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Web-based Supplementary Materials for Parameter Redundancy in Mark-Recovery Models

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Web Appendix A: Extended Models

In the main paper we considered models with a separate survival probability for animals in their first year of life. We may wish to consider the survival probability separately for longer than a year, for example if an animal is juvenile for longer than a year. To include such cases we extend the models examined to consider survival separately for the first J years of their life. The animal would then be considered an adult for years $J+1$ onwards. A similar x/y/z notation is used. Here x represents the first J years of life with the following options:

(i) Dependent on age for years 1 to J ($A_{1:J}$). (Note that $A_{1:1}$ is equivalent to C.)

(ii) Dependent on age and time for years 1 to J ($A_{1:J}$,T). (Note that $A_{1:1}$,T is equivalent to T.)

The options for y, the adult survival probability, are:

(i) Constant (C).

(ii) Dependent on time (T).

(iii) Dependent on age (A).

(iv) Dependent on age and time (A,T).

The options for z, the recovery probability, are:

(i) Constant (C).

(ii) Dependent on time (T).

(iii) Dependent on age (A).

(iv) Dependent on age for years 1 to J with separate adult recovery ($A_{1:J+1}$).

(v) Dependent on time and age for years 1 to J with separate adult recovery ($A_{1:J+1}$,T).

(vi) Dependent on age and time (A,T).

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For example if $J = 3$, survival is considered separately for the first, second and third years of life. Thereafter the animals are considered to be adults. The model $A_{1:3}$/C/C would have survival parameters ϕ_1, ϕ_2, ϕ_3 for the first three years and survival parameter ϕ_a for adult years, as well as constant recovery probability λ. Parameter redundancy results are given for all such models in Table 1 and the estimable parameter combinations are given in Table 2. The following terms are used in the tables $E = n_1 n_2 - \frac{1}{2}n_1^2 + \frac{1}{2}n_1$, $B = \frac{1}{2}J(2n_2 - J + 1)$, $S = \frac{1}{2}[(n_2 - n_1 - 1)^2 + (n_2 - n_1 - 1)]$, $G = S$ if $n_2 < n_1 + J$, $G = \frac{1}{2}(J-1)(2n_2 - J - 2n_1)$ if $n_2 \geq n_1 + J$ (note that in the common case of $n_1 = n_2$ $S = 0$ and $G = 0$).

The same methods are used to determine the rank as described in the main paper. Method 1 uses the extension theorem for full rank models, method 2 uses reparameterisation and in method 3 the rank is limited by the number of unique exhaustive summary terms. Here method 2 is also used for full rank models. Reparameterisation is needed to allow the use of the extension theorem to generalise results for any J.

In the extended x/y/z family of models, survival is considered separately for each year an animal is a juvenile. Models of interest may also include having juvenile survival the same for some or all the juvenile years. More formally this is $A_{1:J}$/*/* models, such that there is $\phi_j = \phi_i$ for at least one i and j (with $i \neq j \leq J$) for $J > 1$, and ($A_{1:J}$,T)/*/* models such that $\phi_{j,t} = \phi_{i,t}$ for at least one i and j (with $i \neq j \leq J$) for $J > 1$ and for all t. Alternatively the model of interest could have some juvenile survival probabilities time dependent and others constant, which are ($A_{1:J}$,T)/*/* models with $\phi_{i,t} = \phi_i$ for at least one but not all $i \leq J$. Results for these models are also given in Table 1 in the last column of the table, labeled K. If the original extended x/y/z model is full rank (has deficiency zero), this calculation is relatively easy using a modified PLUR decomposition (also known as a Turing factorisation) of the derivative matrix, for which Maple has an intrinsic procedure. Here we write the derivative matrix of any full rank model as $\mathbf{D} = \mathbf{PLUR}$, where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ is a lower diagonal matrix with ones on the diagonal, $\mathbf{U}$ is a upper diagonal matrix and $\mathbf{R}$ is in reduced echelon form. As long as the determinant of $\mathbf{U}$ is not equal to zero, any nested models will also be full rank; see Cole et al (2010). If the original extended x/y/z model is parameter redundant further calculation is required, unless method 3 was used, when it is obvious that the rank will still be limited by the number of unique exhaustive summary terms.

The effect on the rank or deficiency for models with one or more of the juvenile survival probabilities being equal is also given in Table 1, in the column labeled K. There are the following options:

(i) $K = 1$ the deficiency is still 0, but the rank will change (the model remains full rank);

(ii) $K = 2$ the rank remains the same, but the deficiency will change;

(iii) $K = 3$ the deficiency decreases to 0, but the rank will change (the model is no longer parameter redundant);

(iv) $K = 4$ the deficiency is at least 1 (the model remains parameter redundant);

(v) $K = 5$ the deficiency is unchanged, but the rank will change.

(vi) $K = 6$ the deficiency remains at 0 if any $\phi_{i,t} = \phi_{k,t}$ (model remains full rank), however the deficiency increases to at least 1 if any $\phi_{i,t} = \phi_i$ (model becomes parameter redundant).

(vii) $K = 7$ the deficiency is at least 1 if any $\phi_{i,t} = \phi_{k,t}$ (the model remains parameter redundant). However if $\phi_{i,t} = \phi_i$ for at least one $2 \leq i \leq \min(3, J)$ the deficiency decreases to 0 (the model is no longer parameter redundant) otherwise the deficiency decreases but the model remains parameter redundant.

For example consider a model with time-dependent first-year survival probability $\phi_{1,t}$, constant second-year survival probability, ϕ_2, constant third-year survival probability, ϕ_3 and constant adult-survival probability, ϕ_a. This is a sub-model of the $A_{1:3}$,T/C/*, which has $K = 1$ if the reporting probability is constant (C) or time dependent (T). Therefore the models with parameters $\phi_{1,t}$, ϕ_2, ϕ_3, ϕ_a and λ or $\phi_{1,t}$, ϕ_2, ϕ_3, ϕ_a and λ_t parameters remain full rank. However when the reporting probability is dependent on age class ($A_{1:4}$) $K = 6$. Therefore the model with parameters $\phi_{1,t}$, ϕ_2, ϕ_3, ϕ_a, λ_1, λ_2, λ_3 and λ_a is no longer full rank.

