Digital Atlases and Difference Forms

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Based on joint work with

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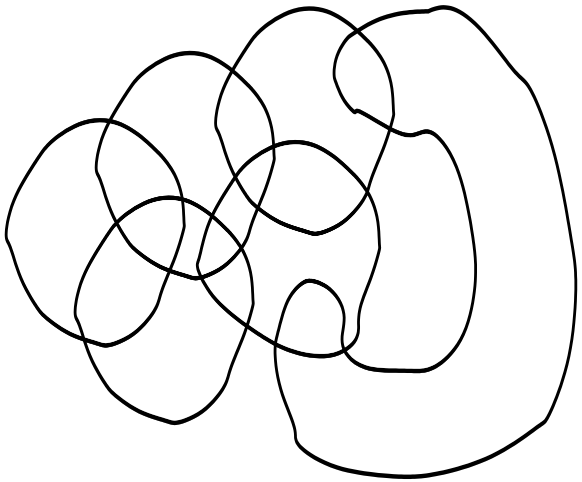
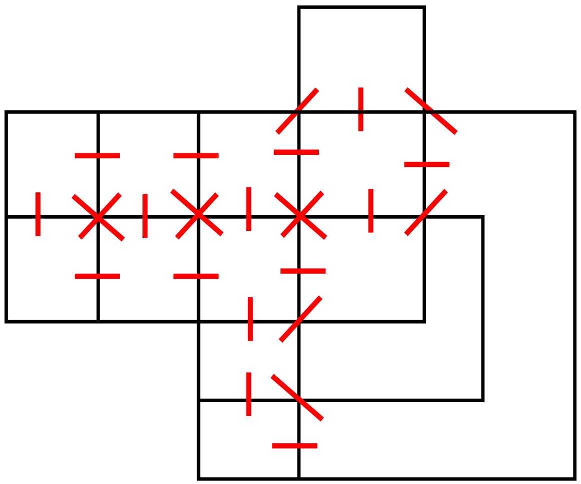
An atlas is a covering of your space by maps.

- overlap



- holes
- $\cap$ need not be connected

- adjacency



Let the space be

$$M = \cup_{\alpha \in \mathcal{A}} U_\alpha$$

Given functions $f_\alpha : U_\alpha \to \mathbb{R}$, when do they piece together to form a function on M? Answer:

$$(f_\alpha - f_\beta)|_{U_\alpha \cap U_\beta} = 0.$$

- Explore this compatibility condition.

Notation: $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, etc. Consider the simplest functions

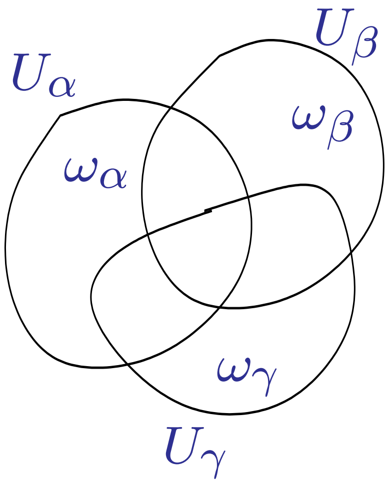
$$\check{C}^i(M) = \text{locally constant functions on the } U_{\alpha_0 \ldots \alpha_i}$$

called Čech co-chains.

Write the compatibility condition as

$$\delta : \check{C}^0(M) \to \check{C}^1(M)$$

Given $\omega \in \check{C}^0$ with value ω_α on U_α, then $(\delta\omega)_{\alpha\beta} = \omega_\alpha - \omega_\beta$. That is, the value of $\delta\omega$ on $U_{\alpha\beta}$ is $\omega_\alpha - \omega_\beta$. Note ordered index now.



Want $\delta : \check{C}^1 \to \check{C}^2$

to be the compatibility condition of the compatibility condition. Given $\zeta \in \check{C}^1$ with value $\zeta_{\alpha\beta}$ on $U_{\alpha\beta}$, define $(\delta\zeta)_{\alpha\beta\gamma} = \zeta_{\beta\gamma} - \zeta_{\alpha\gamma} + \zeta_{\alpha\beta}$.

