Approximating diffusions by piecewise constant parameters

Lothar Breuer

Institute of Mathematics and Statistics, University of Kent, Canterbury CT2 7NF, UK

Abstract

We approximate the resolvent of a one-dimensional diffusion process by the resolvent of a level dependent Brownian motion. This is a diffusion process with piecewise constant parameters. The approximating resolvent is determined for the case of a diffusion on an interval with absorbing barriers. A recursive scheme to solve the two-sided exit problem for the approximating process is provided.

Key words: diffusion process, resolvent, approximation
2000 MSC: 60J60, 60J65

1. Introduction

Diffusion processes serve as models for many phenomena in physics, biology and economics, to name but a few application fields (see e.g. section 15.2 in [4] for some examples). They further arise as approximations for many Markov processes after rescaling and normalising (see [2]). Although many functionals, like first passage probabilities or mean exit times, can be determined explicitly for one-dimensional diffusions, many others still elude an exact solution.

There is of course a good reason for this. Diffusion processes arise as solutions to some stochastic differential equations (SDEs, see section II.4.6-7 in [5]). Even for one-dimensional and time-homogeneous SDEs, their deterministic counterpart, which are ordinary differential equations (ODEs), have explicit solutions only in special cases. Thus a general approach to solve SDEs can only be expected in terms of numerical approximation schemes.

The vast majority of existing numerical schemes mimic the methods applied to deterministic differential equations, resulting in the Euler scheme and refinements thereof. The basic idea is a discretisation of the time axis and the determination of a suitable system of difference equations. Many examples of such schemes can be found in [5].

While time discretisation of a one-dimensional ODE yields a finite sequence of numbers as a result, the same method applied to a SDE results in a random variable for each step in time and each fixed initial value. Hence, functionals like transition probabilities can only be obtained by simulating many path realisations under any fixed initial value (cf. the MCMC approach). This of course may quickly amount to high computational costs.

The present paper aims to explore possibilities of a numerical approximation that is based on a discretisation of the state space rather than the time axis. We shall consider only one-dimensional diffusion processes, which correspond to solutions of one-dimensional and time-homogeneous SDEs. We intend to derive an approximation of the resolvent of a given diffusion process. This preserves to a large degree the distribution of the process and allows to approximate functionals like the stationary distribution without the need to recur to costly MCMC methods.

The approximation of the resolvent is introduced in the following section. In section 3, the approximating process is introduced in more detail. Section 4 contains the determination of the approximating resolvent. The last section provides a recursive scheme to solve the two-sided exit problem.

2. The approximation

Let $\mathcal{X} = (X_t : t \geq 0)$ denote a one-dimensional diffusion process with a state space $E = [l, u]$ where $l < u \in \mathbb{R}$. Denote the infinitesimal drift and variance functions by $\mu : E \to \mathbb{R}$ and $\sigma : E \to \mathbb{R}^+$, respectively. Assume that μ and σ are Lipschitz continuous. Let $T(t)$, $t \geq 0$, denote the transition semi-group of $\mathcal{X}$, acting on a suitable domain $\mathcal{D}$ as

$$T(t)f(x) := \mathbb{E}\left(f(X_t)|X_0 = x\right)$$

for all $f \in \mathcal{D}$ and $x \in E$. The generator of $\mathcal{X}$ is defined as

$$\mathcal{A}f := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left(T(\varepsilon)f - f\right)$$

for all $f \in \mathcal{D}(\mathcal{A})$, the domain of $\mathcal{A}$, and given by

$$\mathcal{A}f(x) = \frac{\sigma^2(x)}{2} f''(x) + \mu(x)f'(x)$$

for all $x \in E^o =]l, u[$, the interior of E. We assume that the boundaries are absorbing.

Let $\mathcal{Y} = (Y_t : t \geq 0)$ denote a diffusion process with the same state space E, but piecewise constant infinitesimal drift and variance functions $\tilde{\mu} : E \to \mathbb{R}$ and $\tilde{\sigma} : E \to \mathbb{R}^+$. Assume that they are given as $\tilde{\mu}(x) := \mu_i$ and $\tilde{\sigma}(x) := \sigma_i$ for $b_{i-1} < x \leq b_i$ and $i = 1, \ldots, N$, where $b_0 := l$ and $b_N := u$. Assume further that

$$\sup_{x \in E^o} \left(|\mu(x) - \tilde{\mu}(x)| + \frac{1}{2} |\sigma^2(x) - \tilde{\sigma}^2(x)| \right) < \varepsilon$$

The boundaries are assumed to be absorbing. Let $\tilde{T}(t)$, $t \geq 0$, denote the transition semi-group of $\mathcal{Y}$ and $\tilde{\mathcal{A}}$ its generator.

