consideration the mass point at 0 for this quantity denoted by $S(\underline{R}_T)$. For that purpose, in the table below we calculated the cdf of the time of ruin and the one for the time of the

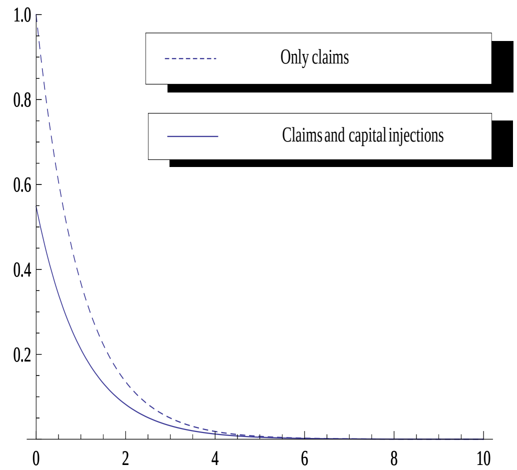

Figure 1: The density of the minimum surplus prior to ruin.

minimum surplus prior to ruin for various values of t and u.

TABLE 5.3 : Finite and infinite time ruin probabilities

	t	$P(T \leq t)$	$P(S(\underline{R}_T) \leq t)$	$P(S(\underline{R}_T) \leq t) + P(S(\underline{R}_T) = 0)$
$u = 1$	1	0.27410	0.21920	0.56873
	10	0.70601	0.45858	0.85388
	100	0.86583	0.53604	0.88557
	∞	0.90389	0.55436	0.90389
$u = 10$	1	0.00041	0.00139	0.00143
	10	0.06759	0.09256	0.09260
	100	0.42352	0.43970	0.43974
	∞	0.57699	0.57695	0.56799

Note that the mass points for the time of the minimum surplus prior to ruin at 0 are 0.34953, when $u = 1$, and 0.00004, when $u = 10$. As expected, one can easily check based on the values obtained in the table that the time of ruin random variable T is greater than the time of the minimum surplus prior to ruin $S(\underline{R}_T)$ under the usual stochastic order, $T \succeq S(\underline{R}_T)$. We also note the significant role played by the mass point at 0 of the time of the minimum prior to ruin for small values of the initial surplus. For large values of the initial surplus this mass point becomes negligible as the probability of dropping below the initial surplus without being ruined increases. As t approaches infinity, the area under the cdfs in both cases is equal, and this is in perfect accordance to the fact that $P(T=\infty)=P(S(\underline{R}_T)=\infty)$. In those sample paths where the surplus drifts to infinity without reaching level 0, the time of the minimum surplus prior to ruin does not appear in a finite horizon either.

5.2 A contagion model with negative jumps

In the second subsection we consider the contagion model from Example 2. We let the premiums to be collected at rates $c_A, c_B > 0$ which is superposed by a MPP for the claims with

$$D_0 = \begin{pmatrix} -\alpha_A-\delta_1 & \alpha_A \\ \alpha_B & -\alpha_B-\delta_1-\delta_2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_1+\delta_2 \end{pmatrix}.$$

Claims occur in two ways. There is at all times a Poisson process with parameter δ_1 of small exponential claims, their mean being $\frac{1}{\mu_1}$. While in environment B, there is in addition a second process, with parameter δ_2 of exponential claims; their expected value is $\frac{1}{\mu_2}$.