Then $0 \equiv (\omega_\beta - \omega_\gamma) - (\omega_\alpha - \omega_\gamma) + (\omega_\alpha - \omega_\beta)$

for any $\omega \in \check{C}^0$. Thus $\delta \circ \delta = 0$.

Keep going! Define

$$\delta : \check{C}^i \to \check{C}^{i+1}$$

by

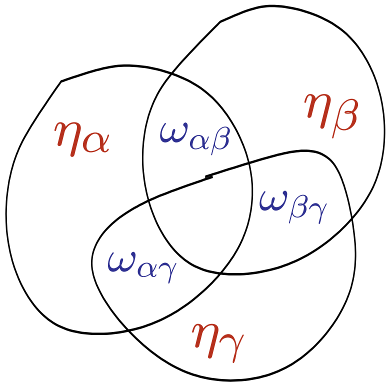
$$(\delta\omega)_{\alpha_0\ldots\alpha_{i+1}} = \sum_{j=0}^{i+1} (-1)^j \omega_{\alpha_0\ldots\widehat{\alpha_j}\ldots\alpha_{i+1}}.$$

Since $\delta^2 = 0$, each $\delta|_{\check{C}^i}$ is the compatibility condition of $\delta|_{\check{C}^{i-1}}$.

•Given $\omega \in \check{C}^i$, $i > 0$, such that $\delta\omega = 0$, we can solve the equation $\omega = \delta\eta$ for η locally.

This generalises the idea of extending the domain of $\omega \in \check{C}^0$ if $\delta\omega = 0$.

$\check{C}^1$:



local means restrict to $U_\alpha \cup U_\beta \cup U_\gamma$

Given $\omega \in \check{C}^1$, so that the value of ω on $U_{\alpha\beta}$ is $\omega_{\alpha\beta}$, such that $\delta\omega = 0$, that is $\omega_{\beta\gamma} - \omega_{\gamma\alpha} + \omega_{\alpha\beta} = 0$. find $\eta \in \check{C}^0$, locally, with value η_α on U_α, such that $\eta_\alpha - \eta_\beta = \omega_{\alpha\beta}$ and so forth, i.e. $\omega = \delta\eta$ locally.

So, let $\eta_\alpha = c$ for some constant c. Then we must have

$$\begin{cases} \eta_\beta &= c - \omega_{\alpha\beta} \\ \eta_\gamma &= c - \omega_{\alpha\gamma} \end{cases}$$

We need only verify the third condition, that $\eta_\beta - \eta_\gamma = \omega_{\beta\gamma}$ but we have that $-\omega_{\alpha\beta} + \omega_{\alpha\gamma} = \omega_{\beta\gamma}$ from $\delta\omega = 0$ on $U_{\alpha\beta\gamma}$.

So far we have

$$0 \longrightarrow \underbrace{\mathbb{R} \stackrel{\text{inc}}{\longrightarrow}}_{\text{global constant fns}} \check{C}^0 \stackrel{\delta}{\longrightarrow} \check{C}^1 \stackrel{\delta}{\longrightarrow} \check{C}^2 \stackrel{\delta}{\longrightarrow} \cdots$$

• $\delta^2 = 0$

• "locally exact", that is, there exist local pre-images for ω such that $\delta\omega = 0$.

Thus, those ω such that $\delta\omega = 0$ and having no global pre-images, that is, there does not exist η s.t. $\delta\eta = \omega$, η globally defined, encode global information of some sort.

•Čech cohomology

•Depends purely on the "combinatorics of the atlas".

•Can be defined for a wide range of spaces, not just smooth.

What is cohomology?

Given a complex $\mathcal{C}$ such that

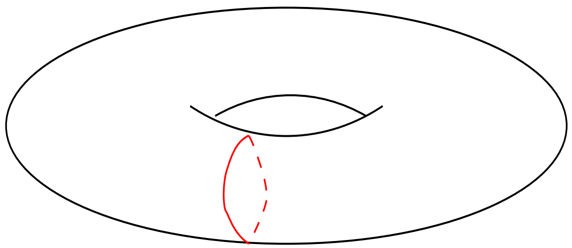
$$\cdots \xrightarrow{\delta_{i-1}} A^i \xrightarrow{\delta_i} A^{i+1} \xrightarrow{\delta_{i+1}} A^{i+2} \xrightarrow{\delta_{i+2}} \cdots$$

where

- the A^i are linear spaces
- the δ_i are linear maps
- $\delta_{i+1} \circ \delta_i = 0$ for all i