The fact that

$$\frac{d}{ds} T(t-s)\tilde{T}(s)f = -T(t-s)\mathcal{A}\tilde{T}(s)f + T(t-s)\tilde{\mathcal{A}}\tilde{T}(s)f$$

for $t > s$ yields

$$\tilde{T}(t)f - T(t)f = \int_0^t T(t-s)(\tilde{\mathcal{A}} - \mathcal{A})\tilde{T}(s)f \; ds \tag{1}$$

cf. [2], lemma 6.2 of chapter 1. We are interested in the λ-resolvents

$$R_\lambda := \int_0^\infty e^{-\lambda s} T(s) \; ds \qquad \text{and} \qquad \tilde{R}_\lambda := \int_0^\infty e^{-\lambda s} \tilde{T}(s) \; ds$$

see proposition 2.1 of chapter 1 in [2]. Equation (1) yields

$$\begin{aligned}
\tilde{R}_\lambda f - R_\lambda f &= \int_0^\infty e^{-\lambda t} \int_0^t T(t-s)(\tilde{\mathcal{A}} - \mathcal{A})\tilde{T}(s)f \; ds \; dt \\
&= \int_0^\infty e^{-\lambda t} \int_0^t T(s)(\tilde{\mathcal{A}} - \mathcal{A})\tilde{T}(t-s)f \; ds \; dt \\
&= \int_{s=0}^\infty e^{-\lambda s} T(s)(\tilde{\mathcal{A}} - \mathcal{A}) \int_{t=s}^\infty e^{-\lambda(t-s)} \tilde{T}(t-s)f \; dt \; ds \\
&= R_\lambda(\tilde{\mathcal{A}} - \mathcal{A})\tilde{R}_\lambda f
\end{aligned}$$

Since

$$\|R_\lambda g\| \leq \int_0^\infty e^{-\lambda s} \|T(s)g\| ds \leq \frac{1}{\lambda} \|g\|$$

for any $g \in \mathcal{D}$, we obtain

$$\|\tilde{R}_\lambda f - R_\lambda f\| \le \frac{1}{\lambda}\|(\tilde{\mathcal{A}} - \mathcal{A})\tilde{R}_\lambda f\|$$

and in particular for $g := \tilde{R}_\lambda f$

$$\begin{aligned}\|\tilde{R}_\lambda f(x) - R_\lambda f(x)\| &\le \frac{1}{\lambda}\left\|\frac{1}{2}(\tilde{\sigma}^2(x) - \sigma^2(x))g''(x) + (\tilde{\mu}(x) - \mu(x))g'(x)\right\| \\ &\le \frac{|g'(x) + g''(x)|}{\lambda}\varepsilon\end{aligned}$$

for all $x \in E^o$. This implies weak convergence of the resolvents. The next two sections are dedicated to determining $\tilde{R}_\lambda$.

3. The approximating process

The approximating process $\mathcal{Y} = (Y_t : t \ge 0)$ may be called a level dependent Brownian motion with a finite number of thresholds $b_0, \dots, b_N$. It is defined by the stochastic differential equation

$$dY_t = \begin{cases} \mu_k\, dt + \sigma_k\, dW_t, & b_{k-1} < Y_t \le b_k, 1 \le k \le N-1 \\ \mu_N\, dt + \sigma_N\, dW_t, & b_{N-1} < Y_t < b_N \end{cases}$$

where $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$ for $i \le N$, and $\mathcal{W} = (W_t : t \ge 0)$ denotes the standard Wiener process. Define the intervals $I_k :=]b_{k-1}, b_k]$ for $k \in \{1, \dots, N-1\}$ and $I_N :=]b_{N-1}, b_N[$. We call I_k together with the parameters (μ_k, σ_k) the kth regime of $\mathcal{Y}$.