Web Appendix B: Combined Adult and Juvenile Models

In some ringing examples birds are ringed as both juveniles and as adults. In such cases a combined model can be fitted combining both juvenile and adult data sets, with common parameters for adult survival. In Table 3 we examine how the deficiency of a model changes if we combine x/y/z models for the juvenile data set with y/z models models for the adult data set. Note that in the table only models that were parameter redundant individually are considered, as if the juvenile and adult models are full rank individually they will obviously still be full rank when combined. Also it is assumed that the adult data consist of adults of unknown age, therefore it is not possible under this framework to fit models combining */A/* models with A/* or models combining */(A,T)/* models with (A,T)/*, nor is it possible to combine */*/A models with */A models, and combine */*/(A,T) models with */(A,T) models. An alternative framework for models providing an age structure to ring-recovery data for animals with unknown age is given in McCrea et al (2010).

Determining the parameter redundancy of a model for two combined data sets is an extension of the standard derivative method. Details are given in Cole (2011).

The models C/C/$A_{1:2}$ with C/C, C/C/$A_{1:2}$,T with C/T and C/T/$A_{1:2}$ with T/C, which are parameter redundant individually, are full rank when combined. In these cases the animals have separate survival probabilities for their first year of life. If the animals have separate survival probabilities for more than one year of life then the models remain parameter redundant. In such cases the deficiency only decreases.

Web Appendix C: Imperfect Data

Here we consider how having only m diagonals of data ($N_{i,j} = 0$ if $j - i + 1 > m$) affects the parameter redundancy of x/y/z models, where the animal has separate survival probabilities for the first $J > 1$ years of life. These results are presented in Table 4.

Only four models never change. One model will only change if $m \leq J - 1$; seventeen models will only change if $m \leq J$, and three models will change if $m \leq J + 1$. The remainder of the models are parameter-redundant models limited by the number of exhaustive summary terms, which now reduces to E_m, where

$$E_m = \begin{cases} E - \frac{1}{2}(n_2 - m)(n_2 - m + 1) & n_2 - n_1 < m - 1 \\ mn_1 & n_2 - n_1 \geq m - 1 \end{cases} .$$

If a model assumes the animal is juvenile for J years, we recommend that there are at least $m = J + 1$ diagonals of perfect data, otherwise the rank of the model will decrease, resulting in more parameter-redundant models. In the lapwing data set (Table 1a of the main paper), the first three diagonals of data are complete. In this case if $J = 2$ all models that are full-rank model in the presence of perfect data will remain full-rank. If $J = 3$ most of these models will remain full rank (but the model $A_{1:3}$,T/T/A will now be parameter redundant). However if $J \geq 4$ many of the models which are full-rank in the presence of perfect data will now be parameter redundant.

Web Appendix D: Parameter redundancy in Conditional Models

The total number of birds ringed in each year may be unknown or unreliable. In such a case a model can be fitted by conditioning on the number of birds recovered from each cohort. Such models are similar to the standard ring-recovery models described in the main paper, but with each of the probabilities of being recovered equal to $P_{c,i,j}$ with,

$$P_{c,i,j} = \frac{\left(\prod_{k=i}^{j-1} \phi_{k-i+1,k}\right)(1 - \phi_{j-i+1,j})\lambda_{j-i+1,j}}{\sum_{h=i}^{n_2} \left(\prod_{k=i}^{h-1} \phi_{k-i+1,k}\right)(1 - \phi_{h-i+1,h})\lambda_{h-i+1,h}},$$

for $i = 1, ..., n_1$ and $j = i, ..., n_2$, where again $\phi_{i,j}$ is the probability of an animal aged i surviving the jth year of the study and where $\lambda_{i,j}$ is the probability of recovering a dead animal aged i in the jth year of the study and n_1 and n_2 are again the number of years of ringing and recovery respectively. The survival and recovery parameters can be estimated using maximum likelihood, using the expression for the likelihood given below

$$L \propto \left(\prod_{i=1}^{n_1} \prod_{j=1}^{n_2} P_{c,i,j}^{N_{i,j}} \right).$$

This now excludes the probabilities of an animal being ringed but never seen again.

The rank and deficiency of this conditional model are given in Tables 5 to 7. Note that if the recovery parameter, λ, is constant then λ disappears from the model completely. It then obviously cannot be estimated and is excluded from the count of parameters in the model. Similarly in other models the parameters λ_{1,n_1} or ϕ_{1,n_1} do not appear in the model if $n_1 = n_2$. In such cases these completely redundant parameters are not counted in the number of parameters. This can sometimes lead to a different result for the rank or deficiency for $n_1 = n_2$, compared to $n_1 < n_2$.

The maximum rank for standard models, $E = n_1 n_2 - \frac{1}{2} n_1^2 + \frac{1}{2} n_1$, is based on the number of terms in the exhaustive summary. This now decreases to $E_c = n_1 n_2 - \frac{1}{2} n_1^2 - \frac{1}{2} n_1$ for conditional models. This is because

$$\mathbf{P}_c = \begin{bmatrix} \frac{P_{1,1}}{\sum_{h=1}^{n_2} P_{1,h}} & \frac{P_{1,2}}{\sum_{h=1}^{n_2} P_{1,h}} & \cdots & \frac{P_{1,n_2}}{\sum_{h=1}^{n_2} P_{1,h}} \\ 0 & \frac{P_{2,2}}{\sum_{h=2}^{n_2} P_{2,h}} & \cdots & \frac{P_{1,n_2}}{\sum_{h=2}^{n_2} P_{2,h}} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \frac{P_{n_1,n_2}}{\sum_{h=n_1}^{n_2} P_{n_1,h}} \end{bmatrix}$$