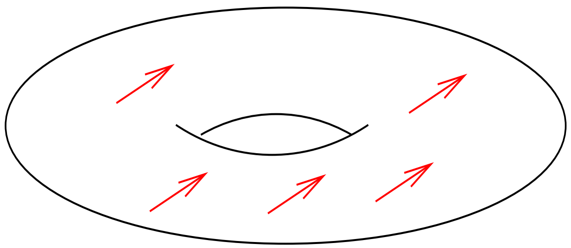
$$\begin{aligned} H^i(\mathcal{C}) &= \frac{\ker \delta_i}{\mathrm{im}\delta_{i-1}} \\ &= \frac{\langle \text{ closed forms : } \delta\omega = 0 \rangle}{\langle \text{ exact forms : } \omega = \delta\eta \rangle} \end{aligned}$$

Global structure of a manifold is often described by either a



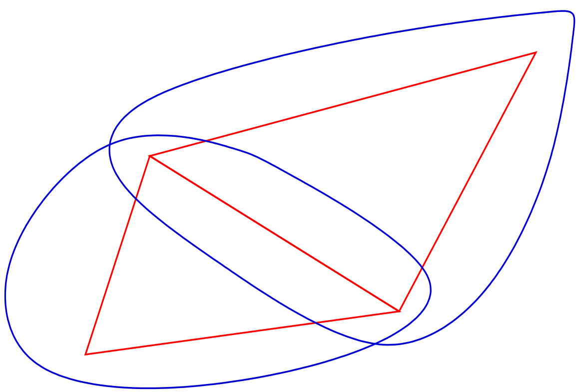
• triangulation: simplicial (co)homology

Do there exist circles that are not boundaries?



• exterior calculus: de Rham cohomology

Is there a zero curl vector field that is not a gradient?

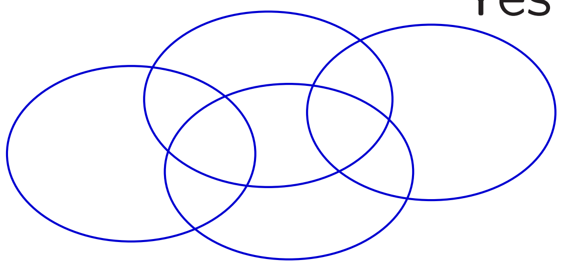


Easy to see that covers and triangulations are related: the global information is the same.

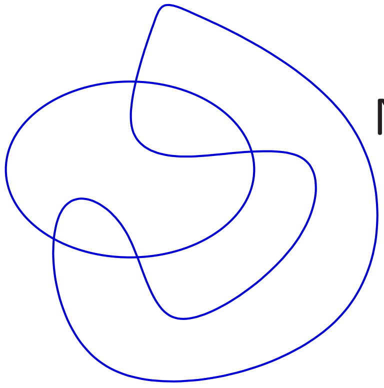
Famous proof by Weil that shows exterior calculus and “good covers” give the same global information: the cohomologies are isomorphic.

A good cover is one where all the U_α, $U_{\alpha\beta}$, $U_{\alpha\beta\gamma}$ are contractible to a point.

Yes



No



- The proof is constructive and 99% algebraic.

WHY is there a chance that de Rham and Čech cohomologies might be isomorphic?

$$0 \longrightarrow \underbrace{\mathbb{R} \xrightarrow{\text{inc}}}_{\text{same initial space:}} \check{C}^0 \xrightarrow{\delta} \check{C}^1 \xrightarrow{\delta} \check{C}^2 \xrightarrow{\delta} \cdots$$

$$0 \longrightarrow \overbrace{\mathbb{R} \xrightarrow{\text{inc}}}^{\text{global constant fns}} \Lambda^0 \xrightarrow{\text{d (grad)}} \Lambda^1 \xrightarrow{\text{d (curl)}} \Lambda^2 \xrightarrow{\text{d (div)}} \cdots$$

- $\delta^2 = 0$ and $\mathrm{d}^2 = 0$: each is the compatibility condition of its previous.
- Each sequence is locally exact.