If $N = 1$, i.e. if there is only one regime, we call the process $\mathcal{Y}$ homogeneous (in space). It is then of course a Brownian motion on an interval with absorbing boundaries. Define

$$u_k^\pm = u_k^\pm(\lambda) := \frac{\pm\mu_k - \sqrt{\mu_k^2 + 2\lambda\sigma_k^2}}{\sigma_k^2}$$

for $k \le N$. Within each regime I_k the behaviour of $\mathcal{Y}$ equals that of a classical Brownian motion with λ-scale function $W_k^{(\lambda)}(x) := e^{-u_k^+ x} - e^{u_k^- x}$. For instance, the exit times

$$\tau(l, u) := \inf\{t \ge 0 : Y_t \notin [l, u]\}$$

from an interval $[l,u] \subset I_k$ are given by their Laplace transforms

$$\begin{aligned}\Psi_k^+(u-l|x-l) &:= \mathbb{E}\left(e^{-\lambda\tau(l,u)}; Y_{\tau(l,u)} = u | Y_0 = x\right)\\ &= \frac{e^{-u_k^+\cdot(x-l)} - e^{u_k^-\cdot(x-l)}}{e^{-u_k^+\cdot(u-l)} - e^{u_k^-\cdot(u-l)}}\end{aligned} \tag{2}$$

for all $x \in [l,u]$. By reflection around the initial value x, we obtain further

$$\begin{aligned}\Psi_k^-(u-l|x-l) &:= \mathbb{E}\left(e^{-\lambda\tau(l,u)}; Y_{\tau(l,u)} = l | Y_0 = x\right)\\ &= \frac{e^{-u_k^-\cdot(u-x)} - e^{u_k^+\cdot(u-x)}}{e^{-u_k^-\cdot(u-l)} - e^{u_k^+\cdot(u-l)}}\end{aligned} \tag{3}$$

using the fact that the roles of u_k^+ and u_k^- are simply reversed for the Brownian motion with parameters $(-\mu_k, \sigma_k^2)$.

We now wish to determine the exit times of $\mathcal{Y}$ from an interval $[b_{k-1}, b_{k+1}]$, given that $Y_0 = b_k$. Abbreviate those by

$$\tau(k) := \inf\{t \geq 0 : Y_t \notin [b_{k-1}, b_{k+1}]\} \tag{4}$$

for $k = 1, \ldots, N-1$ and write $\Delta b_k := b_k - b_{k-1}$ for $k = 1, \ldots, N$.

Theorem 1 *For $k = 1, \ldots, N-1$, the distribution of the exit time $\tau(k)$, given that $Y_0 = b_k$, is given by the Laplace transforms*

$$\begin{aligned}E^+(k) &:= \mathbb{E}\left(e^{-\lambda\tau(k)}; Y_{\tau(k)} = b_{k+1} | Y_0 = b_k\right)\\ &= \frac{u_{k+1}^+ + u_{k+1}^-}{e^{-u_{k+1}^+\cdot\Delta b_{k+1}} - e^{u_{k+1}^-\cdot\Delta b_{k+1}}}\\ &\quad\times\left(\frac{u_k^+ e^{-u_k^+\cdot\Delta b_k} + u_k^- e^{u_k^-\cdot\Delta b_k}}{e^{-u_k^+\cdot\Delta b_k} - e^{u_k^-\cdot\Delta b_k}} + \frac{u_{k+1}^- e^{-u_{k+1}^+\cdot\Delta b_{k+1}} + u_{k+1}^+ e^{u_{k+1}^-\cdot\Delta b_{k+1}}}{e^{-u_{k+1}^+\cdot\Delta b_{k+1}} - e^{u_{k+1}^-\cdot\Delta b_{k+1}}}\right)^{-1}\end{aligned}$$

and

$$\begin{aligned}E^-(k) &:= \mathbb{E}\left(e^{-\lambda\tau(k)}; Y_{\tau(k)} = b_{k-1} | Y_0 = b_k\right)\\ &= \frac{u_k^+ + u_k^-}{e^{-u_k^-\cdot\Delta b_k} - e^{u_k^+\cdot\Delta b_k}}\\ &\quad\times\left(\frac{u_k^+ e^{-u_k^+\cdot\Delta b_k} + u_k^- e^{u_k^-\cdot\Delta b_k}}{e^{-u_k^+\cdot\Delta b_k} - e^{u_k^-\cdot\Delta b_k}} + \frac{u_{k+1}^- e^{-u_{k+1}^+\cdot\Delta b_{k+1}} + u_{k+1}^+ e^{u_{k+1}^-\cdot\Delta b_{k+1}}}{e^{-u_{k+1}^+\cdot\Delta b_{k+1}} - e^{u_{k+1}^-\cdot\Delta b_{k+1}}}\right)^{-1}\end{aligned}$$

for $\lambda \geq 0$.

