Supplementary Material for Parameter Redundancy in Capture-Recapture-Recovery Models

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1. Web Appendix A: Theorem proofs

In this appendix we provide proofs of Theorems 1 and 2. We first provide a variation of the extension theorem of [1, 2], which is required to prove Theorem 1.

1.1. The two-stage extension theorem

In order to derive the simpler exhaustive summary we need to use a modified version of the standard extension theorem [1, 2]. This version of the extension theorem has been used in [4–6], but has not been formally written down.

In the standard extension theorem, [1, 2], we begin with exhaustive summary $\boldsymbol{\kappa}_1^S(\boldsymbol{\theta}_1)$, which has parameters $\boldsymbol{\theta}_1$. Then we extend this model to give the exhaustive summary $\boldsymbol{\kappa}^S(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = [\boldsymbol{\kappa}_1^S(\boldsymbol{\theta}_1), \boldsymbol{\kappa}_2^S(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)]^T$ with parameters $\boldsymbol{\theta}^S = [\boldsymbol{\theta}_1, \boldsymbol{\theta}_2]^T$. If $\partial \boldsymbol{\kappa}_1^S(\boldsymbol{\theta}_1)/\partial \boldsymbol{\theta}_1$ is full rank and $\partial \boldsymbol{\kappa}_2^S(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)/\partial \boldsymbol{\theta}_2$ is full rank, then $\partial \boldsymbol{\kappa}^S(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)/\partial \boldsymbol{\theta}^S$ is also full rank.

In the two-stage extension theorem, we begin with exhaustive summary $\boldsymbol{\kappa}_1^O(\boldsymbol{\theta}_1)$, which can be partitioned as $\boldsymbol{\kappa}_1^O(\boldsymbol{\theta}_1) = [\boldsymbol{\kappa}_1^E(\boldsymbol{\theta}_{1,1}), \boldsymbol{\kappa}_2^{NE}(\boldsymbol{\theta}_{1,1}, \boldsymbol{\theta}_{1,2})]$ with parameters $\boldsymbol{\theta}_1 = [\boldsymbol{\theta}_{1,1}, \boldsymbol{\theta}_{1,2}]^T$. This exhaustive summary is then extended to $\boldsymbol{\kappa}_2^O(\boldsymbol{\theta}_{1,1}, \boldsymbol{\theta}_{1,2}, \boldsymbol{\theta}_{2,2}) = [\boldsymbol{\kappa}_1^E(\boldsymbol{\theta}_{1,1}), \boldsymbol{\kappa}_2^E(\boldsymbol{\theta}_{1,1}, \boldsymbol{\theta}_{1,2}, \boldsymbol{\theta}_{2,2})]$, with parameters

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$\boldsymbol{\theta}_2 = [\boldsymbol{\theta}_{1,1}, \boldsymbol{\theta}_{1,2}, \boldsymbol{\theta}_{2,2}]^T$. Let $\boldsymbol{\theta}_2^E = [\boldsymbol{\theta}_{1,2}, \boldsymbol{\theta}_{2,2}]^T$. Then we have the following result:

Theorem 3. The Two-Stage Extension Theorem *If* $\partial \boldsymbol{\kappa}_1^O(\boldsymbol{\theta}_1)/\partial \boldsymbol{\theta}_1$, $\partial \boldsymbol{\kappa}_1^E(\boldsymbol{\theta}_{1,1})/\partial \boldsymbol{\theta}_{1,1}$, *and* $\partial \boldsymbol{\kappa}_2^E(\boldsymbol{\theta}_{1,1}, \boldsymbol{\theta}_{1,2}, \boldsymbol{\theta}_{2,2})/\partial \boldsymbol{\theta}_2^E$ *are all full rank then* $\boldsymbol{\kappa}_2^O(\boldsymbol{\theta}_{1,1}, \boldsymbol{\theta}_{1,2}, \boldsymbol{\theta}_{2,2})$ *is also full rank.*

The proof follows the same form as the standard extension theorem or can be derived via Theorem 4.2 of [7]. Using induction, Theorem 3 can be used to create general rules in the same way as the standard extension theorem.