which can be rewritten as

$$\mathbf{P}_c = \begin{bmatrix} \frac{1}{1+\sum_{h=2}^{n_2} P_{1,h}} & \frac{P_{1,2}}{1+\sum_{h=2}^{n_2} P_{1,h}} & \cdots & \frac{P_{1,n_2}}{1+\sum_{h=2}^{n_2} P_{1,h}} \\ 0 & \frac{1}{1+\sum_{h=3}^{n_2} P_{2,h}} & \cdots & \frac{P_{1,n_2}}{1+\sum_{h=3}^{n_2} P_{2,h}} \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \frac{P_{n_1,n_2}}{1+\sum_{h=n_1+1}^{n_2} P_{n_1,h}} \end{bmatrix}$$

so that $P_{i,i}$ for $i = 1, \ldots, n_1$ no longer appears in the matrix. There are therefore n_1 fewer terms compared to the standard model.

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Table 1 Table of parameter redundancy for extended x/y/z models. Here d denotes the deficiency, r denotes the number of estimable parameters, and C corresponds to how the parameter redundancy results change if the survival probabilities are the same for any of the J juvenile years (see text for details).

			Valid for			
Model	r	d	$n_1 \geq$	$n_2 \geq$	M	K
$A_{1:J}$/C/C	$J+2$	0	$J+1$	$J+2$	2	1
$A_{1:J}$/C/T	$J+n_2+1$	0	$J+1$	$J+2$	2	1
$A_{1:J}$/C/A	n_2	$J+1$	$J+1$	$J+1$	3	2
$A_{1:J}$/C/$A_{1:J+1}$	$J+2$	J	$J+1$	$J+2$	2	3
$A_{1:J}$/C/($A_{1:J+1}$,T)	$n_1+B-J+1-G$	J	$J+1$	$J+2$	2	3
$A_{1:J}$/C/(A,T)	E	$J+1$	$J+1$	$J+1$	3	2
$A_{1:J}$/T/C	n_2+1	0	$J+1$	$J+2$	2	1
$A_{1:J}$/T/T	$\min(2n_2, n_2+n_1+J-1)$	$\max(0, n_2-n_1-J+1)$	$J+1$	$J+2$	2	5
$A_{1:J}$/T/A	$2n_2-J$	J	$J+3$	$J+3$	2	2
$A_{1:J}$/T/$A_{1:J+1}$	n_2+1	J	$J+1$	$J+2$	2	3
$A_{1:J}$/T/($A_{1:J+1}$,T)	$\begin{cases} B+n_1+n_2-2J-1-S & n_2<n_1+J \\ B-J+2n_1-1-G & n_2 \geq n_1+J \end{cases}$	$\begin{cases} J+1 & n_2<n_1+J \\ n_2-n_1+1 & n_2 \geq n_1+J \end{cases}$	$J+1$	$J+1$	2	4
$A_{1:J}$/T/(A,T)	E	n_2	$J+1$	$J+1$	3	2
$A_{1:J}$/(A,T)/C	$E+n_2-n_1-B+J+G$	1	$J+1$	$J+1$	3	3
$A_{1:J}$/(A,T)/T	$\begin{cases} E & J=1 \\ E-n_1-1+2n_2-B+J+S & J>1, n_2<n_1+J \\ E+n_2-B+G-2+2J & J>1, n_2 \geq n_1+J \end{cases}$	$\begin{cases} n_2-n_1+1 & J=1 \\ 1 & J>1, n_2<n_1+J \\ n_2-n_1-J+2 & J>1, n_2 \geq n_1+J \end{cases}$	$J+1$	$J+1$	3	3
$A_{1:J}$/(A,T)/A	$E+n_2-n_1-B+J+G$	n_2	$J+1$	$J+1$	3	2
$A_{1:J}$/(A,T)/$A_{1:J+1}$	$E+n_2-n_1-B+J+G$	$J+1$	$J+1$	$J+1$	3	2
$A_{1:J}$/(A,T)/($A_{1:J+1}$,T)	E	n_2	$J+1$	$J+1$	3	2
$A_{1:J}$/(A,T)/(A,T)	E	$E+n_2-n_1-B+J+G$	$J+1$	$J+1$	3	2
($A_{1:J}$,T)/C/C	$n_1-n_2+2+B-G$	0	$J+1$	$J+2$	2	1
($A_{1:J}$,T)/C/T	$n_1+B+1-G$	0	$J+2$	$2J+2$	2	1
($A_{1:J}$,T)/C/A	n_1+B-G	1	$J+2$	$2J+2$	2	7
($A_{1:J}$,T)/C/$A_{1:J+1}$	$n_1-n_2+B+J+2-G$	0	$2J$	$2J+1$	1	6
($A_{1:J}$,T)/C/($A_{1:J+1}$,T)	$\begin{cases} n_1+n_2+B-2J-1-S & n_2<n_1+J \\ B-J+2n_1-1-G & n_2 \geq n_1+J \end{cases}$	$\begin{cases} n_1-2n_2+B+J+2-S & n_2<n_1+J \\ B-n_2+2-G & n_2 \geq n_1+J \end{cases}$	$J+1$	$J+1$	2	4
($A_{1:J}$,T)/C/(A,T)	E	$n_1-n_2+B+1-G$	$J+1$	$J+1$	3	2
$A_{1:J}$,T/T/C	$n_1+B-J+1-G$	0	$J+1$	$J+2$	2	1
($A_{1:J}$,T)/T/T	$\begin{cases} n_1+n_2+B-2J-1-S & n_2<n_1+J \\ 2n_1+B-J-1-G & n_2 \geq n_1+J \end{cases}$	$\begin{cases} J+1 & n_2<n_1+J \\ n_2-n_1+1 & n_2 \geq n_1+J \end{cases}$	$J+1$	$J+1$	2	7
($A_{1:J}$,T)/T/A	$n_1+n_2+B-J-G$	0	$J+3$	$2J+3$	2	6
($A_{1:J}$,T)/T/$A_{1:J+1}$	$n_1+B+1-G$	0	$2J+1$	$2J+2$	2	6
($A_{1:J}$,T)/T/($A_{1:J+1}$,T)	$\begin{cases} n_1+n_2+B-2J-1-S & n_2<n_1+J \\ 2n_1+B-J-1-G & n_2 \geq n_1+J \end{cases}$	$\begin{cases} n_1-n_2+B+1-S & n_2<n_1+J \\ B-J+1-G & n_2 \geq n_1+J \end{cases}$	$J+1$	$J+1$	2	4
($A_{1:J}$,T)/T/(A,T)	E	$n_1+B-J-G$	$J+1$	$J+1$	3	2
($A_{1:J}$,T)/A/C	$n_1+B-J+1-G$	0	$J+1$	$J+2$	2	1
($A_{1:J}$,T)/A/T	$n_1+n_2+B-J-G$	0	$J+3$	$2J+3$	2	1
($A_{1:J}$,T)/A/A	n_1+B-G	n_2-J	$J+1$	$2J+1$	2	4
($A_{1:J}$,T)/A/$A_{1:J+1}$	n_1+B-G	1	$2J+1$	$2J+2$	2	4
($A_{1:J}$,T)/A/($A_{1:J+1}$,T)	$\begin{cases} n_1+2n_2+B-3J-3-S & n_2<n_1+J \\ 2n_1+n_2+B-2J-3-G & n_2 \geq n_1+J \end{cases}$	$\begin{cases} n_1-2n_2+B+J+3-S & n_2<n_1+J \\ B-n_2+3-G & n_2 \geq n_1+J \end{cases}$	$2J+2$	$2J+3$	2	4
($A_{1:J}$,T)/A/(A,T)	E	$n_1+B+J-G$	$J+1$	$J+1$	3	2