Proof: We begin with the first statement and consider

$$E(\varepsilon) := \mathbb{E}\left(e^{-\lambda\tau(k)}; Y_{\tau(k)} = b_{k+1} | Y_0 = b_k + \varepsilon\right)$$

First we assume that the regime changes at $b_k - \varepsilon$ for downward crossings of b_k and at $b_k + \varepsilon$ for upward crossings. Then we let $\varepsilon \downarrow 0$. Summing up over the number of down and up crossings of the interval $[b_k - \varepsilon, b_k + \varepsilon]$ before leaving the interval $[b_{k-1}, b_{k+1}]$ at b_{k+1}, we obtain

$$\begin{aligned} E(\varepsilon) &= \sum_{n=0}^{\infty} \left(\Psi_{k+1}^{-}(\Delta b_{k+1} + \varepsilon | 2\varepsilon)\Psi_k^{+}(\Delta b_k + \varepsilon | \Delta b_k - \varepsilon)\right)^n \Psi_{k+1}^{+}(\Delta b_{k+1} + \varepsilon | 2\varepsilon) \\ &= \frac{\Psi_{k+1}^{+}(\Delta b_{k+1} + \varepsilon | 2\varepsilon)}{1 - \Psi_{k+1}^{-}(\Delta b_{k+1} + \varepsilon | 2\varepsilon)\Psi_k^{+}(\Delta b_k + \varepsilon | \Delta b_k - \varepsilon)} \end{aligned}$$

Equations (2) and (3) along with L'Hospital's rule now yield the first statement. The proof of the second statement is completely analogous.
□

4. The resolvent

We consider the level dependent Brownian motion as in section 3 and seek to find an expression for its resolvent density function $\tilde{r}_\lambda(x, y)$, defined by

$$\tilde{r}_\lambda(x, y)dy := \tilde{R}_\lambda(x, dy) = \mathbb{E}\left(\int_0^\infty e^{-\lambda t} 1_{\{Y_t \in dy\}}\, dt \,| Y_0 = x\right)$$

for $\lambda > 0$. Let $\mathcal{E}(\lambda)$ denote an independent exponential time of parameter λ. Then $\tilde{R}_\lambda(x, dy)$ is simply the probability that $\mathcal{Y}$ is located in dy before the exponential time $\mathcal{E}(\lambda)$ has expired. We separate any path of $\mathcal{Y}$ from x to y within the time $\mathcal{E}(\lambda)$ into three parts.

First, if $x \notin \{b_1, \ldots, b_N\}$, denote the probabilities of hitting the next upper/lower grid point before $\mathcal{E}(\lambda)$ by $p_1^{\pm}(x)$. Let $r(x)$ denote the regime of x, i.e. $r(x) = k$ if $x \in I_k$, $k \in \{1, \ldots, N\}$. With $k := r(x)$ and $\Delta b_k := b_k - b_{k-1}$ the values $p_1^{\pm}(x)$ are given by

$$p_1^{\pm}(x) = \Psi_k^{\pm}(\Delta b_k | x - b_{k-1}) \tag{5}$$

cf. (2) and (3).

The second part of the path consists of movements among the grid points $b_0, \ldots, b_N$ before $\mathcal{E}(\gamma)$. To this aim, define the square matrix $P = (p_{ij})_{0 \le i,j \le N}$ of dimension $N + 1$ by

$$p_{ij} := \begin{cases} 1, & j = i \in \{0, N\} \\ E^+(i), & j = i+1, i \in \{1, \ldots, N-1\} \\ E^-(i), & j = i-1, i \in \{1, \ldots, N-1\} \\ 0 & \text{else} \end{cases}$$

Theorem 2 *Define the hitting times $T(x) := \min\{t \ge 0 : Y_t = x\}$ for all $x \in E$. Further define the matrix $M = (m_{ij})_{0 \le i,j \le N}$ by $M := (I - P)^{-1}$. Then*

$$m_{ij} = \mathbb{E}\left(e^{-\lambda T(b_j)} | Y_0 = b_i\right) = \mathbb{P}(T(b_j) < \mathcal{E}(\lambda) | Y_0 = b_i)$$

for all $i, j \le N$.