We further assume that the drift parameters are $\tilde{\mu}_A=-c_A$, $\tilde{\mu}_B=-c_B$. The process $\tilde{\mathcal{X}}$ can be represented as a MAP $(\tilde{\mathcal{X}},\tilde{\mathcal{J}})$ with phase space $\tilde{E}=E_n=\{1_A,1_B\}$ and the generator matrix $\tilde{Q}=\begin{pmatrix} -\alpha_A & \alpha_A \\ \alpha_B & -\alpha_B \end{pmatrix}$ for $\mathcal{J}$. According to section 3.1 we construct the MAP $(\mathcal{X},\mathcal{J})$ as follows. The enlarged phase space E consists of the subsets $E_n=\{1_A,1_B\}$, $E_+=\{(1_A,1_A,1,+),(1_B,1_B,1,+),(1_B,1_B,2,+)\}$, according to (7) and (8). For simplicity we let the subsets E_σ and E_p to be empty. Indeed one can consider a more general case (for e.g. assuming different volatilities in each environments A and B), but this implies tedious calculations that we prefer to omit here. For ease of notation, denote the jump phases by $1_A^+=(1_A,1_A,1,+)$, $1_B^+=(1_B,1_B,1,+)$ and $2_B^+=(1_B,1_B,2,+)$. The parameters for each phase are set according to (9) as

$$(\mu_i,\sigma_i,\nu_i)=\begin{cases} (1,0,\mathbf{0}), & i=1_A^+,1_B^+,2_B^+ \\ (-c_A,0,\mathbf{0}), & i=1_A \\ (-c_B,0,\mathbf{0}), & i=1_B \end{cases}.$$

The generator matrix Q can be easily obtain from equation (15) using the lexicographical order. This completely defines the MAP $(\mathcal{X},\mathcal{J})$. The matrices $A(\gamma)$ and $U(\gamma)$ have dimension 2×3 and 3×3, respectively. According to (17) (ignoring the details), they are determined as the limit $(U(\gamma),A(\gamma))=\lim_{n\to\infty}(U_n,A_n)$, with initial values $A_0=\mathbf{0}$

and $U_0 = \begin{pmatrix} -\mu_1 & 0 & 0 \\ 0 & -\mu_1 & 0 \\ 0 & 0 & -\mu_2 \end{pmatrix}$ and iteration

$$U_{n+1} = U_0 + \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1 \\ 0 & \mu_2 \end{pmatrix} A_n,$$

$$\begin{pmatrix} \frac{\alpha_A+\delta_1+\gamma}{c_A} & -\frac{\alpha_A}{c_A} \\ -\frac{\alpha_B}{c_B} & \frac{\alpha_B+\delta_1+\delta_2+\gamma}{c_B} \end{pmatrix} A_{n+1} - A_{n+1}U_n = \begin{pmatrix} \frac{\delta_1}{c_A} & 0 & 0 \\ 0 & \frac{\delta_1}{c_B} & \frac{\delta_2}{c_B} \end{pmatrix}. \tag{25}$$

Furthermore the system in (25) can be reduced to a matrix Riccati equation in terms of A, of the form

$$\begin{pmatrix} \frac{\delta_1}{c_A} & 0 & 0 \\ 0 & \frac{\delta_1}{c_B} & \frac{\delta_2}{c_B} \end{pmatrix} + [\begin{pmatrix} -\frac{\alpha_A+\delta_1}{c_A} & \frac{\alpha_A}{c_A} \\ \frac{\alpha_B}{c_B} & -\frac{\alpha_B+\delta_1+\delta_2}{c_B} \end{pmatrix} - \gamma I]A + AU_0 + A\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1 \\ 0 & \mu_2 \end{pmatrix} A = 0. \tag{26}$$

Note that equation (26) generalizes equation (12) from [6] for the Laplace transform of a busy period in the sense that the premium rates can vary among the various states of the economy.

In this case, Δ_ϕ given in (13) yields $\begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix}$ according to (20). The transition matrix of phase changes is given by

$$P = \Delta_q^{-1}Q + I = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{\delta_1}{\alpha_1+\delta_1} & 0 & 0 & 0 & \frac{\alpha_1}{\alpha_1+\delta_1} \\ 0 & \frac{\delta_1}{\alpha_2+\delta_1+\delta_2} & \frac{\delta_2}{\alpha_2+\delta_1+\delta_2} & \frac{\alpha_2}{\alpha_2+\delta_1+\delta_2} & 0 \end{pmatrix}.$$