WANT an analogous theory adapted to solving difference equations on globally non-trivial spaces.

- Not just equations with continuum limits
- Not just spaces approximating manifolds

- Conserved quantities relate to global structures, e.g. periodic boundary conditions imply space is a torus.
- Monodromy.
- Topological invariants and obstructions impact solution space: 1) the weather sits on a sphere, and 2) mathematical physics, kinks, monopoles, skyrmions etc . . . .

Is there a difference analogue of de Rham?

Recall the exterior derivative

$$\begin{aligned} \mathrm{d}\, f(x,y) &= f_x \mathrm{d}x + f_y \mathrm{d}y \\ \mathrm{d}(f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y) &= (g_x - f_y)\mathrm{d}x\mathrm{d}y \end{aligned}$$

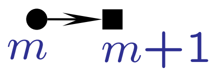
Forward difference analogue:

$$\begin{aligned} \Delta\, f(m,n) &= (S_m - \mathrm{id})f\, \Delta_m + (S_n - \mathrm{id})f\, \Delta_n \\ \Delta(f(m,n)\Delta_m + g(m,n)\Delta_n) &= ((S_m - \mathrm{id})g - (S_n - \mathrm{id})f)\, \Delta_m \Delta_n \end{aligned}$$

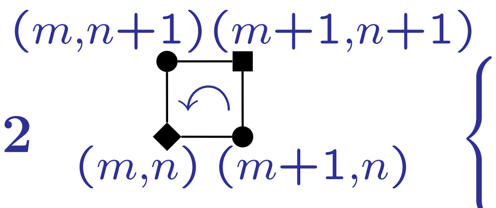
• $\Delta^2 = 0$ • $\Delta(\text{constant fn}) = 0$
but nothing else: Δ is not a derivation nor is there a tangent space of any kind.

Fundamental cubes are unit cubes in $\mathbb{Z}^p$ together with what kinds of forms are defined where.

$\boldsymbol{p = 0}$  only 0-forms i.e. functions.

$\boldsymbol{p = 1}$ 

- arrow indicates orientation.
- 0 and 1-forms defined at m.

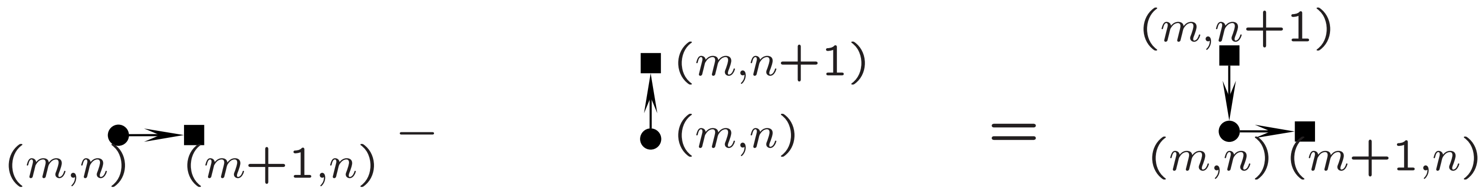
$\boldsymbol{p = 2}$ 

- arrow gives the orientation.
- 0, 1 and 2 forms defined at (m, n).

Can think of lattice varieties as either sums or unions of cubes.

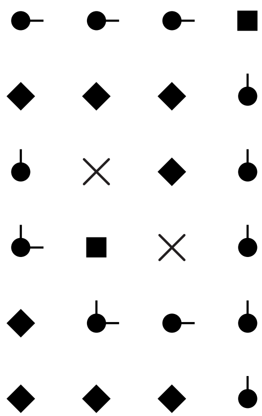
For example,

(1)



where minus means reverse the orientation.