Table 2 Combinations of parameters that can be estimated for extended x/y/z models.

Model	Estimable Parameter Combinations
$A_{1:J}$/C/A	$\{P_{1,j}\}_{j=1,\ldots,n_2}$
$A_{1:J}$/C/$A_{1:J+1}$	$\phi_a, \left(\prod_{i=1}^{J}\phi_i\right)\lambda_a, \{P_{1,j}\}_{j=1,\ldots,J}$
$A_{1:J}$/C/($A_{1:J+1}$,T)	$\phi_a, \{P_{i,i+j-1}\}_{\substack{i=1,\ldots,n_1-j+1\\ j=1,\ldots,J}}, \{(\prod_{j=1}^{J}\phi_j)\lambda_{a,i}\}_{i=J+1,\ldots,n_2}$
$A_{1:J}$/C/(A,T)	$\{P_{i,j}\}_{\substack{j=i,\ldots,n_2\\ i=1,\ldots,n_1}}$
$A_{1:J}$/T/T*	$\{\phi_i\}_{i=1,\ldots,n_1+J-1}, \{\lambda_i\}_{i=1,\ldots,n_1+J-1}, \{P_{i,j}\}_{\substack{j=n_1+J,\ldots,n_2\\ i=1,\ldots,n_1}}$
$A_{1:J}$/T/A	$\{\phi_i\}_{i=J+1,\ldots,n_2}, \{\lambda_i\prod_{k=1}^{J}\phi_k\}_{i=J+1,\ldots,n_2}, \{P_{1,j}\}_{j=1,\ldots,J}$
$A_{1:J}$/T/$A_{1:J+1}$	$\{\phi_i\}_{i=J+1,\ldots,n_2}, \lambda_a\prod_{i=1}^{J}\phi_i, \{P_{1,j}\}_{j=1,\ldots,J}$
$A_{1:J}$/T/($A_{1:J+1}$,T)	$\{\phi_i\}_{i=J+1,\ldots,\min(n_2-1,n_1)}, \{P_{i,i+j-1}\}_{\substack{i=1,\ldots,n_1-j+1\\ j=1,\ldots,J}}, \{\lambda_{a,i}\prod_{k=1}^{J}\phi_k\}_{i=J+1,\ldots,\min(n_2-1,n_1)},$
	$\left\{(1-\phi_{n_j})\lambda_{a,j}\prod_{k=1}^{J}\phi_k\prod_{k=\min(n_2,n_1+1)}^{n_j-1}\phi_k\right\}_{j=\min(n_2,n_1+1),\ldots,n_2}$
$A_{1:J}$/T/(A,T)	$\{P_{i,j}\}_{\substack{j=i,\ldots,n_2\\ i=1,\ldots,n_1}}$
$A_{1:J}$/(A,T)/C	$\{P_{1,j}\}_{j=1,\ldots,J}, \{P_{i,j}\}_{\substack{j=i,\ldots,n_2\\ i=J+1,\ldots,n_1}}$
$A_{1:J}$/(A,T)/T $J=1$	$\{P_{i,j}\}_{\substack{j=n_1+1,\ldots,n_2\\ i=2,\ldots,n_1}}, \{(1-\phi_1)\lambda_i\}_{i=1,\ldots,n_1}, \left\{\frac{1-P_{i,j}}{(1-\phi_1)\lambda_j}\right\}_{\substack{j=i,\ldots,n_1\\ i=2,\ldots,n_1}}$
$A_{1:J}$/(A,T)/T $J>1$	$\{(1-\phi_1)\lambda_i\}_{i=1,\ldots,n_1}, \left\{\frac{1-\prod_{k=1}^{j}\phi_k}{1-\phi_1}\right\}_{j=2,\ldots,J}, \left\{\frac{1-P_{i,j}}{(1-\phi_1)\lambda_j}\right\}_{\substack{j=i,\ldots,n_1\\ i=J+1,\ldots,n_1}}$
$A_{1:J}$/(A,T)/A	$\{P_{1,j}\}_{j=1,\ldots,n_2}, \left\{\frac{P_{i,j}}{P_{1,j-1}}\right\}_{\substack{j=i,\ldots,n_2\\ i=J+1,\ldots,n_1}}$
$A_{1:J}$/(A,T)/$A_{1:J+1}$	$\{P_{1,j}\}_{j=1,\ldots,J}, \{P_{i,j}\}_{\substack{j=J+1,\ldots,n_2\\ i=1,\ldots,n_1}}$
$A_{1:J}$/(A,T)/($A_{1:J+1}$,T)	$\{P_{i,j}\}_{\substack{j=i,\ldots,n_2\\ i=1,\ldots,n_1}}$
$A_{1:J}$/(A,T)/(A,T)	$\{P_{i,j}\}_{\substack{j=i,\ldots,n_2\\ i=1,\ldots,n_1}}$
($A_{1:J}$,T)/C/A	$\{\phi_{i,j}\}_{\substack{j=i,\ldots,n_2\\ i=1,\ldots,J}}, \{\lambda_i\}_{i=1,\ldots,J}, \left\{\phi_a^{j-J-1}(1-\phi_a)\lambda_j\right\}_{j=J+1,\ldots,n_2}$