According to (19) we obtain the matrix $P^{(+,-)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$ and the matrix $P^{(\sim,+)} = \begin{pmatrix} \frac{\delta_1}{\alpha_1+\delta_1} & 0 & 0 \\ 0 & \frac{\delta_1}{\alpha_2+\delta_1+\delta_2} & \frac{\delta_2}{\alpha_2+\delta_1+\delta_2} \end{pmatrix}$. The matrices $\Pi_\sim$ and Π_a^* are determined by the stationary row vector π which satisfies $\pi Q = \mathbf{0}$ and $\pi\mathbf{1} = 1$. We thus obtain

$$\pi_{1_A^+} = \frac{\alpha_B\delta_1\mu_2}{\alpha_A\delta_2\mu_1 + (\alpha_a+\alpha_b)(\delta_1+\mu_1)\mu_2}, \qquad \pi_{1_B^+} = \pi_{1_A^+}\frac{\alpha_A}{\alpha_B}, \qquad \pi_{2_B^+} = \pi_{1_B^+}\frac{\mu_1\delta_2}{\mu_2\delta_1},$$

$$\pi_{1_A} = \pi_{1_A^+} \frac{\mu_1}{\delta_1}, \qquad \pi_{1_B} = \pi_{1_A} \frac{\alpha_A}{\alpha_B},$$

which determines $\Pi_\sim = diag(0, 0, 0, \pi_{1_A}, \pi_{1_B})$ and $\Pi_a^* = diag(1/\pi_{1_A}, 1/\pi_{2_A})$ according to (21). Similar steps will lead to the calculation of the matrices $A^*(\gamma^*)$ and $U^*(\gamma^*)$, but to avoid repetition we omit them here. We further pursue to the next numerical example.

Example 6 We assume that the standard claims will occur according to a Poisson process at rate $\delta_1 = 1$ and infectious claims will occur at rate $\delta_2 = 10$ during the contagion periods. Standard claim amounts have mean $1/\mu_1 = 1/5$, while the mean size of an infectious claim is $1/\mu_2 = 15/\mu_1 = 3$. It is assumed that the rate at which the system enters the infectious environment is $\alpha_A = 0.02$ and the return rate to the standard environment is $\alpha_B = 1$, so that in 98% of the cases the system will be in the standard environment. We consider nine different scenarios for the finite and infinite time ruin probabilities in order to illustrate the effect of changing the premium rates in each environment, as well as changing the initial phase distribution of the environment denoted here by $\beta = (\beta_A, \beta_B)$. The numerical results are presented in the following table.

TABLE 5.4 : Finite and infinite time ruin probabilities for $u = 1$

$\{\beta_A,\ \beta_B\}$	t	$c_A = 1,\ c_B = 1$	$c_A = 1, c_B = 10$	$c_A = 10, c_B = 1$
$\{0.5, 0.5\}$	1	0.45065	0.22929	0.44124
	10	0.52062	0.47475	0.44814
	100	0.72879	0.65028	0.47204
	1000	0.83439	0.71527	0.47251
	∞	0.84665	0.71643	0.47251
$\{0.9, 0.1\}$	1	0.10454	0.05222	0.08955
	10	0.21905	0.17552	0.10140
	100	0.55820	0.44851	0.13744
	1000	0.73016	0.55092	0.14105
	∞	0.75013	0.55274	0.14105
$\{0.1, 0.9\}$	1	0.79675	0.40636	0.79293
	10	0.82219	0.77403	0.79408
	100	0.89947	0.85206	0.80313
	1000	0.93863	0.87962	0.80396
	∞	0.94317	0.88011	0.80396

We first remark that for $c_A = 1$ and $c_B = 1$ and $\beta = \{0.5, 0.5\}$ we recover the results obtained in [6]. Furthermore, if we keep the initial probability vector the same, but we increase the rate at which premiums will be collected in the infectious environment by a factor of 10 the ruin probabilities will decrease as expected. A similar effect can be observed when we choose $c_A = 10$ and $c_B = 1$. An interesting thing is the fact that the increase in premium rate will produce a wider range of values for the ruin probabilities when the increase happens in the infectious phase rather than the standard phase. This shows once again the importance of the contagion environment despite the rare occurrences of this environment (only 2% of the time). Finally, a change in the initial phase distribution plays an important role in the value of the ruin probabilities. As it can be seen from the values illustrated in Table 5.4, a start of the system in the infectious environment with a higher probability will produce a higher finite time ruin probability.