Proof: We first observe that

$$E^-(k) = \mathbb{P}\left(\tau(k) < \mathcal{E}(\lambda), Y_{\tau(k)} = b_{k-1} | Y_0 = b_k\right)$$

and

$$E^+(k) = \mathbb{P}\left(\tau(k) < \mathcal{E}(\lambda), Y_{\tau(k)} = b_{k+1} | Y_0 = b_k\right)$$

Thus P is sub-stochastic with the kth row sum

$$e_k' P \mathbf{1} = 1 - \mathbb{P}(\tau(k) > \mathcal{E}(\lambda) | Y_0 = b_k) < 1$$

for $k = 2, \ldots, N-1$, where e_k' denotes the kth canonical row base vector and $\mathbf{1}$ the column vector with all entries being 1. Due to the memoryless property of the exponential distribution, the matrix $M = (I - P)^{-1} = \sum_{n=0}^{\infty} P^n$ contains the probabilities m_{ij} of hitting state b_j before $\mathcal{E}(\lambda)$, given that $Y_0 = b_i$.
□

The last part is a movement from a grid point to y before $\mathcal{E}(\lambda)$ without hitting another grid point in between. Thus we are interested in the occupation measures

$$l_k(y)dy := \mathbb{E}\left(\int_0^{\tau(k)} e^{-\lambda t} \mathbb{I}_{\{Y_t \in dy\}} \, dt | Y_0 = b_k\right)$$

where $b_{k-1} < y < b_{k+1}$.

Theorem 3 *Define*

$$m_k(y) = \frac{e^{-u_{k+1}^+\cdot(b_{k+1}-y)} - e^{u_{k+1}^-\cdot(b_{k+1}-y)}}{e^{-u_{k+1}^+\cdot\Delta b_{k+1}} - e^{u_{k+1}^-\cdot\Delta b_{k+1}}}\,\mathbb{I}_{\{y>b_k\}} + \frac{e^{-u_k^-\cdot(y-b_{k-1})} - e^{u_k^+\cdot(y-b_{k-1})}}{e^{-u_k^-\cdot\Delta b_k} - e^{u_k^+\cdot\Delta b_k}}\,\mathbb{I}_{\{y<b_k\}}$$

and further

$$h_k^+ = \frac{u_k^+ e^{-u_k^+\cdot\Delta b_k} + u_k^- e^{u_k^-\cdot\Delta b_k}}{e^{-u_k^+\cdot\Delta b_k} - e^{u_k^-\cdot\Delta b_k}}, \quad h_{k+1}^- = \frac{u_{k+1}^- e^{-u_{k+1}^-\cdot\Delta b_{k+1}} + u_{k+1}^+ e^{u_{k+1}^+\cdot\Delta b_{k+1}}}{e^{-u_{k+1}^-\cdot\Delta b_{k+1}} - e^{u_{k+1}^+\cdot\Delta b_{k+1}}}$$

Then

$$l_k(y) = -\frac{m_k(y)}{h_k^+ + h_{k+1}^-}$$

for $b_{k-1} < y < b_{k+1}$.

Proof: The density function $l_k(y)$ is the product of the occupation density at b_k before $\min\{\tau(k), \mathcal{E}(\lambda)\}$ and the occupation density at y before returning to b_k and before $\min\{\tau(k), \mathcal{E}(\lambda)\}$. Thus we write $l_k(y) = d_k \cdot m_k(y)$.

The first factor, the occupation density at b_k, is obtained as

$$d_k := \lim_{\varepsilon\downarrow 0} 2\varepsilon(1 - h_{k+1}^-(\varepsilon)h_k^+(\varepsilon))^{-1}$$

cf. [3], section 2.4, where

$$h_{k+1}^-(\varepsilon) = \Psi_{k+1}^-(\Delta b_{k+1} + \varepsilon|2\varepsilon) \quad \text{and} \quad h_k^+(\varepsilon) = \Psi_k^+(\Delta b_k + \varepsilon|\Delta b_k - \varepsilon)$$

Since $\lim_{\varepsilon\downarrow 0} h_{k+1}^-(\varepsilon) = \lim_{\varepsilon\downarrow 0} h_k^+(\varepsilon) = 1$, we can write

$$\lim_{\varepsilon\downarrow 0} 2\varepsilon(1 - h_{k+1}^-(\varepsilon)h_k^+(\varepsilon))^{-1} = -2\left(\frac{d}{d\varepsilon}h_{k+1}^-(\varepsilon)h_k^+(\varepsilon)\bigg|_{\varepsilon=0}\right)^{-1}$$

where

$$\frac{d}{d\varepsilon}h_{k+1}^-(\varepsilon)\bigg|_{\varepsilon=0} = 2\frac{u_{k+1}^- e^{-u_{k+1}^-\cdot\Delta b_{k+1}} + u_{k+1}^+ e^{u_{k+1}^+\cdot\Delta b_{k+1}}}{e^{-u_{k+1}^-\cdot\Delta b_{k+1}} - e^{u_{k+1}^+\cdot\Delta b_{k+1}}} \tag{6}$$

and

$$\frac{d}{d\varepsilon}h_k^+(\varepsilon)\bigg|_{\varepsilon=0} = 2\frac{u_k^+ e^{-u_k^+\cdot\Delta b_k} + u_k^- e^{u_k^-\cdot\Delta b_k}}{e^{-u_k^+\cdot\Delta b_k} - e^{u_k^-\cdot\Delta b_k}} \tag{7}$$

Thus $d_k = -\left(h_k^+ + h_{k+1}^-\right)^{-1}$. Note that $d_k > 0$ for $\lambda > 0$, since $u_k^{\pm} < 0$ for all $k \leq N$.

