A Piecewise–Deterministic Model for Brownian Motion

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Abstract

In the present paper, the classical Brownian motion of a particle suspended in an homogeneous liquid is modeled as a piecewise–deterministic Markov process with state space inculding position as well as velocity of the particle in motion. This model is less idealized than the classical Wiener or Ornstein–Uhlenbeck processes. It leads to more complex expressions for the transition densities, but also to path properties consistent with Newtonian physics. It is shown that the process introduced in this paper converges to the Wiener process as the intensity of collisions between the particle and the molecules of the liquid tends to infinity. This describes the point of idealization in the classical models.

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1 Introduction

The Wiener process, developed as a model for Brownian motion (see Wiener [11, 12]), is the most important and classical example for a stochastic process with continuous paths. It has become important for many application fields even before its existence as a mathematical concept (e.g. Bachelier [2] for stock prices or Einstein [6] for molecular–kinetic heat theory). During its 80 years of existence, it has become the basic stochastic process for models in ever more fields of application.

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Its derivation as a model for one–dimensional Brownian motion is easily described as follows: Consider some small particle suspended in a homogeneous liquid. The particle will show a phenomenon called Brownian motion, which is caused by collisions of molecules of the liquid with the particle due to thermal motion of the molecules. In a stochastic model of this phenomenon, the impulses of the molecules are regarded as random and independent of the position of the particle, since the liquid is homogeneous. Further, the different molecules can be regarded as independent of each other. Neglecting the current impulse of the particle and regarding only the limiting process as the inter–collision times tend to infinitesimal small intervals, the classical derivation concludes that the position process of a particle under Brownian motion has independent increments, continuous paths and is homogeneous in time. Then it follows that the position process is a Wiener process.

The modeling by a Wiener process has two sides: On the one hand it yields a reasonable approximation of the physical phenomenon, resulting in very simple transition density functions of the process. On the other hand, the Wiener process necessarily shows some very odd path properties like the non–differentiability at any time, which means that the particle cannot be ascribed a velocity. The latter is a consequence of the idealizing assumptions that the impulse of the particle can be neglected in the description of its motion process and that the model is approximated by the limiting process for inter–collision tending to infinitesimal small intervals.

The non–existence of a velocity of the particle has been overcome in the later Ornstein–Uhlenbeck model of Brownian motion (Uhlenbeck and Ornstein [10], see also Nelson [9], chapters 9–10). This model takes the velocity of the particle under consideration (thus coming closer to a classical Newtonian perspective on Brownian motion), but still is based on the transition to the limit of infinitesimal small inter–collision times. This results in non–differentiable paths of the velocity process, which breaks with a Newtonian view, too. It further turned out that the Ornstein–Uhlenbeck process describing the velocity of the particle can be realized by a Wiener process with deterministic change of the time clock and rescaling of the state variable (see Karlin and Taylor [8], p.218).

In the present paper, a strictly Newtonian view on the phenomenon of Brownian motion will be maintained. The motion process of the particle is modelled by a Markov process which is less idealized, since it takes into account the position as well as the impulse of the particle. Furthermore, in the model presented in this paper the particle will have real inter–collision times allowing it to move according to classical kinetic laws during non–vanishing intervals between collisions.

This leads to a piecewise–linear path behaviour if there are no additional forces and to piecewise–deterministic paths otherwise.

The main mathematical concepts used in the present paper are piecewise–deterministic Markov processes and Markov–additive processes. The development of the former has been initiated by Kovalenko and later Kalashnikov and Tien under the concept of piecewise–linear Markov processes (see Gnedenko and Kovalenko [7], section 3.3). Later the concept has been generalized by Davis [4, 5]. A recent treatise on Markov–additive processes is given in Asmussen [1], section II.5.

In section 2, a piecewise–deterministic model for Brownian motion will be introduced, and its basic properties, such as expressions for transition kernels, infinitesimal generator and a law of large numbers, will be derived. Section 3 contains a derivation of the Fourier transforms and by such means a convergence theorem. This, the main property to be proven in this paper, states that the present piecewise–deterministic model will converge to the classical Wiener process as the intensity of collisions tends to infinity.

