Discrete Variational Methods

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Context and Background

Can we use symbolic algebra to study numerical methods?

Examples

- Can you design a numerical method, automatically, to inherit a variational principle and selected conservation laws?

- Can one obtain symmetries and hence conservation laws, automatically, of variational numerical methods?

Noether's Theorem

links symmetries and conservation laws for Euler Lagrange Systems.

A conservation law is a divergence expression which is zero on solutions of the system.

The heat equation $u_t + (-u_x)_x = 0$ is its own conservation law. Integrating,

$$\frac{\partial}{\partial t}\int_{\Omega} u + (-u_x)]_{\partial\Omega} = 0$$

Rate of change of total heat in Ω	=	Net of comings and goings across the boundary

no sources or sinks

The usual examples:

Symmetry	Conserved Quantity
leaves $L\mathrm{d}x$ invariant	the quantity behind $\frac{\mathrm{D}}{\mathrm{D}t}$ in Div
$\begin{cases} t^* = t + c \\ \text{translation in time} \end{cases}$	Energy
$\begin{cases} x_i^* = x_i + c \\ \text{translation in space} \end{cases}$	Linear Momenta
$\begin{cases} \mathbf{x}^* = \mathcal{R}\mathbf{x} \\ \text{rotation in space} \end{cases}$	Angular Momenta
$\begin{cases} a^* = \phi(a,b), b^* = \psi(a,b) \\ \phi_a\psi_b - \phi_b\psi_a \equiv 1 \\ \text{Particle relabelling} \end{cases}$	Potential vorticity

How to prove Noether's Theorem?

Step 1: the Euler Lagrange operator

$$\begin{aligned}
\hat{\mathrm{d}}(L\mathrm{d}x) &= \hat{\mathrm{d}}\left(\tfrac{1}{2}\left(u_x^2+u_{xx}^2\right)\mathrm{d}x\right) \\
&= (u_x\mathrm{d}u_x + u_{xx}\mathrm{d}u_{xx})\mathrm{d}x \\
&= (-u_{xx}\mathrm{d}u + u_{xxxx}\mathrm{d}u)\mathrm{d}x \\
&\quad +\tfrac{\mathrm{D}}{\mathrm{D}x}\left(u_x\mathrm{d}u - 2u_{xx}\mathrm{d}u_x + \tfrac{\mathrm{D}}{\mathrm{D}x}\left(u_{xx}\mathrm{d}u\right)\right) \\
&= E(L)\mathrm{d}u\mathrm{d}x + \tfrac{\mathrm{D}}{\mathrm{D}x}\eta_L
\end{aligned}$$

General Formula, explicit, exact, symbolic, for η_L known.

$E = \pi \circ \hat{\mathrm{d}}$, where π projects out the divergence term.

More than one dependent variable:

$$\hat{\mathrm{d}}L(x,u,v,\ldots)\mathrm{d}x = E^u(L)\mathrm{d}u\mathrm{d}x + E^v(L)\mathrm{d}v\mathrm{d}x + \tfrac{\mathrm{D}}{\mathrm{D}x}\eta_L$$

Step 2: Variational Symmetries

Symmetries arise from Lie group actions.

EXAMPLE: $G = (\mathbb{R}, +)$

$$\epsilon \cdot x = x^* = \frac{x}{1-\epsilon x}, \qquad \epsilon \cdot u = u^*(x^*) = \frac{u(x)}{1-\epsilon x}$$

Group Action Property

$$\delta \cdot (\epsilon \cdot x) = \delta \cdot \left(\frac{x}{1-\epsilon x}\right) = \frac{\frac{x}{1-\delta x}}{1-\epsilon\frac{x}{1-\delta x}} = \frac{x}{1-(\epsilon+\delta)x} = (\epsilon+\delta)\cdot x$$

Prolonged Group Action

$$\epsilon \cdot u_x = u^*_{x^*} = \frac{\partial u^*(x^*)}{\partial x} \Big/ \frac{\partial x^*}{\partial x} = \frac{u_x}{(1-\epsilon x)^2}$$

and

$$\delta \cdot (\epsilon \cdot u_x) = \frac{\delta \cdot u_x}{(1-\epsilon(\delta \cdot x))^2} = \frac{u_x}{(1-(\delta+\epsilon)x)^2}$$

Action on Integrals

$$\epsilon \cdot \int_\Omega L(x, u, u_x, \ldots)\,\mathrm{d}x$$

def'n of $\epsilon\cdot$ $$= \int_{\epsilon\cdot\Omega} L(\epsilon\cdot x, \epsilon\cdot u, \epsilon\cdot u_x, \cdots)\,\mathrm{d}\epsilon\cdot x$$

change of variable $$= \int_\Omega L(\epsilon\cdot x, \epsilon\cdot u, \epsilon\cdot u_x, \cdots)\frac{\mathrm{d}\epsilon\cdot x}{\mathrm{d}x}\,\mathrm{d}x$$