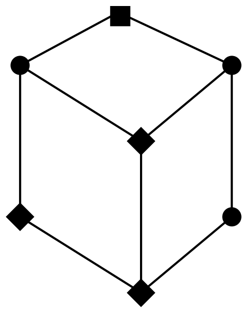
(2) Lattice with two points removed



Key:

- ◆ two form defined
- • one form defined in direction indicated
- ■ zero form defined
- × point removed

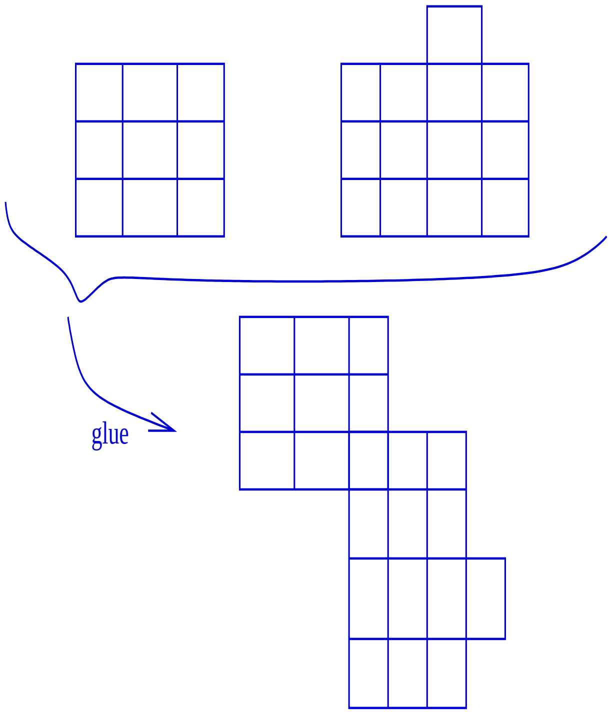
(3) A corner of a cube: a sum of three 2-cubes. If a form is defined at a point, it stays defined after adding.



From atlases to the global object, glue pieces together.