($A_{1:J}$,T)/C/($A_{1:J+1}$,T)	$\{P_{i,j}\}_{\substack{j=i,\ldots,i+J-1\\ i=1,\ldots,n_1}}, \{P_{1,j}\}_{j=J+1,\ldots,n_2}, \left\{\frac{(\prod_{j=1}^{J}\phi_{j,i+j})}{(\prod_{k=1}^{J}\phi_{k,k})\phi_a^i}\right\}_{i=1,\ldots,J}$
($A_{1:J}$,T)/C/(A,T)	$\{P_{i,j}\}_{\substack{j=i,\ldots,n_2\\ i=1,\ldots,n_1}}$
($A_{1:J}$,T)/T/T	$\{P_{i,j}\}_{\substack{j=i,\ldots,\min(i+J+2,n_2)\\ i=1,\ldots,n_1}}$
($A_{1:J}$,T)/T/($A_{1:J+1}$,T)	$\{P_{i,j}\}_{\substack{j=i,\ldots,i+J-1\\ i=1,\ldots,n_1}}, \{P_{1,j}\}_{j=J+1,\ldots,n_2}, \left\{\frac{\prod_{i=1}^{J}\phi_{i,i+j}}{\prod_{i=1}^{J}\phi_{i,i}\prod_{k=1}^{j}\phi_{k+J}}\right\}_{j=1,\ldots,\min(n_2-J-1,n_1-J)}$
($A_{1:J}$,T)/T/(A,T)	$\{P_{i,j}\}_{\substack{j=i,\ldots,n_2\\ i=1,\ldots,n_1}}$
($A_{1:J}$,T)/A/A	$\{\phi_{i,j}\}_{\substack{j=i,\ldots,J\\ i=1,\ldots,n_1}}, \{\lambda_i\}_{i=1,\ldots,J}, \{\prod_{k=J+1}^{j-1}\phi_k(1-\phi_j)\lambda_j\}_{j=J+1,\ldots,n_2}$
($A_{1:J}$,T)/A/$A_{1:J+1}$	$\{\phi_{i,j}\}_{\substack{i=j,\ldots,J\\ j=1,\ldots,n_1}}, \{\lambda_j\}_{j=1,\ldots,J}, \{\prod_{k=J+1}^{j-1}\phi_k(1-\phi_j)\lambda_a\}_{j=J+1,\ldots,n_2}$
($A_{1:J}$,T)/A/($A_{1:J+1}$,T)	$\{P_{i,j}\}_{\substack{i=j,\ldots,J\\ j=1,\ldots,n_1}}, \{P_{1,j}\}_{j=J+1,\ldots,n_2}, \left\{\frac{P_{i,i+J}}{P_{1,i+J-1}}\right\}_{i=2,\ldots,n_2-J}, \left\{\frac{\lambda_{a,i}\lambda_{a,i+2}}{\lambda_{a,i+1}^2}\right\}_{i=J+1,\ldots,\min(n_2-2,n_1-1)}$
($A_{1:J}$,T)/A/(A,T)	$\{P_{i,j}\}_{\substack{j=i,\ldots,n_2\\ i=1,\ldots,n_1}}$

* for $A_{1:J}$/T/T the estimable parameters are valid if $n_2 > n_1+J-1$. If $n_2 \leq n_1+J-1$ the deficiency is 0 ; by definition all of the parameters are estimable.

Table 3 Table of parameter redundancy for combined x/y/z and y/z models. We use + as shorthand for combined with. d denotes the deficiency, r denotes the number of estimable parameters, n_1 is the number of years of ringing, and n_2 is the number of years of recovery. In all cases the results are only valid for $n_1 \geq J+1$ and $n_2 \geq J+1$. $E = n_1 n_2 - \frac{1}{2}n_1^2 + \frac{1}{2}n_1$, $B = \frac{1}{2}J(2n_2 - J - 1)$.

$J = 1$

Model	r	d
C/C/$\mathrm{A}_{1:2}$ + C/C	4	0
C/C/($\mathrm{A}_{1:2}$,T) + C/T	$n_1 + n_2 + 2$	0
C/T/T + T/T	$\min(2n_2 + J, n_2 + n_1 + 2J - 1)$	$n_2 - n_1$
C/T/$\mathrm{A}_{1:2}$ + T/C	$n_2 + 3$	0
C/T/($\mathrm{A}_{1:2}$,T) + T/T	$\min(n_1 + 2n_2, 2n_1 + n_2 + 1)$	$\max(1, n_2 - n_1)$
T/C/($\mathrm{A}_{1:2}$,T) + C/T	$\begin{cases} n_1 + 2n_2 & n_2 = n_1 \\ 2n_1 + n_2 + 1 & n_2 > n_1 \end{cases}$	$\begin{cases} 1 & n_2 = n_1 \\ 0 & n_2 > n_1 \end{cases}$
T/T/T + T/T	$\min(n_1 + 2n_2 - 1, 2n_1 + n_2)$	$\max(1, n_2 - n_1)$
T/T/($\mathrm{A}_{1:2}$,T) + C/T	$\min(2n_1 + 2n_2 - 2, 3n_1 - n_2 - 1)$	$\max(2, n_2 - n_1 + 1)$