Use L^2 theory to get that a variational symmetry of a Lagrangian is a group action such that

$$L(x, u, u_x, \ldots) = L(\epsilon\cdot x, \epsilon\cdot u, \epsilon\cdot u_x, \cdots)\frac{\mathrm{d}\epsilon\cdot x}{\mathrm{d}x}$$

Assuming the identity transformation occurs for $\epsilon = 0$, Infinitesimal Actions are obtained by applying $\frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}$. If

$$\phi^\alpha = \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}\epsilon \cdot u^\alpha, \quad \xi_i = \frac{\mathrm{d}}{\mathrm{d}\epsilon}|_{\epsilon=0}\epsilon \cdot x_i$$

then for

$$Q^\alpha = \phi^\alpha - \sum_i \xi_i u^\alpha_{x_i}$$

we can finally state Noether's theorem precisely:

$$Q \cdot E(L) = \sum_\alpha Q^\alpha E^\alpha(L) = \mathrm{Div}(\mathcal{A}(Q, L))$$

On the simplest level, the proof involves a manipulation of the expressions involved. Need to dig deeper to translate the theorem to a discrete setting.

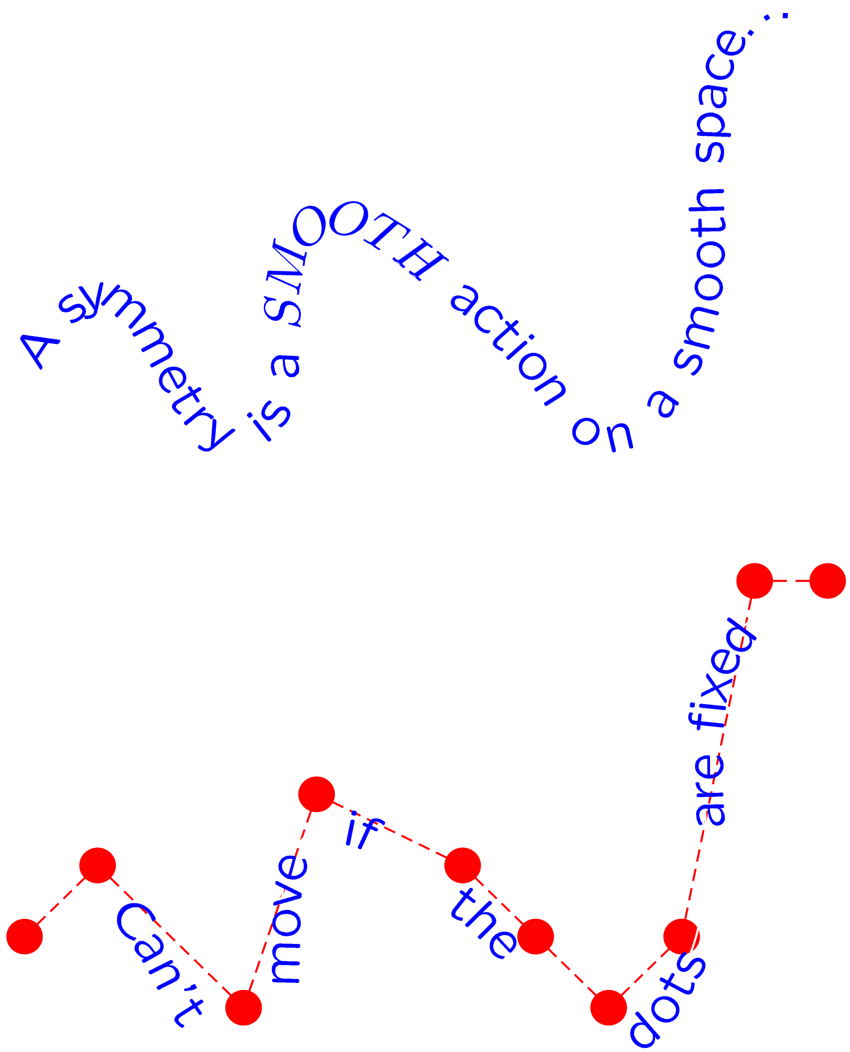


Figure 1: First Challenge

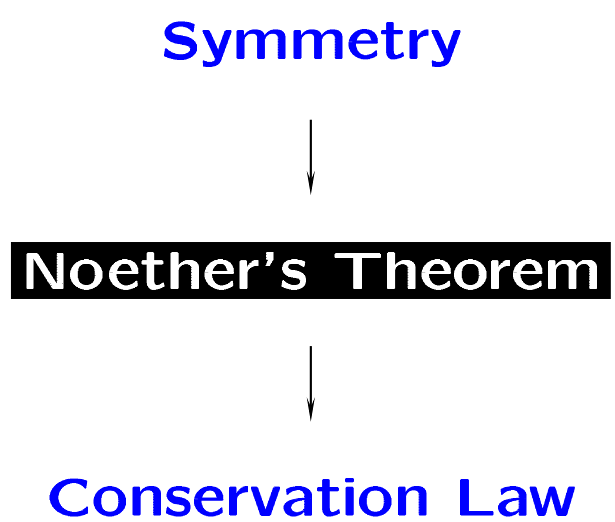


Figure 2: Second Challenge. What replaces the black box for numerical data?