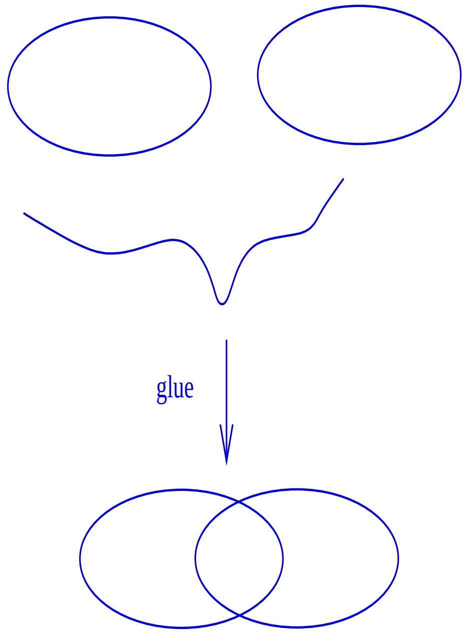
for lattice varieties

must maintain adjacency

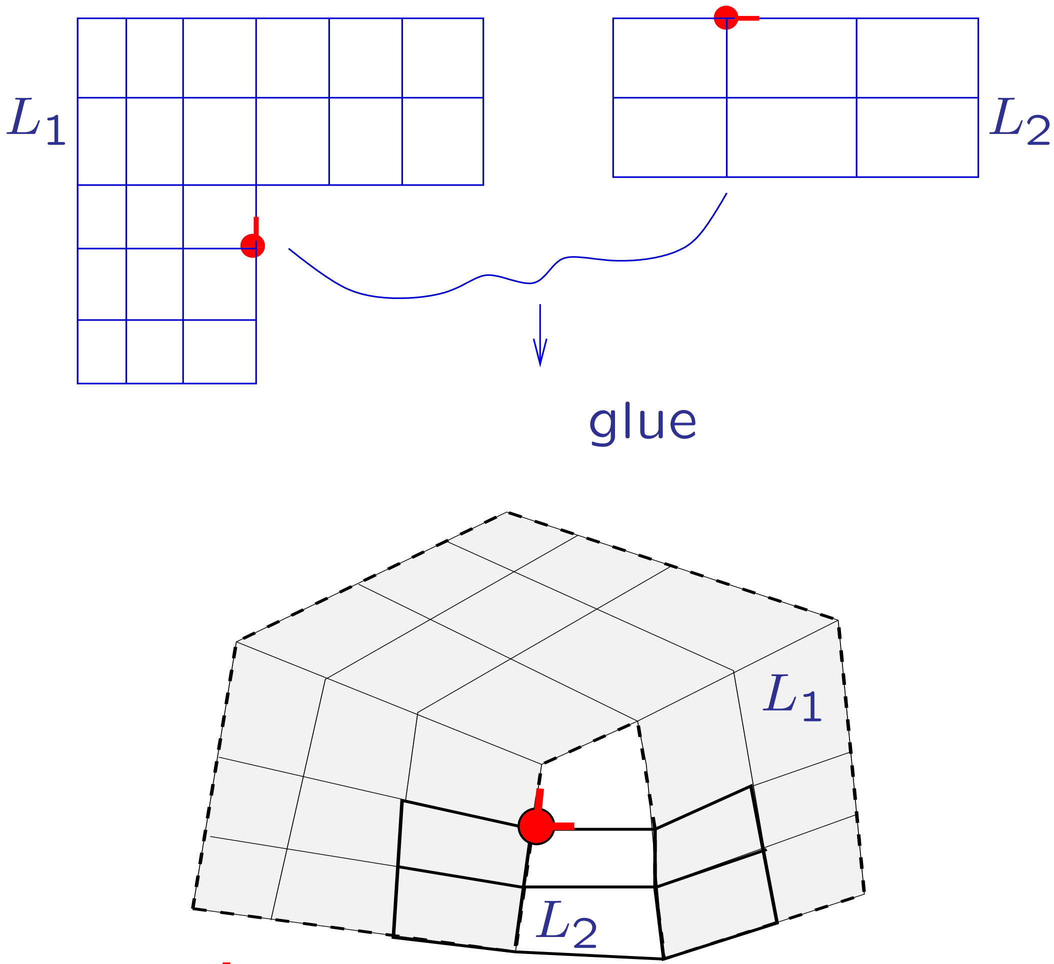


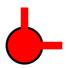
for manifolds

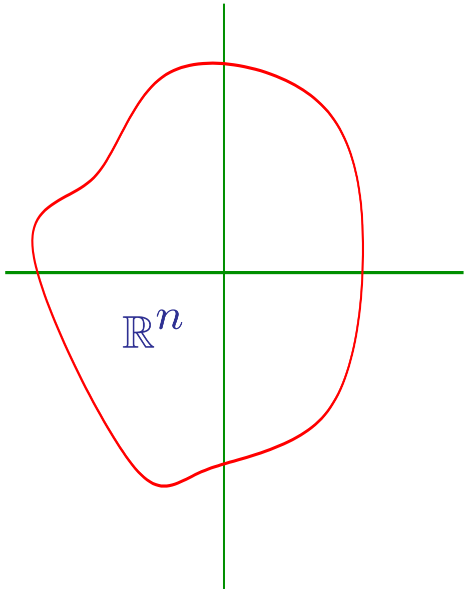
must maintain smoothness



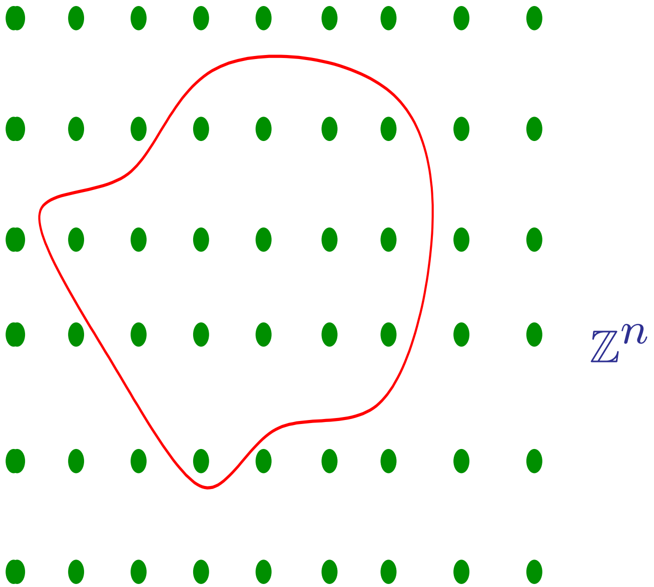
(4)



 two-form undefined, as not defined in either L_1 nor L_2



open set, continuity



ordering on $\mathbb{Z}$, adjacency

- Continuity is an illusion.