The second factor in $l_k(y)$ is the occupation density at y before returning to b_k and before $\min\{\tau(k), \mathcal{E}(\lambda)\}$. Denote this by $m_k(y)$. Exploiting the Siegmund duality (see [1]), we consider the time-reversed process $\mathcal{Y}^r$. This is defined as a Brownian motion with parameters $(-\mu_k, \sigma_k)$ in the kth regime. Define

$$u_k^{\pm,r} := \frac{\mp\mu_k - \sqrt{\mu_k^2 + 2\lambda\sigma_k^2}}{\sigma_k^2} = u_k^{\mp} \tag{8}$$

for $k = 1, \ldots, N$. Then

$$m_k(y) = \begin{cases} \Psi_{k+1}^{-,r}(\Delta b_{k+1} | y - b_k), & y > b_k \\ \Psi_k^{+,r}(\Delta b_k | y - b_{k-1}), & y < b_k \end{cases} \tag{9}$$

where

$$\Psi_{k+1}^{-,r}(\Delta b_{k+1} | y - b_k) = \frac{e^{-u_{k+1}^{-,r}\cdot(b_{k+1}-y)} - e^{u_{k+1}^{+,r}\cdot(b_{k+1}-y)}}{e^{-u_{k+1}^{-,r}\cdot\Delta b_{k+1}} - e^{u_{k+1}^{+,r}\cdot\Delta b_{k+1}}}$$

and

$$\Psi_k^{+,r}(\Delta b_k | y - b_{k-1}) = \frac{e^{-u_k^{+,r}\cdot(y-b_{k-1})} - e^{u_k^{-,r}\cdot(y-b_{k-1})}}{e^{-u_k^{+,r}\cdot\Delta b_k} - e^{u_k^{-,r}\cdot\Delta b_k}}$$

Now the second equality in (8) yields the statement.
□

If $r(x) = r(y)$, then it is possible to reach y from x before $\mathcal{E}(\lambda)$ without hitting any threshold b_k, $k \leq N$.

Theorem 4 *Define*

$$m(x,y) = \frac{e^{-u_k^+\cdot(b_k-y)} - e^{u_k^-\cdot(b_k-y)}}{e^{-u_k^+\cdot(b_k-x)} - e^{u_k^-\cdot(b_k-x)}}\,\mathbb{I}_{\{y>x\}} + \frac{e^{-u_k^-\cdot(y-b_{k-1})} - e^{u_k^+\cdot(y-b_{k-1})}}{e^{-u_k^-\cdot(x-b_{k-1})} - e^{u_k^+\cdot(x-b_{k-1})}}\,\mathbb{I}_{\{y<x\}},$$

$$h^-(x) := \frac{u_k^- e^{-u_k^-\cdot(b_k-x)} + u_k^+ e^{u_k^+\cdot(b_k-x)}}{e^{-u_k^-\cdot(b_k-x)} - e^{u_k^+\cdot(b_k-x)}},$$

and

$$h^+(x) := \frac{u_k^+ e^{-u_k^+\cdot(x-b_{k-1})} + u_k^- e^{u_k^-\cdot(x-b_{k-1})}}{e^{-u_k^+\cdot(x-b_{k-1})} - e^{u_k^-\cdot(x-b_{k-1})}}$$

Then

$$p_0(x,y)dy := \mathbb{E}\left(\int_0^{\tau(b_{k-1},b_k)} e^{-\lambda t}\mathbb{I}_{\{Y_t\in dy\}}\,dt|Y_0=x\right) = -\frac{m(x,y)}{h^+(x)+h^-(x)}dy$$

for $r(x) = r(y) = k$.

Proof: This is shown by the same arguments as theorem 3. We can determine $p_0(x,y)$ as the product of $d(x)$ and $m(x,y)$, where $d(x)$ is the occupation density at x before $\min\{\tau(b_{k-1},b_k),\mathcal{E}(\lambda)\}$ and $m(x,y)$ is the occupation density at y before returning to x and before $\min\{\tau(b_{k-1},b_k),\mathcal{E}(\lambda)\}$.

