Autumn Term 2014 | University of Kent
Exercises on Stochastic Processes | L. Breuer / F. Leisen

Assessment 1

NOTE: Please return your solutions to the SMSAS general office before noon on Tuesday, the 18th of November.

Exercise 1: 8 Marks

Let $\mathcal{X} = (X_n : n \in \mathbb{N}_0)$ denote a Markov chain with state space $E = \{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 1/4 & 1/2 & 0 & 1/4 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/5 & 4/5 \end{pmatrix}$$

(a) Draw a transition graph for $\mathcal{X}$.
(b) Determine the conditional probability $\mathbb{P}(X_2 = 4 | X_0 = 1)$.

Exercise 2: 8 Marks

Let $\mathcal{X} = (X_n : n \in \mathbb{N}_0)$ denote a Markov chain with state space $E = \{1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

Determine the stationary distribution for $\mathcal{X}$. Why is this unique?

Exercise 3: 16 Marks

(a) Let $\mathcal{X} = (X_n : n \in \mathbb{N}_0)$ be a Markov chain with transition matrix

$$P = \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 2/3 \\ 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

(i) Determine the communication classes for $\mathcal{X}$ and state which of them are recurrent and which are transient.
(ii) Determine the stationary distribution of $\mathcal{X}$.
(b) A Markov chain $\mathcal{X}$ on a finite state space E is called doubly stochastic if its transition matrix $P = (p_{ij})_{i,j\in E}$ satisfies the condition

$$\sum_{i\in E} p_{ij} = 1$$

for all $j \in E$, i.e. if all its columns sum up to one.
(i) Give an example of a doubly stochastic Markov chain with state space $E = \{1, 2, 3, 4\}$.
(ii) For $E = \{1, \ldots, n\}$ with $n \in \mathbb{N}$, show that $\pi = (1/n, \ldots, 1/n)$ is a stationary distribution of $\mathcal{X}$.
(c) For the Markov chain from question 2, determine the mean recurrence time $\mathbb{E}(\tau_1|X_0 = 1)$, where $\tau_1 := \min\{n \geq 1 : X_n = 1\}$.

Exercise 4: 8 Marks

Let $\mathcal{Y} = (Y_t : t \geq 0)$ denote a Markov process with state space $E = \{1, 2, 3\}$ and generator matrix

$$G = \begin{pmatrix} -2 & 2/3 & 4/3 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

Determine the holding time parameters and the transition matrix $P = (p_{ij})_{i,j\in E}$ of the embedded Markov chain. You may assume that $p_{ii} = 0$ for all $i \in E$.