1.2. Proof of Theorem 1a.

In this section we provide a proof for Theorem 1a. The theorem states that a simpler exhaustive summary for the capture-recapture model consists of the terms $s_{i,j} = \phi_{i,j} p_{i+1,j+1}$ (for $i = 1, \ldots, n_2$ and $j = i, \ldots, \min(n_1+i-1, n_2)$), and $t_{i,j} = \phi_{i,j}(1-p_{i+1,j+1})$ (for $i = 1, \ldots, n_2-1$ and $j = i, \ldots, \min(n_1+i-1, n_2-1)$). The proof of Theorem 1a. is split into three parts:

- In part one, we show that the original exhaustive summary consisting of the capture-histories can be reparameterised in terms of $[\mathbf{s}^T, \mathbf{t}^T]$.
- In part two, we create a new exhaustive summary, denoted as $\boldsymbol{\kappa}_{uvw}$, utilising Theorem 8 of [2]. This is created so that the extension theorem of [1, 2] can be applied in order for results to be extended to any dimension.
- In part three, we show that the reparameterisation $[\mathbf{s}^T, \mathbf{t}^T]$ is an exhaustive summary, again utilising Theorem 8 of [2].

We assume that none of the parameters are on boundary values, so that our parameter space is restricted to $0 < \phi_{i,j} < 1$ and $0 < p_{i,j} < 1$ for all values of i and j for this theorem to apply.

Theorem 8 of [2] states that if the derivative matrix $\partial \boldsymbol{\kappa}(\boldsymbol{\kappa}_{\text{new}})/\partial \boldsymbol{\kappa}_{\text{new}}$ is full rank then $\boldsymbol{\kappa}_{\text{new}}$ is a new exhaustive summary.

Part one:

We show that any capture-history can be expressed as parameters $s_{i,j}$ and $t_{i,j}$. The probability of a particular capture-history, h, is

$$Pr(h) = \prod_{k=a+1}^{b} \phi_{k-a,k-1} \left(\delta_k p_{k-a+1,k} + \bar{\delta}_k \bar{p}_{k-a+1,k}\right) \chi_{b-a+1,b},$$

where an animal is first recaptured at time a and last recaptured at time b, with individual history entry δ_k at time k, and $\bar{x} = 1 - x$. These probabilities can be reparameterised in terms of $s_{i,j} = \phi_{i,j} p_{i+1,j+1}$ and $t_{i,j} = \phi_{i,j}(1 - p_{i+1,j+1})$ to give

$$Pr(h) = \prod_{k=a+1}^{b} \left(\delta_k s_{k-a+1,k} + \bar{\delta}_k t_{k-a+1,k}\right) \chi_{b-a+1,b}.$$

The probability of an animal never being seen again, $\chi_{i,j} = (1 - \phi_{i,j}) + \phi_{i,j}(1 - p_{i+1,j+1})\chi_{i+1,j+1}$, with $\chi_{i,n_2+1} = 1$, can be shown to be a function of $s_{i,j}$ and $t_{i,j}$, by first expanding $\chi_{i,j}$. This gives:

$$\begin{aligned} \chi_{i,j} &= (1 - \phi_{i,j}) + \phi_{i,j}(1 - p_{i+1,j+1}) \cdot [(1 - \phi_{i+1,j+1}) + \\ &\quad \phi_{i+1,j+1}(1 - p_{i+2,j+2}) \cdot [(1 - \phi_{i+2,j+2}) + \phi_{i+2,j+2} \cdot \\ &\quad (1 - p_{i+3,j+3}) \cdot [\cdots [((1 - \phi_{n_2,n_2}) + \phi_{n_2,n_2}(1 - p_{n_2+1,n_2+1})] \cdots]]]. \end{aligned}$$

By noting that $(1 - \phi_{i,j}) = (1 - s_{i,j} - t_{i,j})$, we can write $\chi_{i,j}$ as

$$\begin{aligned} \chi_{i,j} &= (1 - s_{i,j} - t_{i,j}) + t_{i,j}[(1 - s_{i+1,j+1} - t_{i+1,j+1}) + t_{i+1,j+1} \cdot \\ &\quad [(1 - s_{i+2,i+2} - t_{i+2,j+2}) + t_{i+3,j+3}[\cdots [(1 - s_{n_2,n_2})] \cdots]]]. \end{aligned}$$

Therefore all capture-histories can be parameterised in terms of **s** and **t** only.

Part two:

We derive a new exhaustive summary which consists of the following terms:

- $u_j = \prod_{k=1}^{n_2-j+1} \phi_{k,k+j-1} p_{k+1,k+j},$
  for all $j = 1, \ldots, n_2$;

- $v_{i,j} = \dfrac{(1 - p_{i+1,j+1})}{p_{i+1,j+1}}$,
  for all $i = 1, \ldots, n_2 - 1$ and $j = i, \ldots, \min(i + n_1 - 1, n_2)$;

- and $w_{i,j} = \dfrac{\chi_{i+1,j+1}}{\left(\prod_{k=1}^{n_2 - j} \phi_{k+i,k+j} p_{k+i+1,k+j+1} \right)}$,
  for all $i = 1, \ldots, n_2 - 1$ and $j = i, \ldots, \min(i + n_1 - 1 \leq n_2)$.