The first factor is obtained as $d(x) = \lim_{\varepsilon\downarrow 0} 2\varepsilon(1-h^-(x,\varepsilon)h^+(x,\varepsilon))^{-1}$, where

$$h^-(x,\varepsilon) = \Psi_k^-(b_k - x + \varepsilon|2\varepsilon) \quad \text{and} \quad h^+(x,\varepsilon) = \Psi_k^+(x-b_{k-1}+\varepsilon|x-b_{k-1}-\varepsilon)$$

Since $\lim_{\varepsilon\downarrow 0} h^-(x,\varepsilon) = \lim_{\varepsilon\downarrow 0} h^+(x,\varepsilon) = 1$, we can write

$$\lim_{\varepsilon\downarrow 0} 2\varepsilon(1-h^-(x,\varepsilon)h^+(x,\varepsilon))^{-1} = -2\left(\frac{d}{d\varepsilon}h^-(x,\varepsilon)h^+(x,\varepsilon)\Big|_{\varepsilon=0}\right)^{-1}$$

where

$$\frac{d}{d\varepsilon}h^-(x,\varepsilon)\Big|_{\varepsilon=0} = 2\frac{u_k^- e^{-u_k^-\cdot(b_k-x)} + u_k^+ e^{u_k^+\cdot(b_k-x)}}{e^{-u_k^-\cdot(b_k-x)} - e^{u_k^+\cdot(b_k-x)}} = 2h^-(x)$$

and

$$\frac{d}{d\varepsilon}h^+(x,\varepsilon)\Big|_{\varepsilon=0} = 2\frac{u_k^+ e^{-u_k^+\cdot(x-b_{k-1})} + u_k^- e^{u_k^-\cdot(x-b_{k-1})}}{e^{-u_k^+\cdot(x-b_{k-1})} - e^{u_k^-\cdot(x-b_{k-1})}} = 2h^+(x)$$

This yields $d(x) = -(h^-(x)+h^+(x))^{-1}$.

The second factor $m(x,y)$ is given in terms of the time-reversed process $\mathcal{Y}^r$, namely

$$m(x,y) = \begin{cases} \Psi_k^{-,r}(b_k - x|y-x), & y > x \\ \Psi_k^{+,r}(x-b_{k-1}|y-b_{k-1}), & y < x \end{cases}$$

Now the formulas (2), (3), and the second equality in (8) yield the statement.
□

Assembling the above results, the resolvent density $\tilde{r}(x,y)$ can now be given explicitly as

$$\tilde{r}(x,y) = p_1^-(x) \sum_{k=2}^{N-1} m_{r(x)-1,k}\, l_k(y) + p_1^+(x) \sum_{k=2}^{N-1} m_{r(x),k}\, l_k(y) \\ + \mathbb{I}_{\{r(x)=r(y)\}} p_0(x,y)$$

for $y \notin \{b_0, \ldots, b_N\}$ and

$$\tilde{r}(x,y) = p_1^+(x) m_{r(x),i} + p_1^-(x) m_{r(x)-1,i}$$

for $y = b_i$ and $i \in \{0, \ldots, N\}$. The ingredients are given in (5) and theorems 2-4.

5. Exit problems

We wish to derive Laplace transforms for the exit times

$$\tau(l,u) := \inf\{t \geq 0 : Y_t \notin [l,u]\}$$

from an interval $[l,u] \subset]b_0, b_N[$ in the form of

$$E^+(l,a,u) := \mathbb{E}\left(e^{-\lambda\tau(l,u)}; Y_{\tau(l,u)} = u | Y_0 = a\right)$$

where $l < a < u$ and $\lambda \geq 0$. The derivations for

$$E^-(l,a,u) := \mathbb{E}\left(e^{-\lambda\tau(l,u)}; Y_{\tau(l,u)} = l | Y_0 = a\right)$$

are analogous.