Difference Systems

Difference Euler-Lagrange operator

$$\begin{aligned}
\hat{\mathsf{d}}(L_n) &= \hat{\mathsf{d}}\left(\tfrac{1}{2}u_n^2 + u_n u_{n+1}\right) \\
&= (u_n \mathsf{d}u_n + u_{n+1}\mathsf{d}u_n + u_n \mathsf{d}u_{n+1}) \\
&= (u_n + u_{n+1} + u_{n-1})\mathsf{d}u_n + (S - \mathsf{id})(\cdots) \\
&= E(L_n)\mathsf{d}u_n + (S - \mathsf{id})(\eta_{L_n})
\end{aligned}$$

General formula, explicit, exact, symbolic, for η_{L_n} known.

$E = \pi \circ \hat{\mathsf{d}}$, where π projects out the total difference term.

More than one dependent variable:

$$\hat{\mathsf{d}}(L_n \Delta_n) = E^u(L_n)\mathsf{d}u_n + E^v(L_n)\mathsf{d}v_n + (S - \mathsf{id})(\eta_{L_n})$$

- Since n cannot vary in a smooth way, the "mesh variables" x_n are treated as dependent variables.

- The group action commutes with shift:

$$\epsilon \cdot S^j(u_n) = \epsilon \cdot u_{n+j} = S^j \epsilon \cdot u_n$$

For example,

$$\epsilon \cdot u_n = \frac{u_n}{1 - \epsilon x_n} \Longrightarrow \epsilon \cdot u_{n+j} = \frac{u_{n+j}}{1 - \epsilon x_{n+j}}$$

The symmetry condition is:

$$\begin{aligned} &L_n(x_n, \cdots x_{n+j}, u_n, \cdots u_{n+k}) \\ &\quad = L_n(x_n^*, \cdots x_{n+j}^*, u_n^*, \cdots u_{n+k}^*) \end{aligned}$$

where $(\,)^* \equiv \epsilon \cdot (\,)$.

Setting

$$Q_n^x = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} x_n^*, \qquad Q_n^u = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} u_n^*$$

then the difference Noether's theorem is

$$Q \cdot E(L_n) = (S - \mathrm{id})(\mathcal{A}_n(Q_n, L_n)),$$

so a symmetry yields a total difference expression which is zero on solutions of the difference Euler Lagrange system. Explicit formulae for $\mathcal{A}_n(Q_n, L_n)$ known. The result is independent of any continuum limit.

Note the similarity of the formula to that of the smooth case!!

Elementary example T.D. Lee, Difference Equations and Conservation Laws, J. Stat. Phys., **46** (1987)

A difference model for $\int(\frac{1}{2}\dot{x}^2 - V(x))\,\mathrm{d}t$

Define

$$\bar{V}(n) = \frac{1}{x_n - x_{n-1}} \int_{x_{n-1}}^{x_n} V(x)\,\mathrm{d}x$$

and take

$$L_n = \left[\frac{1}{2}\left(\frac{x_n - x_{n-1}}{t_n - t_{n-1}}\right)^2 - \bar{V}(n)\right](t_n - t_{n-1})$$

The group action is $t_n^* = t_n + \epsilon$, with x_n invariant. The conserved quantity is thus "energy". Now, $Q_n^t = 1$ for all n, and $Q_n^x = 0$. By definition of difference EL operator,

$$0 \;=\; E^t(L_n) \;=\; \frac{\partial}{\partial t_n} L_n + S\left(\frac{\partial}{\partial t_{n-1}} L_n\right)$$

A difference model for $\int(\frac{1}{2}\dot{x}^2 - V(x))\,\mathrm{d}t$ (cont.)

Since L_n is a function of $(t_n - t_{n-1})$,

$$0 = E^t(L_n) = (S - \mathrm{id})\left(\frac{\partial}{\partial t_n} L_n\right)$$

and thus

$$\frac{1}{2}\left(\frac{x_n - x_{n-1}}{t_n - t_{n-1}}\right)^2 + \bar{V}(n) = c$$

Note that the energy in the smooth case is

$$\frac{1}{2}\dot{x}^2 + V.$$

Can regard the EL eqn for the mesh variables as an equation for a variable mesh.

INTERLUDE

If we know the group action for a particular conservation law, we can "design in" that conservation law into a discretisation by taking a Lagrangian composed of invariants. The Fels and Olver formulation of moving frames is particularly helpful here: a sample theorem is

Discrete rotation invariants in $\mathbb{Z}^2$

Let (x_n, y_n), (x_m, y_m) be two points in the plane. Then

$$I_{n,m} = x_n y_n + x_m y_m, \quad J_{n,m} = x_n y_m - x_m y_n$$

are rotation invariants. Moreover, any discrete rotation invariant is a function of these.