2 The Joint Process of Position and Velocity

The model of Brownian motion, which will be introduced in this section, first is developed under negligence of any additional forces besides the influence of the collisions upon the motion of the particle. In remark 1, it is sketched how additional forces like friction or external force fields can be incorporated into the model without changing its characteristics.

Under this convention, the movement of the particle will be linear (following its present impulse) between collisions. Since the molecules of the suspending liquid are assumed as independent, the times of collisions can be modelled as a Poisson process with some (high) intensity λ. At an instant of collision the particle immediately receives an additional impulse by the colliding molecule, which leads to a (usually minor) change of its velocity. This change will be represented by some distribution function F, which reasonably is to be assumed as a normal distribution. Without restriction of generality, we can asssume that the particle has unit mass, thus identifying the values of impulse and velocity.

This model leads to a piecewise–linear path of the particle, the future of which at any time is completely determined by its present position and velocity (which equals its impulse). Hence we can model the motion as a piecewise–linear Markov process (see Gnedenko and Kovalenko [7], section 3.3) with states denoting the

present position and velocity of the particle.

The velocity process V can be modeled by a compound Poisson process (see e.g. Karlin and Taylor [8], pp.426–440), i.e.

$$V_t = \sum_{k=1}^{N_t} Y_k$$

where the process $N = (N_t : t \in I\!R_0^+)$ is a Poisson process with some intensity λ and $(Y_k : k \in I\!N)$ are iid with common distribution function F and characteristic function φ_Y. Then the characteristic function φ_{V_t} of V_t is given by

$$\varphi_{V_t}(u) = \exp\left(-\lambda t(1 - \varphi_Y(u))\right)$$

for all $u \in I\!R$. Further, the distribution function of V_t is given by

$$P(V_t \leq x) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} F^{(n)}(x)$$

for all $x \in I\!R$, with $F^{(n)}$ denoting the n–fold convolution of the distribution function F.

Let $Z = (X, V) = ((X_t, V_t) : t \in I\!R_0^+)$ denote the joint process of position X_t and velocity V_t, denoting position and velocity of the particle at time $t \in I\!R_0^+$. Assuming that besides the impulses that the particle receives there is no other force influencing the path of the particle, we know that between consecutive collisions the position of the particle changes in a deterministic (even in a linear) way.

Hence the joint process Z can be modelled as a piecewise–deterministic process (PDMP) without intrinsic jumps (see Davis [5], and Breuer [3] for this special class). The state space of Y is $E = I\!R^2$ with its usual Borel σ–algebra, denoted by $\mathcal{B}^2$. The process Z has jump rate λ, which is independent of the state of the process, and a jump transition measure Q defined by

$$Q((x, v), A \times]-\infty, w]) := 1_A(x) \cdot F(w - v)$$

for all $v, w, x \in I\!R$ and $A \in \mathcal{B}$. Clearly, Q is translation–invariant in the second dimension. Between jumps, we have a deterministic motion according to the flow function defined by

$$\Phi_t(x, v) := (x + t \cdot v, v)$$

for all $t \in I\!R_0^+$ and $x, v \in I\!R$. The value $\Phi_t(x, v)$ gives the state of the process at time t provided that no jump (i.e. here: no collision) has occured and the

process was in state (x, v) at time 0. We note that $\Phi_s(\Phi_t(x, v)) = \Phi_{t+s}(x, v)$ for all $s, t \in \mathbb{R}_0^+$ and $x, v \in \mathbb{R}$. The definition of the PDMP model is concluded by stating that the set Δ which induces intrinsic jumps is empty and thus $t_*(x) = \infty$ for all $x \in E$.

Clearly, the process Z is homogeneous. Let $P(t; x, A)$ denote the conditional probability that Z is in some state $y \in A \in \mathcal{B}^2$ at time $t \in \mathbb{R}_0^+$ given that Z is in state $x \in E$ at time 0. Further, let $P(t)$ denote the kernel with entries $P(t)(x, A) = P(t; x, A)$ for all $x \in E$ and $A \in \mathcal{B}^2$. Define the identity kernel on E by $I((x, v), A \times B) := 1_A(x)1_B(v)$ for all $x, v \in \mathbb{R}$ and $A, B \in \mathcal{B}$. Further define the generator kernel $G := \lambda(Q - I)$. Define for a kernel $K : E \times \mathcal{E} \to \mathbb{R}$ and a function $f : E \to E$ the operation $K \circ f(x, A) := K(f(x), A)$ for all $x \in E$ and $A \in \mathcal{E}$.