We can reparameterise the original exhaustive summary consisting of the capture-histories, when there are $n_1 = n_2 = 2$ years of marking and recapture, as

$$\boldsymbol{\kappa} = \begin{bmatrix} Pr(111) \\ Pr(101) \\ Pr(011) \\ Pr(110) \\ Pr(100) \\ Pr(010) \end{bmatrix} = \begin{bmatrix} \phi_{1,1} p_{2,2} \phi_{2,2} p_{3,3} \\ \phi_{1,1} \bar{p}_{2,2} \phi_{2,2} p_{3,3} \\ \phi_{1,2} p_{2,3} \\ \phi_{1,1} p_{2,2} \chi_{2,2} \\ \chi_{1,1} \\ \chi_{1,2} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_1 v_{1,1} \\ u_2 \\ u_1 w_{1,1} \\ \bar{u}_1 - u_1(v_{1,1} + w_{1,1}) \\ \bar{u}_2 \end{bmatrix}.$$

The reparameterisation is $\boldsymbol{\kappa}_{uvw} = [u_1, u_2, v_{1,1}, w_{1,1}]^T$. The derivative matrix,

$$\left[\frac{\partial \boldsymbol{\kappa}}{\partial \boldsymbol{\kappa}_{uvw}} \right] = \begin{bmatrix} 1 & v_{1,1} & 0 & w_{1,1} & -1 - v_{1,1} - w_{1,1} & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & u_1 & 0 & 0 & -u_1 & 0 \\ 0 & 0 & 0 & u_1 & -u_1 & 0 \end{bmatrix},$$

has full rank 4. A modified PLUR decomposition of $\partial \boldsymbol{\kappa} / \partial \boldsymbol{\kappa}_{uvw}$ shows that this is valid for all values of $u_1, u_2, v_{1,1}$ and $w_{1,1}$ as long as $u_1 = \phi_{1,1} p_{2,2} \phi_{2,2} p_{3,3}$ is non-zero. This only occurs at a boundary and the parameter space has already been restricted to exclude boundary values. Therefore by Theorem 8 of [2], when $n_1 = n_2 = 2$, $\boldsymbol{\kappa}_{uvw}$ is an alternative exhaustive summary for the model. Now consider extending the model firstly by adding another year of recapture so that $n_2 = 3$, while keeping $n_1 = 2$. The original exhaustive summary is then

$$\boldsymbol{\kappa} = \begin{bmatrix} Pr(1111) \\ Pr(1011) \\ Pr(0111) \\ Pr(1100) \\ Pr(1000) \\ Pr(0101) \\ Pr(1110) \\ Pr(1010) \\ Pr(0110) \\ Pr(1101) \\ Pr(1001) \\ Pr(0100) \end{bmatrix} = \begin{bmatrix} \phi_{1,1}p_{2,2}\phi_{2,2}p_{3,3}\phi_{3,3}p_{4,4} \\ \phi_{1,1}\bar{p}_{2,2}\phi_{2,2}p_{3,3}\phi_{3,3}p_{4,4} \\ \phi_{1,2}p_{2,3}\phi_{2,3}p_{3,4} \\ \phi_{1,1}p_{2,2}\chi_{2,2} \\ \chi_{1,1} \\ \phi_{1,2}\bar{p}_{2,3}\phi_{2,3}p_{3,4} \\ \phi_{1,1}p_{2,2}\phi_{2,2}p_{3,3}\chi_{2,2}) \\ \phi_{1,1}\bar{p}_{2,2}\phi_{2,2}p_{3,3}\chi_{2,2} \\ \phi_{1,2}\chi_{2,3} \\ \phi_{1,1}p_{2,2}\phi_{2,2}\bar{p}_{3,3}\phi_{3,3}p_{4,4} \\ \phi_{1,1}\bar{p}_{2,2}\phi_{2,2}\bar{p}_{3,3}\phi_{3,3}p_{4,4} \\ \bar{\phi}_{1,2} + \phi_{1,2}\bar{p}_{2,3}\chi_{2,3} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_1 v_{1,1} \\ u_2 \\ u_1 w_{1,1} \\ k_4 \\ u_2 v_{1,2} \\ u_1 w_{2,2} \\ u_1 v_{1,1} w_{2,2} \\ u_2 w_{1,2} \\ u_1 v_{2,2} \\ u_1 v_{1,1} v_{2,2} \\ \bar{u}_2 - u_2(v_{1,2} + w_{1,2}) \end{bmatrix},$$