Made up example

Suppose

$$L_n = \frac{1}{2}J_{n,n+1}^2 = \frac{1}{2}(x_n y_{n+1} - x_{n+1} y_n)^2$$

then

$$\begin{cases} E_n^x & = & J_{n,n+1} y_{n+1} - J_{n-1,n} y_{n-1} \\ E_n^y & = & -J_{n,n+1} x_{n+1} + J_{n-1,n} x_{n-1} \end{cases}$$

Now, $Q_n = (Q_n^x, Q_x^y) = (-y_n, x_n) = \frac{\mathrm{d}}{\mathrm{d}\theta}\Big|_{\theta=0}(x_n^*, y_n^*)$ and thus

$$\begin{aligned} Q_n \cdot E_n &= J_{n,n+1}(-y_n y_{n+1} - x_n x_{n+1}) \\ &\quad + J_{n-1,n}(y_n y_{n-1} + x_n x_{n-1}) \\ &= -J_{n,n+1} I_{n,n+1} + J_{n-1,n} I_{n-1,n} \\ &= -(S - \mathrm{id})(J_{n-1,n} I_{n-1,n}) \end{aligned}$$

gives the conserved quantity.

Similarity of formulae arises as proofs can be given a common algebraic foundation.

Variational Complexes

$\mathcal{SMOOTH}$ e.g. P.J. Olver, Applications . . .

$$
\begin{array}{ccccccccccc}
\overset{\text{Curl}}{\longrightarrow} & \Lambda^2 & \overset{\text{Div}}{\longrightarrow} & \Lambda^3 & \overset{\hat{\text{d}}}{\longrightarrow} & \hat{\Lambda}_1 & \overset{\hat{\text{d}}}{\longrightarrow} & \hat{\Lambda}_2 & \overset{\hat{\text{d}}}{\longrightarrow} \\
 & & & & \searrow E & \downarrow \pi & & \downarrow \pi & \\
 & & & & & {\Lambda^1}_* & \overset{\delta}{\longrightarrow} & {\Lambda^2}_* & \overset{\delta}{\longrightarrow}
\end{array}
$$

DIFFERENCE Hydon and ELM, J. FoCM

$$
\begin{array}{ccccccccccc}
\overset{\Delta}{\longrightarrow} & \mathbf{Ex}^2 & \overset{\Delta}{\longrightarrow} & \mathbf{Ex}^3 & \overset{\hat{\text{d}}}{\longrightarrow} & \hat{\Lambda}_1 & \overset{\hat{\text{d}}}{\longrightarrow} & \hat{\Lambda}_2 & \hat{\text{d}}\longrightarrow \\
 & & & & \searrow E & \downarrow \pi & & \downarrow \pi & \\
 & & & & & \Lambda^1_* & \overset{\delta}{\longrightarrow} & \Lambda^2_* & \overset{\delta}{\longrightarrow}
\end{array}
$$

A variational complex is a tool which helps to formulate precisely, and answer, questions such as;

Is my system variational? and if so, what is the Lagrangian?

Is an expression a divergence? and if so, of what?

What about Finite Element Method? that is, moment based approximations calculated relative to a triangulation?

Not all choices of Finite Element are suited to variational methods Consider the projection to piecewise constant functions,

$$\Pi(u) = \sum_n \alpha_n \chi_{e_n}$$

where

$$\alpha_n = \frac{1}{x_{n+1} - x_n} \int_{x_n}^{x_{n+1}} u \, \mathrm{d}x.$$

For $L[u] = \frac{1}{2} u_x^2$ we have $\Pi(u)_x = \sum (\alpha_n - \alpha_{n-1}) \delta(x - x_n)$ taking the weak or distributional meaning of the derivative, and thus