5.1. The two-sided exit problem for two regimes

We first restrict our considerations to the case $N = 2$, i.e. one threshold b_1 dividing two regimes $I_1 =]b_0, b_1]$ and $I_2 =]b_1, b_2[$. There are some simple cases, namely

$$\mathbb{E}\left(e^{-\lambda\tau(l,u)}; Y_{\tau(l,u)} = u | Y_0 = a\right) = \begin{cases} \Psi_1^+(u-l|a-l), & l < a < u < b_1 \\ \Psi_2^+(u-l|a-l), & b_1 < l < a < u \end{cases}$$

For the other cases we obtain, by path continuity,

$$\mathbb{E}\left(e^{-\lambda\tau(l,u)}; Y_{\tau(l,u)} = u | Y_0 = a\right) = \Psi_1^+(b-l|a-l)\, \mathbb{E}\left(e^{-\lambda\tau(l,u)}; Y_{\tau(l,u)} = u | Y_0 = b_1\right)$$

for $l < a < b_1 < u$, and

$$\mathbb{E}\left(e^{-\lambda\tau(l,u)}; Y_{\tau(l,u)} = u | Y_0 = a\right) = \Psi_2^+(u - b_1 | a - b_1) \\ + \Psi_2^-(u - b_1 | a - b_1)\, \mathbb{E}\left(e^{-\lambda\tau(l,u)}; Y_{\tau(l,u)} = u | Y_0 = b_1\right)$$

for $l < b_1 < a < u$. Thus it suffices to determine $E^+(l, b_1, u)$. The same arguments as in theorem 1 yield with $k = 1$

$$E^+(l, b_k, u) = \frac{u_{k+1}^+ + u_{k+1}^-}{e^{-u_{k+1}^+ \cdot (u-b_k)} - e^{u_{k+1}^- \cdot (u-b_k)}} \\ \times \left(\frac{u_k^+ e^{-u_k^+ \cdot (b_k - l)} + u_k^- e^{u_k^- \cdot (b_k - l)}}{e^{-u_k^+ \cdot (b_k - l)} - e^{u_k^- \cdot (b_k - l)}} + \frac{u_{k+1}^- e^{-u_{k+1}^+ \cdot (u-b_k)} + u_{k+1}^+ e^{u_{k+1}^- \cdot (u-b_k)}}{e^{-u_{k+1}^+ \cdot (u-b_k)} - e^{u_{k+1}^- \cdot (u-b_k)}} \right)^{-1}$$

and

$$E^-(l, b_k, u) = \frac{u_k^+ + u_k^-}{e^{-u_k^- \cdot (b_k - l)} - e^{u_k^+ \cdot (b_k - l)}} \\ \times \left(\frac{u_k^+ e^{-u_k^+ \cdot (b_k - l)} + u_k^- e^{u_k^- \cdot (b_k - l)}}{e^{-u_k^+ \cdot (b_k - l)} - e^{u_k^- \cdot (b_k - l)}} + \frac{u_{k+1}^- e^{-u_{k+1}^+ \cdot (u-b_k)} + u_{k+1}^+ e^{u_{k+1}^- \cdot (u-b_k)}}{e^{-u_{k+1}^+ \cdot (u-b_k)} - e^{u_{k+1}^- \cdot (u-b_k)}} \right)^{-1}$$

for $\lambda \geq 0$.

5.2. The two-sided exit problem for $N > 2$ regimes

For $h := \min\{n \geq 1 : b_n > l\}$, the matrix $E^+(l, b_h, b_{h+1})$ has been determined in the previous subsection. Define $k := \max\{n \geq 1 : b_n < u\}$. If $k = h$, then $E^+(l, a, u)$ is given by the previous results. Thus assume that $k > h \geq 1$. We obtain by path continuity

$$E^+(l, a, u) = E^+(l, a, b_k) E^+(l, b_k, u)$$

where

$$E^+(l, b_k, u) = E^+(b_{k-1}, b_k, u) + E^-(b_{k-1}, b_k, u) E^+(l, b_{k-1}, b_k) E^+(l, b_k, u)$$

This yields

$$E^+(l, b_k, u) = \left(I - E^-(b_{k-1}, b_k, u) E^+(l, b_{k-1}, b_k)\right)^{-1} E^+(b_{k-1}, b_k, u)$$

Since the matrices $E^+(b_{k-1}, b_k, u)$ and $E^-(b_{k-1}, b_k, u)$ have been determined in the previous subsection, this provides a recursion scheme for $E^+(l, a, u)$.

References

[1] J. Cox and U. Rösler. A duality relation for entrance and exit laws for Markov processes. *Stochastic Processes and their Applications*, 16:141–156, 1984.

[2] S. N. Ethier and T. G. Kurtz. *Markov processes. Characterization and convergence.* New York etc.: John Wiley & Sons, 1986.

[3] K. Ito and H. P. McKean. *Difffusion Processes and their Sample Paths*. Springer, 1996.

[4] S. Karlin and H. M. Taylor. *A second course in stochastic processes.* New York etc.: Academic Press, 1981.

[5] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer, 1992.