$J > 1$

Model	r	d
$\mathrm{A}_{1:J}$/C/$\mathrm{A}_{1:J+1}$ + C/C	$J + 3$	$J - 1$
$\mathrm{A}_{1:J}$/C/($\mathrm{A}_{1:J+1}$,T) + C/T	$\min(E + B + J - \frac{1}{2}n_2^2 + \frac{1}{2}n_2 + 2, 2 + n_2 + Jn_1)$	$J - 1$
$\mathrm{A}_{1:J}$/T/T + T/T	$\min(2n_2 + J, n_2 + n_1 + 2J - 1)$	$\max(0, n_2 - n_1 - J + 1)$
$\mathrm{A}_{1:J}$/T/$\mathrm{A}_{1:J+1}$ + T/C	$n_2 + J + 2$	$J - 1$
$\mathrm{A}_{1:J}$/T/$\mathrm{A}_{1:J+1}, T$ + T/T	$\min(E + B - \frac{1}{2}n_2^2 + \frac{3}{2}n_2, (J+1)n_2 + n_2 + J)$	$\max(J, n_2 - n_1)$
($\mathrm{A}_{1:J}$,T)/C/$\mathrm{A}_{1:J+1}, T$ + C/T	$\begin{cases} E + B + 2J - \frac{1}{2}n_2^2 + \frac{3}{2}n_2 + 1 & n_2 \leq n_1 + J - 1 \\ (J+1)n_1 + n_2 + 1 & n_2 > n_1 + J - 1 \end{cases}$	$\begin{cases} E + B + 2J - \frac{1}{2}n_2^2 - \frac{3}{2}n_2 & n_2 \leq n_1 + J - 1 \\ (J-1)n_1 & n_2 > n_1 + J - 1 \end{cases}$
($\mathrm{A}_{1:J}$,T)/T/T + T/T	$\min\{E + B + J + \frac{3}{2}n_2 - \frac{1}{2}n_2^2 - 1,$ $3n_1 + (J-1)n_2 + J - 1\}$	$\max(1, n_2 - n_1 - J + 1)$
($\mathrm{A}_{1:J}$,T)/T/($\mathrm{A}_{1:J+1}, T$) + T/T	$\begin{cases} 2n_1 + B - n_2 - S & n_1 \leq n_2 + J - 1 \\ J + n_1 + n_2 + 2n_1 - 1 & n_1 > n_2 + J - 1 \end{cases}$	$\begin{cases} B + n_2 + 2J - S & n_1 \leq n_2 + J - 1 \\ n_2 + (J-2)n_1 + 1 & n_1 > n_2 + J - 1 \end{cases}$

Table 4 The effect of missing values on parameter redundancy of x/y/z models with x representing the first J years. Here we compare if the rank of the models changes when there are only m $\{1 \leq a < \min(n_1, n_2)\}$ diagonals of data, compared to perfect data. Models in bold are full-rank models if there is perfect data. E_m is the length of the exhaustive summary and the maximum possible rank with missing values.

Rank Changes if	Rank (if changes)	Models
never	-	**A$_{1:J}$/C/C** **A$_{1:J}$/C/T** A$_{1:J}$/C/A A$_{1:J}$/C/A$_{1:J+1}$
$m \leq J-1$	E_m	**(A$_{1:J}$,T)/C/A$_{1:J+1}$**
$m \leq J$	n_2	**A$_{1:J}$/T/C**
$m \leq J$	$2n_2-1$	A$_{1:J}$/T/T
$m \leq J$	$\min(n_2, n_1+m)$	A$_{1:J}$/T/A$_{1:J+1}$ A$_{1:J}$/(A,T)/A$_{1:J+1}$
$m \leq J$	E_m	A$_{1:J}$/C/(A$_{1:J+1}$,T) **(A$_{1:J}$,T)/C/C** **(A$_{1:J}$,T)/C/T** (A$_{1:J}$,T)/C/A (A$_{1:J}$,T)/C/A$_{1:J+1}$, T **(A$_{1:J}$,T)/T/C** T^J/T/T **(A$_{1:J}$,T)/T/A$_{1:J+1}$** (A$_{1:J}$,T)/T/(A$_{1:J+1}$,T) **(A$_{1:J}$,T)/A/C** **(A$_{1:J}$,T)/A/T** (A$_{1:J}$,T)/A/A$_{1:J+1}$ A$_{1:J}$/T/(A$_{1:J+1}$,T)
$m \leq J+1$	E_m	**(A$_{1:J}$,T)/T/A** (A$_{1:J}$,T)/A/T (A$_{1:J}$,T)/A/(A$_{1:J+1}$,T)
$m \leq J+1$	$\begin{cases} n_2 & a \leq J \\ 2n_2-J-1 & a = J+1 \end{cases}$	A$_{1:J}$/T/A
$m \leq n_2-2$	E_m	A$_{1:J}$/C/(A,T) A$_{1:J}$/T/(A,T) A$_{1:J}$/(A,T)/C A$_{1:J}$/(A,T)/T A$_{1:J}$/(A,T)/A A$_{1:J}$/(A,T)/(A$_{1:J+1}$,T) A$_{1:J}$/(A,T)/(A,T) (A$_{1:J}$,T)/C/(A,T) (A$_{1:J}$,T)/T/(A,T) (A$_{1:J}$,T)/A/(A,T)

Table 5 Parameter redundancy of conditional y/z models. Here r denotes the model rank (or number of estimable parameters), d denotes the deficiency ($d > 0$ means the model is parameter redundant), n_1 is the number of years of ringing and n_2 is the number of years of recovery. E and E_c are maximum possible rank for standard and conditional models respectively with $E = n_1 n_2 - \frac{1}{2}n_1^2 + \frac{1}{2}n_1$ and $E_c = n_1 n_2 - \frac{1}{2}n_1^2 - \frac{1}{2}n_1$.