$$\Pi : \int u_x^2 \mapsto \Pi(L) = \sum_n (\alpha_n - \alpha_{n-1})^2.$$

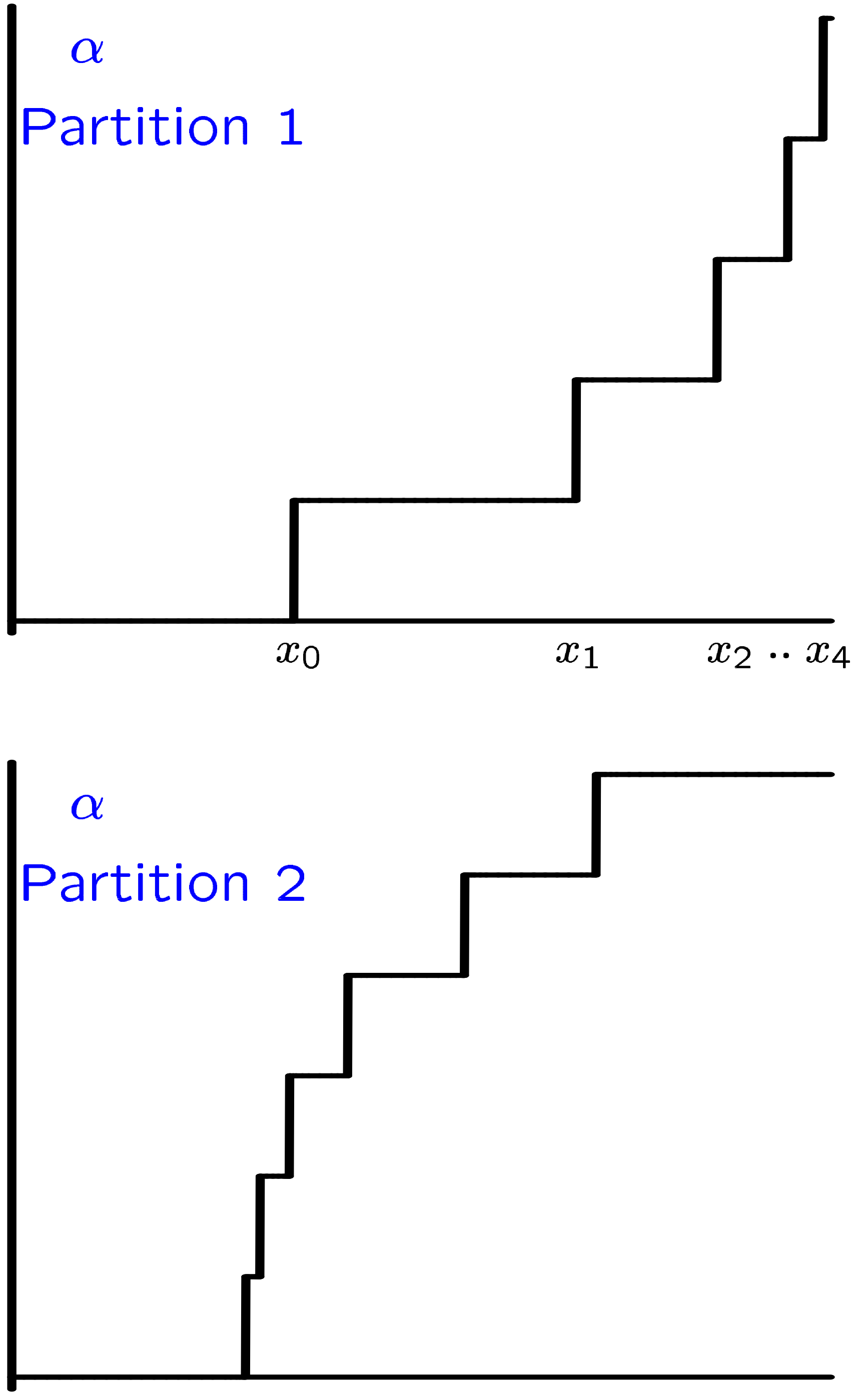
Taking the variational derivative with respect to the moments yields

$$\begin{aligned} \hat{\mathrm{d}}\Pi(L) &= \textstyle\sum_n 2(\alpha_n - \alpha_{n-1})(\mathrm{d}\alpha_n - \mathrm{d}\alpha_{n-1}) \\ &= \textstyle\sum_n 2(2\alpha_n - \alpha_{n-1} - \alpha_{n+1})\mathrm{d}\alpha_n \\ &\quad + \text{boundary terms} \end{aligned}$$

The discrete Euler Lagrange equation is then $2\alpha_n - \alpha_{n-1} - \alpha_{n+1} = 0$ or

$$\alpha_{n+1} - \alpha_n = \kappa, \qquad \kappa \in \mathbb{R}$$

This is not good!! In fact, the solution to the discrete EL eqn is correct only for the regular partition.



α
Partition 1
x0
x1
x2 ·· x4
α
Partition 2
x0 ·· x3
x4
x5

The Quispel fix We can alter the previous unsatisfactory scheme as follows. Use the zeroth moments for u on (x_n, x_{n+1}) and (x_{n+1}, x_{n+2}) to create a piecewise linear approximation for u on (x_n, x_{n+2}) with the same 2 moments:

$$\begin{aligned}
\Pi(u)_n &= x \mapsto A_n x + B_n \\
A_n &= 2\left(\frac{\alpha_n - \alpha_{n+1}}{x_n - x_{n+2}}\right) \\
B_n &= \left(\frac{x_{n+1} + x_{n+2}}{x_{n+2} - x_n}\right)\alpha_n - \left(\frac{x_{n+1} + x_n}{x_{n+2} - x_n}\right)\alpha_{n+1}
\end{aligned}$$

for an approximation on the partition

$$\cdots x_{n-4},\ x_{n-2},\ x_n,\ x_{n+2},\ x_{n+4}, \cdots$$

and do the same calculations as before. The resulting EL eqn "integrates" to the correct equation,

$$\frac{\alpha_n - \alpha_{n+1}}{x_n - x_{n+2}} = \kappa, \qquad \kappa \in \mathbb{R}$$

- Approximations need to involve as many moments (per element) as the order of the resulting EL eqn (twice the order of the Lagrangian).

- The data need to involve information from either the boundary of the element, or from nearby elements.

Problems disappear if the approximation data fit an exact scheme, à la Douglas Arnold. Moreover, such a scheme yields stability!!