According to Breuer [3], theorem 2 (with G instead of Q), the transition probability kernel $P(t)$ can be expressed as

$$P(t) = \sum_{n=0}^{\infty} \int_0^t \int_0^{t_n} \dots \int_0^{t_2} (G \circ \Phi_{t_1})(G \circ \Phi_{t_2 - t_1}) \dots \\ \dots (G \circ \Phi_{t_n - t_{n-1}})(I \circ \Phi_{t - t_n})\, d_{t_1} \dots d_{t_n}$$

for all $t \in \mathbb{R}_0^+$, the sum entry for $n = 0$ being the kernel $I \circ \Phi_t$. This can be computed as the limit $P(t) = \lim_{n\to\infty} P_n(t)$ of the sequence $P_0(t) := I \circ \Phi_t$ for all $t \in \mathbb{R}_0^+$, and

$$P_{n+1}(t) := \int_0^t (G \circ \Phi_u) P_n(t - u)\, du + I \circ \Phi_t$$

for all $t \in \mathbb{R}_0^+$ and $n \in \mathbb{N}_0$.

Likewise, according to Breuer [3], theorem 1, another expression for the transition probability kernel $P(t)$ is given by

$$\begin{aligned} P(t) &= \sum_{n=0}^{\infty} \int_0^t \int_0^{t_n} \dots \int_0^{t_2} e^{-\lambda t_1} \lambda (Q \circ \Phi_{t_1}) e^{-\lambda(t_2 - t_1)} \lambda (Q \circ \Phi_{t_2 - t_1}) \dots \\ &\qquad \dots e^{-\lambda(t_n - t_{n-1})} \lambda (Q \circ \Phi_{t_n - t_{n-1}}) e^{-\lambda(t - t_n)} (I \circ \Phi_{t - t_n})\, d_{t_1} \dots d_{t_n} \\ &= \sum_{n=0}^{\infty} e^{-\lambda t} \lambda^n \int_0^t \int_0^{t_n} \dots \int_0^{t_2} (Q \circ \Phi_{t_1})(Q \circ \Phi_{t_2 - t_1}) \dots \\ &\qquad \dots (Q \circ \Phi_{t_n - t_{n-1}})(I \circ \Phi_{t - t_n})\, d_{t_1} \dots d_{t_n} \end{aligned}$$

for all $t \in I\!R_0^+$, the sum entry for $n = 0$ being the kernel $e^{-\lambda t}(I \circ \Phi_t)$. A computation of this expression as a limit of an iteration sequence is similar as above. We further obtain by substitution

$$\begin{aligned} P(t) &= \sum_{n=0}^{\infty} e^{-\lambda t}\lambda^n \int_0^t \int_0^{t-s_1} \dots \int_0^{t-s_1-\dots-s_n} (Q \circ \Phi_{s_1})(Q \circ \Phi_{s_2}) \dots \\ &\qquad \dots (Q \circ \Phi_{s_n})(I \circ \Phi_{s_{n+1}})\, d_{s_1} \dots d_{s_{n+1}} \\ &= \sum_{n=0}^{\infty} e^{-\lambda t}\lambda^n (Q \circ \Phi)_t^{*n} \end{aligned}$$

defining $(Q \circ \Phi)_t^{*0} := I \circ \Phi_t$ for all $t \in I\!R_0^+$, and iteratively

$$(Q \circ \Phi)_t^{*n} := \int_0^t (Q \circ \Phi_u)(Q \circ \Phi)_{t-u}^{*(n-1)}\, du$$

for all $t \in I\!R_0^+$ and $n \in I\!N$.

Remark 1 The above considerations are valid for any flow function Φ on E. Thus other factors like a frictional force can be taken into account as well by simply changing the definition of Φ. In particular, a Brownian motion under influence of a deterministic external force field can be represented by the same model.