In the end we also consider the density of the minimum of the surplus prior to ruin, under this MPP contagion model. To better asses the effect of change in the premium rates between the environmental phases we fixed the initial probability vector to be $\beta = \{0.5, 0.5\}$. The following graphs present density of the minimum surplus prior to ruin for $u = 1$ and $u = 10$.

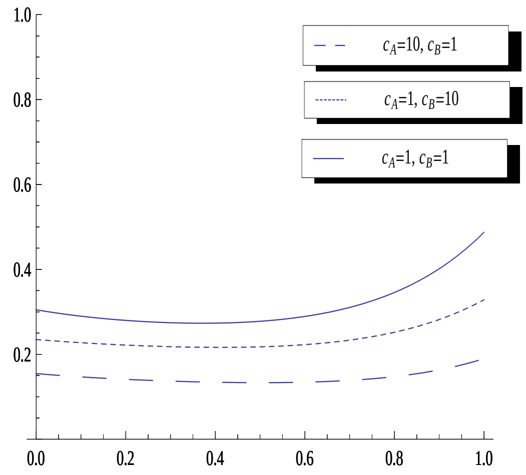

Figure 2: The density of the minimum surplus prior to ruin, $u = 1$.

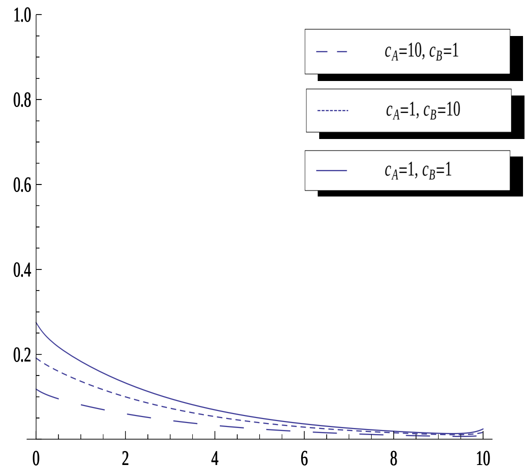

Figure 3: The density of the minimum surplus prior to ruin, $u = 10$.

In both graphs the upper curve corresponds to the case $c_A = 1$, $c_B = 1$, the middle one to $c_A = 1$, $c_B = 10$ and the one below to the case $c_A = 10$, $c_B = 1$. As expected, when the premium rates are at the smallest level, the probability associated to the minimum surplus prior to ruin has the higher scale. Increasing the premium rates in each environment will decrease the associated infinite time ruin probabilities and implicitly the probability of obtaining the same level of the minimum surplus prior to ruin.

When $u = 1$ we observe an increasing behavior in the density of the minimum surplus prior to ruin in contrast to the decreasing behavior obtained in the case $u = 10$. This is somehow expected and can be explained due to the change in the ratio between the area under the continuous part density and the mass point part, as one moves from $u = 1$ to $u = 10$. The last table illustrates how the cdf of the minimum surplus prior to ruin can be decomposed in two parts: the area under the continuous density and the mass point at u, for both values $u = 1$ and $u = 10$.

TABLE 5.4 : Decomposition of the cdf of the minimum surplus prior to ruin

		$c_A = 1,\ c_B = 1$	$c_A = 1, c_B = 10$	$c_A = 10, c_B = 1$
$u = 1$	continuous part	0.31277	0.23672	0.14447
	mass point	0.53387	0.47970	0.32803
$u = 10$	continuous part	0.75830	0.57297	0.34712
	mass point	0.02653	0.02384	0.01631

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