Differential Complexes and FE: D. Arnold, Beijing ICM Plenary talk

Choose a system of moments and sundry other data, aka degrees of freedom, that yield projection operators such that the diagram commutes:

$$
\begin{array}{ccccccccccccc}
0 \to \mathbb{R} & \to \Lambda^0 & \xrightarrow{\mathrm{d}} & \Lambda^1 & \xrightarrow{\mathrm{d}} & \Lambda^2 & \xrightarrow{\mathrm{d}} & \Lambda^3 & \to 0 \\
& \Pi_0 \downarrow & & \Pi_1 \downarrow & & \Pi_2 \downarrow & & \Pi_3 \downarrow & \\
0 \to \mathbb{R} & \to \mathcal{F}^0 & \xrightarrow{\mathrm{d}} & \mathcal{F}^1 & \xrightarrow{\mathrm{d}} & \mathcal{F}^2 & \xrightarrow{\mathrm{d}} & \mathcal{F}^3 & \to 0
\end{array}
$$

all relative to some triangulation.

Λ^p	$\equiv$	p-forms with coefficients smooth functions in x;
		that is, integrands of integrals on p-surfaces
d	$\equiv$	exterior derivative, $\mathrm{d}^2 = 0$
Π	$\equiv$	projection operator, $\Pi \circ \mathrm{d} = \mathrm{d} \circ \Pi$

A Lagrangian is composed of wedge products of 1-, 2- and 3- forms. Choose the discretisation of each to be in the relevant $\mathcal{F}_i$. Then commutativity implies conditions for Brezzi's theorem to hold.

In one dimension: with $e_n = (x_n, x_{n+1})$, Π_0 to piecewise linear, Π_1 to piecewise constant with moment

$$\alpha_n = \int_{x_n}^{x_{n+1}} u(x)\psi_n(x)\,\mathrm{d}x$$

Commutativity of the diagram

$$\begin{array}{ccc} u & \overset{\mathrm{d}}{\mapsto} & u_x \mathrm{d}x \\ \Pi_0 \downarrow & & \downarrow \Pi_1 \\ u|_{e_n} = A_n x + B_n & \mapsto & A_n = \int_{x_n}^{x_{n+1}} u'(x)\psi_n(x)\,\mathrm{d}x \end{array}$$

implies

$$A_n = u(x)\psi_n(x)]_{x_n}^{x_{n+1}} - \int_{x_n}^{x_{n+1}} u(x)\psi_n'(x)\,\mathrm{d}x$$

Note that

$$\int_{x_n}^{x_{n+1}} \psi_n(x)\,\mathrm{d}x = 1.$$

is required by the projection property.

A finite element Lagrangian is built up of wedge products of forms in $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$. Call this resulting space $\tilde{\mathcal{F}}_3$. In each top-dimensional simplex, denoted τ, integrate to get

$$L = \sum_\tau L_\tau(\alpha_\tau^1, \cdots \alpha_\tau^p)$$

where α_τ^j are the degrees of freedom in τ. L can also depend on mesh data. Can now take $\hat{\mathrm{d}}$, the variation with respect to the α_τ^j.

- $$\Pi(fg) \neq \Pi(f)\Pi(g)$$
- $$\Pi(u_x) \neq \Pi(u)_x$$

The left hand side is how a "top down" version of a complex would project a Lagrangian. The right hand side is how a Lagrangian is projected in practice.

Noether's Theorem for 1-D FEM

$$L[u] = \frac{1}{2}\left(\frac{u_x}{u}\right)^2$$

Using the usual PL interpolation, we have

$$\Pi : \int L[u]\,\mathrm{d}x \mapsto \sum_n L_n = \sum_n \frac{(u_{n+1} - u_n)^2}{2u_{n+1}u_n(x_{n+1} - x_n)}$$

Now the discrete EL eqn is

$$E(L) = \partial_{u_n} L_n + S^{-1}\partial_{u_{n+1}} L_n \qquad (1)$$

where $S := n \mapsto n + 1$. The scaling symmetry, $u^* = \lambda u$ of $L[u]$ translates to $u_n^* = \lambda u_n$, so $Q_n = u_n$ and the conservation law is

$$(S - \mathrm{id})(u_n \partial_{u_n} L_n) = 0.$$

Noether's Theorem for 1-D FEM (cont.) $L[u] = \frac{1}{2}(u_x/u)^2$

Setting $u_n \partial_{u_n} L_n = \kappa$, this yields

$$\left(\frac{u_{n+1}}{u_n}\right)^2 - \kappa(x_{n+1} - x_n)\left(\frac{u_{n+1}}{u_n}\right) - 1 = 0$$

or

$$u_{n+1} = H_n u_n, \qquad H_n \sim (\mp 1 + \tfrac{\kappa}{2}(x_{n+1} - x_n))$$

If $(x_{n+1} - x_n) \sim x/n$ this integrates to

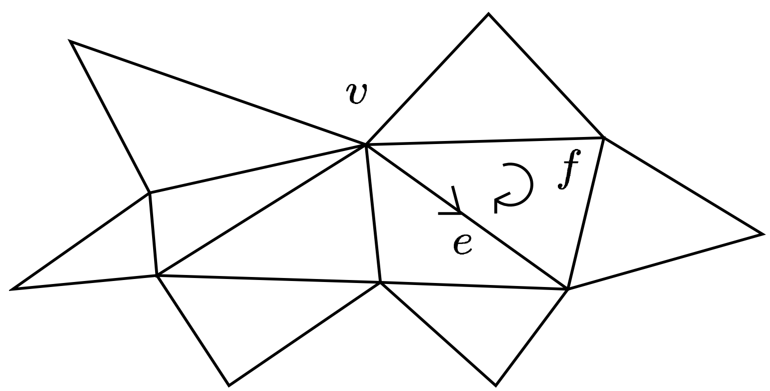
$$u_n = \left(1 + \frac{\kappa x}{2n}\right)^n u_0 \sim u_0 \exp(\kappa x/2)$$

which is the correct result.