Model	r	d
C/C	1	0
C/T	$n_2 - 1$	2
C/A	$n_2 - 1$	2
C/(A,T)	E_c	$\begin{cases} n_1 & n_1 = n_2 \\ n_1 + 1 & n_1 < n_2 \end{cases}$
T/C	$n_2 - 1$	1
T/T	$n_2 - 1$	$n_2 + 1$
T/A	$2n_2 - 3$	3
T/(A,T)	E_c	$\begin{cases} n_1 + n_2 - 1 & n_1 = n_2 \\ n_1 + n_2 & n_1 < n_2 \end{cases}$
A/C	$n_2 - 1$	1
A/T	$2n_2 - 3$	3
A/A	$n_2 - 1$	$n_2 + 1$
A/(A,T)	E_c	$\begin{cases} n_1 + n_2 - 1 & n_1 = n_2 \\ n_1 + n_2 & n_1 < n_2 \end{cases}$
(A,T)/C	E_c	$\begin{cases} n_1 - 1 & n_1 = n_2 \\ n_1 & n_1 < n_2 \end{cases}$
(A,T)/T	E_c	$\begin{cases} n_1 + n_2 - 1 & n_1 = n_2 \\ n_1 + n_2 & n_1 < n_2 \end{cases}$
(A,T)/A	E_c	$\begin{cases} n_1 + n_2 - 1 & n_1 = n_2 \\ n_1 + n_2 & n_1 < n_2 \end{cases}$
(A,T)/(A,T)	E_c	$\begin{cases} n_1 + E - 2 & n_1 = n_2 \\ n_1 + E & n_1 < n_2 \end{cases}$

Table 6 Parameter redundancy results for conditional x/y/z models part A.

Model	r	d
$A_{1:J}$/C/C	$J+1$	0
$A_{1:J}$/C/T	$J+n_2-1$	2
$A_{1:J}$/C/A	n_2-1	$J+2$
$A_{1:J}$/C/$A_{1:J+1}$	$J+1$	$J+1$
$A_{1:J}$/C/($A_{1:J+1}$,T)†	$\begin{cases} n_1+B-2J-1 & n_1=n_2 \\ n_1+B-2J & n_1+1=n_2 \end{cases}$	$2J+1$
$A_{1:J}$/C/(A,T)	E_c	$\begin{cases} n_1+J & n_1=n_2 \\ n_1+J+1 & n_1<n_2 \end{cases}$
$A_{1:J}$/T/C	n_2	0
$A_{1:J}$/T/T‡	$\begin{cases} \min(2n_2-3, n_2+n_1-2) & J=1 \\ \min(2n_2-1, n_2+n_1+J-2) & J>1 \end{cases}$	$\begin{cases} \max(3, n_2-n_1+2) & J=1 \\ \max(1, n_2-n_1-J+2) & J>1 \end{cases}$
$A_{1:J}$/T/A	$2n_2-J-1$	$J+1$
$A_{1:J}$/T/$A_{1:J+1}$	n_1	$J+1$
$A_{1:J}$/T/($A_{1:J+1}$,T)†	$\begin{cases} n_1+B-2J-1 & n_1=n_2 \\ n_1+B-2J & n_1=n_2-1 \end{cases}$	n_2+J
$A_{1:J}$/T/(A,T)	E_c	$\begin{cases} n_2+n_1-1 & n_1=n_2 \\ n_2+n_1 & n_1=n_2-1 \end{cases}$
$A_{1:J}$/(A,T)/C	$E_c+n_2-B+J-1+G$	1
$A_{1:J}$/(A,T)/T	$\begin{cases} E_c & J=1 \\ E_c+2n_2-B+J+S-3 & J>1, n_2<n_1+J \\ E_c+n_1-B+n_2+2J-4+G & J>1, n_2\geq n_1+J \end{cases}$	$\begin{cases} n_2+1 & J=1 \\ 3 & J>1, n_2-n_1<J \\ n_2-n_1-J+4 & J>1, n_2-n_1\geq J \end{cases}$
$A_{1:J}$/(A,T)/A	$\begin{cases} E_c & J=1 \\ E_c+n_2-B+J+G & J>1, n_1=n_2 \\ E_c+n_2-B+J-1+G & J>1, n_1<n_2 \end{cases}$	n_2+1
$A_{1:J}$/(A,T)/$A_{1:J+1}$	$E_c+n_2-B+J+G-1$	$J+2$
$A_{1:J}$/(A,T)/($A_{1:J+1}$,T)	E_c	$\begin{cases} n_1+n_2-1 & n_1=n_2 \\ n_1+n_2 & n_1<n_2 \end{cases}$
$A_{1:J}$/(A,T)/(A,T)	E_c	$\begin{cases} E+n_2-B-1 & n_1=n_2 \\ E+n_2-B+G & n_1<n_2 \end{cases}$
($A_{1:J}$,T)/C/C	$\begin{cases} B-J+1 & n_1=n_2 \\ B-J+1-S & n_1<n_2<n_1+J \\ B-n_2+n_1+1-G & n_2\geq n_1+J \end{cases}$	$\begin{cases} \max(0, J-1) & n_1=n_2 \\ \max(0, n_1-n_2+J) & n_1<n_2<n_1+J \\ 0 & n_2\geq n_1+J \end{cases}$
($A_{1:J}$,T)/C/T	$\begin{cases} B+n_2-2J-S & n_2\leq n_1+J \\ B-J+n_1-1-G & n_2>n_1+J \end{cases}$	$\begin{cases} \min(2J+1, 2J+n_1-n_2+2) & n_2\leq n_1+J \\ J+2 & n_2>n_1+J \end{cases}$
($A_{1:J}$,T)/C/A	$\begin{cases} B+n_2-2J-S & n_2\leq n_1+J \\ B-J+n_1-1-G & n_2>n_1+J \end{cases}$	$\begin{cases} \min(2J+1, 2J+n_1-n_2+2) & n_2\leq n_1+J \\ J+2 & n_2>n_1+J \end{cases}$
($A_{1:J}$,T)/C/$A_{1:J+1}$	$\begin{cases} B+n_1-n_2-J+1 & n_2=n_1 \\ \min(B-J+1-G, B+n_1-n_2+1-G) & n_2>n_1 \end{cases}$	$\begin{cases} 2J & n_2=n_1 \\ \max(2J-n_2+n_1+1, J+1) & n_2>n_1 \end{cases}$
($A_{1:J}$,T)/C/($A_{1:J+1}$,T)†	$\begin{cases} B+2n_1-n_2-2J-1 & n_1=n_2 \\ B+2n_1-n_2-2J+1 & n_1=n_2-1 \end{cases}$	$B+J$
($A_{1:J}$,T)/C/(A,T)	E_c	$\begin{cases} 2n_1-n_2+B-1 & n_1=n_2 \\ 2n_1-n_2+B+1-G & n_1<n_2 \end{cases}$