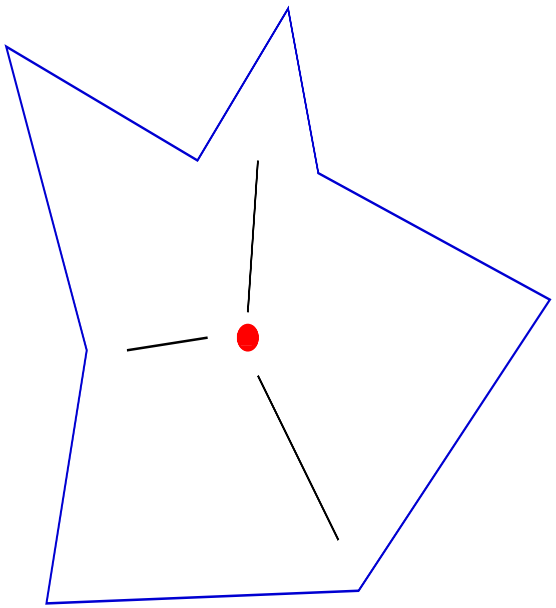
Poincaré Lemma

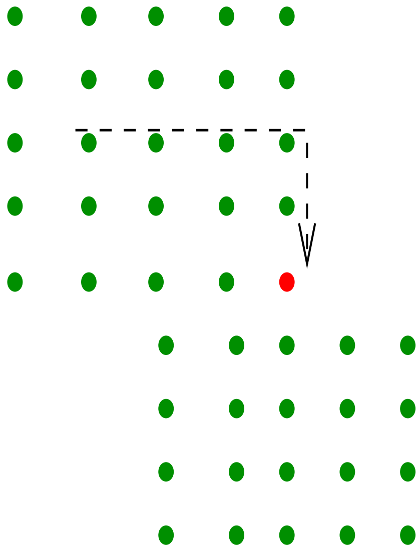
Local exactness of de Rham proved on starshaped domains

both proofs constructive

ELM and Hydon

Local exactness of difference complex proved on projectible domains





all paths first horizontal, then vertical via adjacent points

Write the difference complex on the lattice variety L as

$$0 \longrightarrow \mathbb{R} \overset{\text{inc}}{\longrightarrow} \mathbf{Ex}^0 \overset{\Delta}{\longrightarrow} \mathbf{Ex}^1 \overset{\Delta}{\longrightarrow} \mathbf{Ex}^2 \overset{\Delta}{\longrightarrow} \cdots$$

where **Ex** is for Exterior Algebra. Again, this is locally but not globally exact.

Major result 2 (ELM and Hydon): Given L, the Čech and Δ cohomologies are isomorphic, provided that the Čech cohomology is calculated with respect to a cover $L = \cup L_\alpha$ that satisfies

- L_α, $L_{\alpha\beta} = L_\alpha \cap L_\beta$ etc are all projectible
- any form at $P \in L$ is defined in at least one of the L_α such that $P \in L_\alpha$.

Why is this important?

- If you take a cover of your manifold and match it one-to-one with a cover of your lattice approximation, you guarantee matching global forms.

- The proofs are indpendent of any continuum limit, so get results for inherently discrete spaces; can calculate globally closed ($\Delta\omega = 0$) but not exact ($\omega \neq \Delta\eta$) forms simply by knowing the pattern of intersections.