In two dimensions, what is the equivalence class / cancelling sums / boundary terms needed to calculate the Euler Lagrange operator?

"coboundary of a 1-cochain"*

Take a triangulation: vertices v, oriented edges e and oriented faces f.



0 cochains are maps $< v_i > \to \mathbb{R}$

1 cochains are maps $< e_i > \to \mathbb{R}$

2 cochains are maps $< f_i > \to \mathbb{R}$

All maps linear.

Change orientation

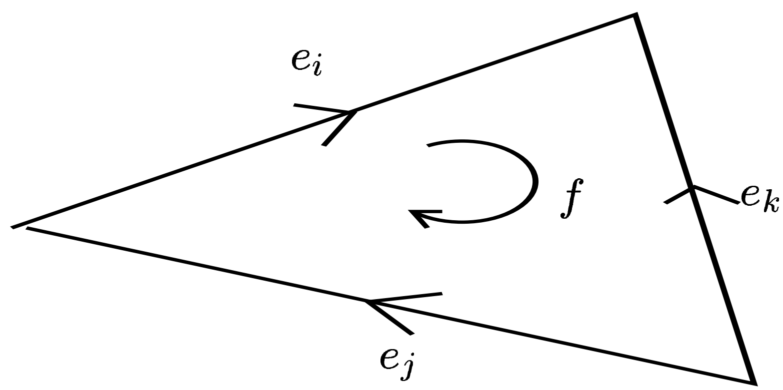
$\equiv$ change sign.

*classical simplicial algebra

Given $F :< e_i > \rightarrow \mathbb{R}$, define $\delta F :< f_i > \rightarrow \mathbb{R}$ by (for f as in the diagram),

$$(\delta F)(f) = F(e_i) + F(e_j) - F(e_k)$$

and extended linearly. Note: the signs are according to whether the orientations match or not.

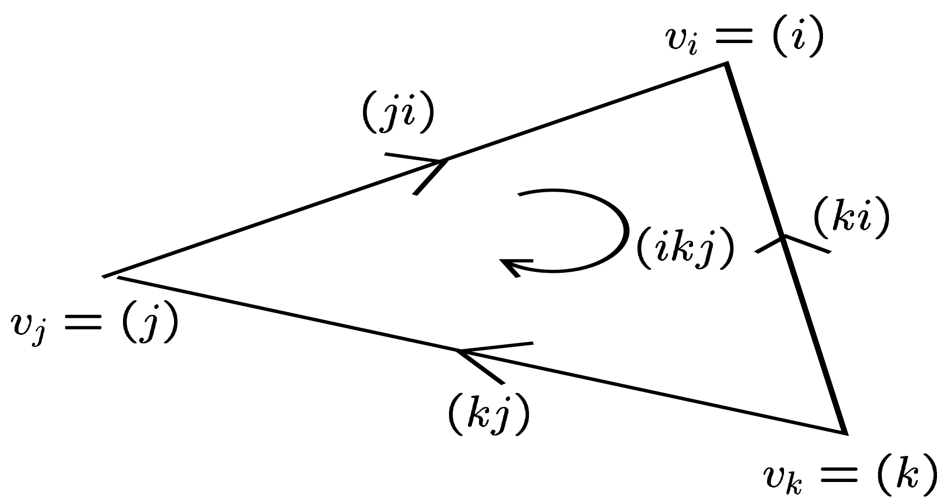


For the face in the diagram, the boundary is $\partial(f) = e_i + e_j - e_k$.

$$(\delta F)(f) = F(\partial f)$$

The map δ is called the coboundary operator. If you are using an interpolation scheme, all the data lie on the vertices. The set of faces is then a set $\mathcal{F}$ of ordered triples of indices and the set of edges $\mathcal{E}$ is a set of ordered pairs of indices.

The ordering gives the orientation.