Assuming that F has a Lebesgue density function f, we can represent the transition density of Z by

$$\begin{aligned} p_t((x,v),(y,w)) &= \sum_{n=2}^{\infty} e^{-\lambda t}\lambda^n \int\cdots\int_{\substack{\sum s_i < t, \sum v_i = w-v, \\ \sum s_i v_{i-1} + s_{n+1} v_n = y-x}} \prod_{i=1}^{n} f(v_i)\, ds_1 \dots ds_n\, dv_1 \dots dv_n \\ &= \sum_{n=2}^{\infty} e^{-\lambda t}\lambda^n \int\cdots\int_{\sum v_i = w-v} \prod_{i=1}^{n} f(v_i)\, \mu_L^{n-1}(L(v_0,\dots,v_n))\, dv_1 \dots dv_n \end{aligned}$$

for all $t \in I\!R_0^+$ and $x, y, v, w \in I\!R$. Here, we have defined $v_0 := v$, $s_{n+1} := t - \sum_{i=1}^n s_i$ and denoted the n–dimensional Lebesgue measure by μ_L^n. Further, $L(v_0,\dots,v_n)$ denotes the set of solution vectors $(s_1,\dots,s_n)$ to the equation

$$\sum_{i=1}^{n} s_i v_{i-1} + s_{n+1} v_n = y - x \quad \Longleftrightarrow \quad \sum_{i=1}^{n} s_i (v_{i-1} - v_n) = y - x - t v_n$$

with $s_i > 0$ for all $i \leq n$ and $\sum_{i=1}^{n} s_i < t$. The cases $n = 0, 1$ can be neglected, since for a model of Brownian motion the parameter λ will be very large and thus $e^{-\lambda t}$ very small.

Theorem 1 *For the marginal process X, the convergence*

$$\lim_{t\to\infty} \frac{X_t}{t} = E(F)$$

holds almost certainly, independent of the initial velocity V_0.

Proof: Let T_i and V_i denote the time and the velocity of the particle between the $i-1$st and the ith change of velocity (i.e. collision of the particle), respectively. Clearly, the sequences $(T_i : i \in I\!N)$ and $(V_i : i \in I\!N)$ are iid with common distributions $Exp(\lambda)$ and F, respectively. Furthermore, the sequence $(T_i \cdot V_i : i \in I\!N)$ is iid with mean $E(T_i \cdot V_i) = E(F)/\lambda$.

Since asymptotically the initial velocity and the time after the last jump do not matter, we obtain

$$\lim_{t\to\infty} \frac{X_t}{t} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} T_i \cdot V_i}{\sum_{i=1}^{n} T_i} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} T_i \cdot V_i}{n} \Big/ \lim_{n\to\infty} \frac{\sum_{i=1}^{n} T_i}{n} = \frac{E(F)/\lambda}{1/\lambda}$$

almost certainly, according to the strong law of large numbers.
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Since Z is a piecewise–deterministic process, its infinitesimal generator is given as in Davis [4], theorem 5.5, by

$$\mathcal{A}g(x,v) = v \cdot \frac{\partial g(x,v)}{\partial x} + \lambda \cdot \int_{I\!R} (g(x, v+w) - g(x,v)) dF(w)$$

for all functions $g : I\!R^2 \to I\!R$ differentiable in the first argument and measurable with respect to $\mathcal{B}$ in the second. Since Z has no intrinsic jumps, we do not need a boundary condition as in Davis [4], equation (5.4), for the domain of the generator.

Specifying g as a density function $g(t,x,v)$ on $I\!R^2$ and depending on time t, we can obtain the probability law of the process by setting the initial value function $g(0,.,.)$ to the density at time 0 and then solving the partial differential equation

$$\frac{\partial g(t,x,v)}{\partial t} = v \cdot \frac{\partial g(t,x,v)}{\partial x} + \lambda \cdot \int_{I\!R} (g(t,x, v+w) - g(t,x,v)) dF(w)$$

with the above initial value. This provides a determination of the density transition function by non–stochastic means and thus a purely analytical treatment of stochastic processes having such infinitesimal generators.

3 Convergence to the Wiener Process

The process $Z = (X, V)$ consists of a simple Markov jump component V, which is independent, and of the component X, which is a uniform translation process conditional on V. This structure allows an analysis of Z along the lines of the theory of Markov–additive processes (see Asmussen [1], section II.5 and references therein).