Glimpse of the proof

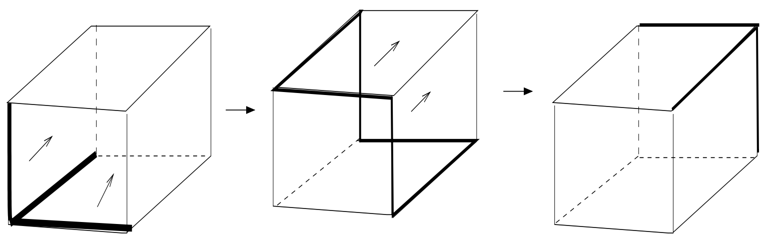
$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \uparrow \boldsymbol{\Delta} & & \uparrow \boldsymbol{\Delta} & & \uparrow \boldsymbol{\Delta} & & \uparrow \boldsymbol{\Delta} \\
0 & \longrightarrow & \mathbf{Ex}^2(L) & \xrightarrow{r} & \bigoplus \mathbf{Ex}^2(L_\alpha) & \xrightarrow{\delta} & \bigoplus \mathbf{Ex}^2(L_{\alpha\beta}) & \xrightarrow{\delta} & \bigoplus \mathbf{Ex}^2(L_{\alpha\beta\gamma}) & \xrightarrow{\delta} \\
 & & \uparrow \boldsymbol{\Delta} & & \uparrow \boldsymbol{\Delta} & & \uparrow \boldsymbol{\Delta} & & \uparrow \boldsymbol{\Delta} \\
0 & \longrightarrow & \mathbf{Ex}^1(L) & \xrightarrow{r} & \bigoplus \mathbf{Ex}^1(L_\alpha) & \xrightarrow{\delta} & \bigoplus \mathbf{Ex}^1(L_{\alpha\beta}) & \xrightarrow{\delta} & \bigoplus \mathbf{Ex}^1(L_{\alpha\beta\gamma}) & \xrightarrow{\delta} \\
 & & \uparrow \boldsymbol{\Delta} & & \uparrow \boldsymbol{\Delta} & & \uparrow \boldsymbol{\Delta} & & \uparrow \boldsymbol{\Delta} \\
0 & \longrightarrow & \mathbf{Ex}^0(L) & \xrightarrow{r} & \bigoplus \mathbf{Ex}^0(L_\alpha) & \xrightarrow{\delta} & \bigoplus \mathbf{Ex}^0(L_{\alpha\beta}) & \xrightarrow{\delta} & \bigoplus \mathbf{Ex}^0(L_{\alpha\beta\gamma}) & \xrightarrow{\delta} \\
\hline
 & & \uparrow & & \uparrow s & & \uparrow s & & \uparrow s \\
 & & 0 & \longrightarrow & \check{\mathcal{C}}^0 & \xrightarrow{\delta} & \check{\mathcal{C}}^1 & \xrightarrow{\delta} & \check{\mathcal{C}}^2 & \xrightarrow{\delta} \\
 & & & & \uparrow & & \uparrow & & \uparrow \\
 & & & & 0 & & 0 & & 0
\end{array}
$$

Applications to studying global solution spaces

Want to solve, globally, a 4-point scheme on a lattice sphere

$0 = au_{m,n} + bu_{m+1,n} + cu_{m,n+1} + du_{m+1,n+1} = 0$, $abcd \neq 0$.

Start with initial data as shown on the left, solve for u on front, left and lower surfaces...



$\sharp$initial conditions $-$ $\sharp$compatibility conditions $\equiv 2$.

The number 2 is famously associated with the sphere.

$$\begin{aligned} 2 &= \text{Betti number of the sphere} \\ &= \textstyle\sum_i (-1)^i \dim \check{C}^i(S^2) \\ &= \textstyle\sum_i (-1)^i \dim \Lambda^i(S^2) \end{aligned}$$

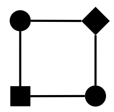
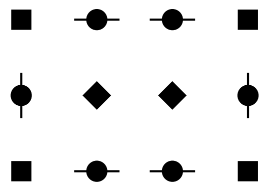
Conjecture

for linear systems on boundary free lattice varieties L,

$$\sharp\text{initial conditions} - \sharp\text{compatibility conditions} \equiv \sum_i (-1)^i \dim \mathbf{Ex}^i(L).$$

Reminiscent of Morse Index Theorem: there is a discrete Morse theory by Forman which has no connection.

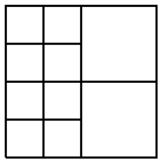
Not just Forward Difference On the left is shown a fundamental 2-cube for a backward difference, while on the right an example is shown of a fundamental 2-cell for a collocation scheme.

Hydon: fundamental cubes exist for Gauss-Legendre, Marker and Cell and Preissman schemes.

Open Problems

The main open problem is to include localised refinements. These violate the adjacency condition stipulated in the lattice variety construction.



Another is to consider non-orientable lattice varieties, such as a lattice Möbius band.

THANK YOU!!!