All results are valid for the same n_1 and n_2 as the standard model, except ‡ which is valid for $n_1\geq J+2, n_2\geq J+3$.
† indicates result is only valid for $n_2\leq n_1+1$

Table 7 Parameter redundancy results for conditional x/y/z models part B.

Model	r	d
($A_{1:J}$,T)/T/C	$\begin{cases} B+n_2-2J-1 & n_1=n_2 \\ B+n_2-2J-1-S & n_1<n_2<n_1+J \\ B+n_1-J-1-G & n_2\geq n_1+J \end{cases}$	$\begin{cases} J & n_1=n_2 \\ J+n_1-n_2+1 & n_1<n_2<n_1+J \\ \max(1,J+n_1-n_2+1) & n_2\geq n_1+J \end{cases}$
($A_{1:J}$,T)/T/T	$\begin{cases} B+n_2-2J-1-S & n_2<n_1+J \\ B-J+n_1-1-G & n_2\geq n_1+J \end{cases}$	$\begin{cases} n_1+J & n_2=n_1 \\ n_1+J+1 & n_1<n_2<n_1+J \\ n_2+1 & n_2\geq n_1+J \end{cases}$
($A_{1:J}$,T)/T/A	$\begin{cases} B+2n_2-3J-3-S & n_2<n_1+J \\ B+n_1+n_2-2J-3-S & n_2\geq n_1+J \end{cases}$	$\begin{cases} 2J-n_2+n_1+2 & n_2=n_1 \\ 2J-n_2+n_1+3 & n_1<n_2<n_1+J \\ J+3 & n_2\geq n_1+J \end{cases}$
($A_{1:J}$,T)/T/$A^{\dagger}_{1:J+1}$	$\begin{cases} n_1+B-2J-1 & n_1=n_2 \\ n_1+B-2J & n_1=n_2-1 \end{cases}$	$2J+1$
($A_{1:J}$,T)/T/($A_{1:J+1}$,T)†	$\begin{cases} 2n_1-n_2+B-2J-1 & n_1=n_2 \\ 2n_1-n_2+B-2J+1 & n_1=n_2-1 \end{cases}$	$B+n_2-1$
($A_{1:J}$,T)/T/(A,T)†	E_c	$\begin{cases} 2n_1+B-J-2 & n_1=n_2 \\ 2n_1+B-J & n_1=n_2-1 \end{cases}$
($A_{1:J}$,T)/A/C	$\begin{cases} B+n_2-2J-1-S & n_2\leq n_1+J \\ B+n_1-J-1-G & n_2>n_1+J \end{cases}$	$\begin{cases} J-n_2+n_1 & n_2=n_1 \\ J-n_2+n_1+1 & n_1<n_2\leq n_1+J \\ \max(1,J-n_2+n_1+1) & n_2>n_1+J \end{cases}$
($A_{1:J}$,T)/A/T	$\begin{cases} B+2n_2-3J-3-S & n_2\leq n_1+J \\ B+n_1+n_2-2J-3-S & n_2>n_1+J \end{cases}$	$\begin{cases} 2J-n_2+n_1+2 & n_2=n_1 \\ 2J-n_2+n_1+3 & n_1<n_2\leq n_1+J \\ J+3 & n_2>n_1+J \end{cases}$
($A_{1:J}$,T)/A/A	$\begin{cases} B+n_2-2J-1-S & n_2\leq n_1+J \\ B+n_1-J-1-G & n_2>n_1+J \end{cases}$	$\begin{cases} n_1+J & n_2=n_1 \\ n_1+J+1 & n_1<n_2\leq n_1+J \\ n_2+1 & n_2>n_1+J \end{cases}$
($A_{1:J}$,T)/A/$A^{\dagger}_{1:J+1}$	$\begin{cases} n_1+B-2J-1 & n_1=n_2 \\ n_1+B-2J & n_1=n_2-1 \end{cases}$	$2J+1$
($A_{1:J}$,T)/A/($A_{1:J+1}$,T)†	$\begin{cases} 2n_1+B-3J-3 & n_1=n_2 \\ 2n_1+B-3J-1 & n_1=n_2-1 \end{cases}$	$B+J+1$
($A_{1:J}$,T)/A/(A,T)	E_C	$\begin{cases} 2n_1+B-J-2 & n_2=n_1 \\ 2n_1+B-J-G & n_2>n_2 \end{cases}$

All results are valid for the same n_1 and n_2 as the standard model, except ‡ which is valid for $n_1\geq J+2, n_2\geq J+3$.

† indicates result is only valid for $n_2\leq n_1+1$