For every $t \in I\!R_0^+$, define the conditional Fourier transform kernel X_t^* of X given V by its entries

$$X_t^*(s)(v, W) := E\left(e^{isX_t} \cdot 1_W(V_t) | V_0 = v\right) \tag{1}$$

for all $s \in I\!R$, $v \in I\!R$ and $W \in \mathcal{B}$. Analogous to proposition 5.2 in chapter 2 of Asmussen [1], we obtain

Theorem 2 *Assume that F is Lebesgue dominated with density function $f : I\!R \to I\!R_0^+$. Define $\Lambda(v, W) := \lambda \cdot (F(W - v) - 1_W(v))$ and $U(v, W) := v \cdot 1_W(v)$ for all $v \in I\!R$ and $W \in \mathcal{B}$. Then for every $t \in I\!R_0^+$ the conditional Fourier transform kernel X_t^* of X given V can be expressed by*

$$X_t^*(s) = e^{t(\Lambda + isU)}$$

for all $s \in I\!R$.

Proof: For all $v \in I\!R$, let $S^v = (S_t^v : t \in I\!R_0^+)$ denote a uniform translation process with drift v, i.e. $P(S_t^v \in A | S_0^v = x) = 1_A(x + vt)$ for all $x \in I\!R$ and $A \in \mathcal{B}$. Since F is Lebesgue dominated and X is conditional upon V (hence dominated by it), the Lebesgue densities

$$f_t(s, v, w) := E\left(e^{isX_t}, V_t = w | V_0 = v\right)$$

of the conditional expectations in definition (1) exist. For them we obtain the linear approximation

$$\begin{aligned} f_{t+h}(s, v, w) &= f_t(s, v, w)(1 - \lambda h)e^{isw} + \int_{I\!R} f_t(s, v, u)\lambda h f(w - u)\, du \\ &= f_t(s, v, w)(1 + h(isw)) + h\lambda \left(\int_{I\!R} f_t(s, v, u) f(w - u)\, du - f_t(s, v, w) \right) \\ &= f_t(s, v, w) \left(1 + h(isw) + h \int_{I\!R} f_t(s, v, u)\lambda (f(w - u) - f_t(s, v, w))\, du \right) \end{aligned}$$

omitting the terms of size $o(h)$. This implies

$$\frac{\partial X_t^*(s)}{\partial t} = X_t^*(s)(\Lambda + isU)$$

for all $t \in I\!R$. Since Λ is bounded, the operator $e^{t(\Lambda+isU)} = e^{t\Lambda}e^{istU}$ does exist for all $t \in I\!R_0^+$ and $s \in I\!R$. Because of $X_0^*(s) = I$ it solves the above differential equation. If $A(t)$ were another solution with the same initial condition, then the difference $\Delta(t) := X_t^* - A(t)$ would be differentiable with derivative $\Delta'(t) = 0$, hence $\Delta(t)$ would be constant in t. Since $\Delta(0) = 0$, the only solution is given by $X_t^* : s \to e^{t(\Lambda+isU)}$.
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Theorem 3 *Let $F(\lambda) = N(0, \sigma^2/\lambda)$ denote the distribution function of the normal distribution with mean 0 and variance σ^2/λ. Further, denote the marginal position process of $Z(\lambda)$ with jump rate λ and impulse distribution $F(\lambda)$ by $X(\lambda)$. Then the limit X of the processes $X(\lambda)$ with initial velocity 0 for $\lambda \to \infty$ is the Wiener process with mean 0 and variance σ^2 under a deterministic change of the time scale.*

Proof: The marginal Fourier transform of $X(\lambda)$ under initial velocity 0 is given by

$$X_t^*(\lambda, s)(0, I\!R) = \delta_0 e^{t\Lambda} e^{tisU} 1_{I\!R}$$

with $1_{I\!R}$ denoting the constant function on $I\!R$ with value 1. Since for $\lambda \to \infty$, the measure $\delta_0 e^{t\Lambda}$ tends to the normal distribution Φ with mean 0 and variance $\sigma^2 t$, and $\delta_x U 1_{I\!R} = x$ for all $x \in I\!R$, we obtain

$$X_t^*(s)(0, I\!R) = \int_{I\!R} e^{tisx} dN_{(0,\sigma^2 t)}(x) = \Phi_{t^2}^*(s)$$

which is the Fourier transform of the Wiener process with mean 0 and variance σ^2 at time t^2.
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