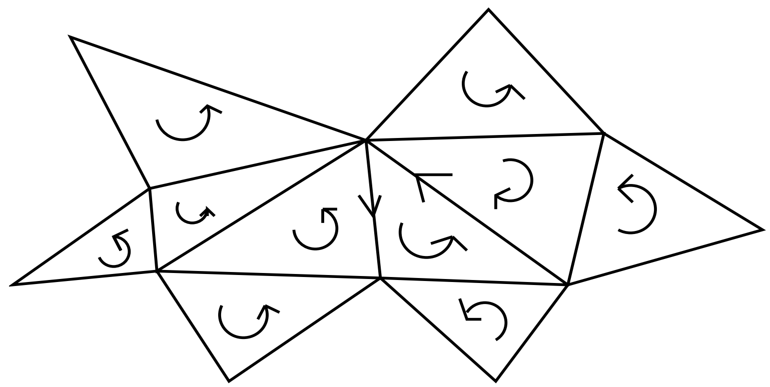
A telescoping/coboundary sum looks like cyclic sums,

$$(\delta F)(ikj) = F(kj) + F(ji) + F(ik)$$

The projection of $\int_X L[u]\,\mathrm{d}x$ is

$$\sum_{f_i} (-1)^{|f_i|} \int_{f_i} \Pi(L[u])\,\mathrm{d}x \qquad (2)$$

where $|f_i| = 1$ if f_i has the anti-clockwise orientation, and $|f_i| = -1$ for the reverse orientation.



If $\Pi(L)$ is a coboundary, the sums over the internal edges will cancel: (2) will depend only on the boundary data.

Well known: we have a discrete Stokes' theorem,

$$\int_X \delta F = \int_{\partial X} F.$$

The simplicial theory is attractive. It allows us to use results and intuition from classical work on triangulations. It generalizes to n dimensions.

From FE forms to simplicial cochains Let the top, i.e. n dimensional simplices be denoted by τ. Given a piecewise defined n-form on the τ, a simplicial n-cochain is achieved by integrating the form on the τ. This map is the de Rham map. We will denote it by $\int$:

$$\omega \in \mathcal{F}^n, \quad (\int \omega)(\tau) = \int_\tau \omega.$$

Theorem

The FE variational complex, shown here for a three dimensional base space,

$$0 \to \mathbb{R} \to \widetilde{\mathcal{F}}_0 \xrightarrow{\mathrm{d}} \widetilde{\mathcal{F}}_1 \xrightarrow{\mathrm{d}} \widetilde{\mathcal{F}}_2 \xrightarrow{\mathrm{d}} \widetilde{\mathcal{F}}_3 \xrightarrow{\mathcal{EL}} \mathcal{F}^1_* \xrightarrow{\mathrm{d}^*} \mathcal{F}^2_* \xrightarrow{\mathrm{d}^*} \cdots$$

is locally exact.

$\widetilde{\mathcal{F}}_*$ is the algebra generated by the $\mathcal{F}_i$ with unevaluated degrees of freedom

$\mathcal{EL} = \pi \circ \widehat{\mathrm{d}} \circ \int$

$\mathrm{d}^* = \pi \circ \widehat{\mathrm{d}}$ is the analogue of the vertical exterior derivative, modulo boundary terms

$\mathcal{F}^*_*$ is the algebra of vertical forms tensored with the space of n-dimensional simplicial co-chains

Group actions on moments

The clue is the variational symmetry group action on $\int_\Omega L(x,u,\cdots)\,\mathrm{d}x$: define

$$\epsilon\cdot\int_\mathcal{T} u(x)\psi_\mathcal{T}(x)\,\mathrm{d}x$$

$$=\int_\mathcal{T}\epsilon\cdot u(x)\psi_\mathcal{T}(\epsilon\cdot x)\frac{\mathrm{d}\epsilon\cdot x}{\mathrm{d}x}\,\mathrm{d}x$$

Example Recall the projective action

$$\epsilon\cdot x=\frac{x}{1-\epsilon x},\qquad \epsilon\cdot u(x)=\frac{u(x)}{1-\epsilon x}$$

Then the induced action on the moments

$$\alpha_n=\int_{x_n}^{x_{n+1}}\frac{u(x)}{x^3}\,\mathrm{d}x,\qquad \beta_n=\int_{x_n}^{x_{n+1}}\frac{u(x)}{x^4}\,\mathrm{d}x$$

is

$$\epsilon\cdot\alpha_n=\alpha_n,\qquad\qquad \epsilon\cdot\beta_n=\beta_n-\epsilon\alpha_n$$

Example (cont.) In general for the projective action,

$$\epsilon \cdot \int_{x_n}^{x_{n+1}} x^m u(x)\,\mathrm{d}x$$

$$= \int_{x_n}^{x_{n+1}} \frac{x^m}{(1-\epsilon x)^m} \frac{u(x)}{1-\epsilon x} \frac{\mathrm{d}x}{(1-\epsilon x)^2}$$

$$= \int_{x_n}^{x_{n+1}} \frac{x^m u(x)}{(1-\epsilon x)^{m+3}}\,\mathrm{d}x$$

THINK: if you want a coherent scheme which maps to itself under this projective action, and involves only a finite amount of data, then take your moments to be

$$u(x) \mapsto \int_{x_n}^{x_{n+1}} \frac{u(x)}{x^m}\,\mathrm{d}x, \qquad m = 3, 4, \ldots N.$$

CONCLUSIONS

- The underlying algebraic pattern of the exact variational complexes provide a framework for generalisations of Noether's Theorem and conservation laws in general.

- Symmetry-adapted moments would appear to be necessary.

- Moving frames yield invariant discrete Lagrangians.

- Open: how do coboundaries etc look in the usual FEM data representations?