where $k_4 = 1 - u_1 - u_1 v_{1,1} - u_1 w_{1,1} - u_1 w_{2,2} - u_1 v_{1,1} w_{2,2} - u_1 v_{2,2} - u_1 v_{1,1} v_{2,2}$. This uses the reparameterisation $\boldsymbol{\kappa}_{uvw} = [u_1, u_2, v_{1,1}, v_{1,2}, v_{2,2}, w_{1,1}, w_{1,2}, w_{2,2}]^T$. We now use the two-stage extension theorem of Section 1.1. The first stage involves the exhaustive summary terms

$$\boldsymbol{\kappa}_1 = \begin{bmatrix} u_1 \\ u_1 v_{1,1} \\ u_2 \\ u_1 w_{1,1} \end{bmatrix},$$

with parameters $\boldsymbol{\theta}_1 = [u_1, u_2, v_{1,1}, w_{1,1}]$. The derivative matrix

$$\left[\frac{\partial \boldsymbol{\kappa}_1}{\partial \boldsymbol{\theta}_1}\right] = \begin{bmatrix} 1 & v_{1,1} & 0 & w_{1,1} \\ 0 & 0 & 1 & 0 \\ 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & u_1 \end{bmatrix}$$

has full rank 4. The second stage examines the remaining exhaustive summary terms

$$\boldsymbol{\kappa}_2 = \begin{bmatrix} k_4 \\ u_2 v_{1,2} \\ u_1 w_{2,2} \\ u_1 v_{1,1} w_{2,2} \\ u_2 w_{1,2} \\ u_1 v_{2,2} \\ u_1 v_{1,1} v_{2,2} \\ 1 - u_2 - u_2 v_{1,2} - u_2 w_{1,2} \end{bmatrix},$$

with parameters $\boldsymbol{\theta}_2 = [v_{1,2}, v_{2,2}, w_{1,2}, w_{2,2}]^T$. The derivative matrix

$$\left[\frac{\partial \boldsymbol{\kappa}_2}{\partial \boldsymbol{\theta}_2}\right] = \begin{bmatrix} 0 & u_2 & 0 & 0 & 0 & 0 & 0 & -u_2 \\ -u_1 - u_1 v_{1,1} & 0 & 0 & 0 & 0 & u_1 & u_1 v_{1,1} & 0 \\ 0 & 0 & 0 & 0 & u_2 & 0 & 0 & -u_2 \\ -u_1 - u_1 v_{1,1} & 0 & u_1 & u_1 v_{1,1} & 0 & 0 & 0 & 0 \end{bmatrix}$$

has full rank 4. Therefore by the two-stage extension theorem, the model can be extended in terms of years of recapture. Adding a year of marking so $n_1 = 3$ while $n_2 = 3$ adds the following exhaustive summary terms

$$\boldsymbol{\kappa}_3 = \begin{bmatrix} Pr(0011) \\ Pr(0010) \end{bmatrix} = \begin{bmatrix} \phi_{1,3} p_{2,4} \\ 1 - \phi_{1,3} p_{2,4} \end{bmatrix} = \begin{bmatrix} u_3 \\ 1 - u_3 \end{bmatrix}.$$

As there is only one additional parameter in $\boldsymbol{\kappa}_3$, this is trivially full rank and means that the original model can be extended for a greater number years of marking. Therefore $\boldsymbol{\kappa}_{uvw}$ is an exhaustive summary for any dimension.

Part three:

This part involves checking whether the derivative matrix $\partial \boldsymbol{\kappa}_{uvw}(\mathbf{s}, \mathbf{t}) / \partial [\mathbf{s}, \mathbf{t}]$ is full rank and then using the two-stage theorem to show it is always full rank for larger dimensions. Starting with $n_1 = n_2 = 2$ we can reparameterise $\boldsymbol{\kappa}_{uvw}$

in terms of $\mathbf{s}$ and $\mathbf{t}$ to get

$$\boldsymbol{\kappa}_{uvw}(\mathbf{s},\mathbf{t}) = \begin{bmatrix} u_1 \\ v_{1,1} \\ w_{1,1} \\ u_2 \end{bmatrix} = \begin{bmatrix} s_{1,1}s_{2,2} \\ t_{1,1}/s_{1,1} \\ (1-s_{2,2})/s_{2,2} \\ s_{1,2} \end{bmatrix}, \tag{1}$$

with parameter set $\boldsymbol{\kappa}_{st} = [s_{1,1}, s_{1,2}, s_{2,2}, t_{1,1}]^T$. The derivative matrix

$$\left[\frac{\partial \boldsymbol{\kappa}_{uvw}(\mathbf{s},\mathbf{t})}{\partial \boldsymbol{\kappa}_{st}}\right] = \begin{bmatrix} s_{2,2} & -\frac{t_{1,1}}{s_{1,1}^2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ s_{1,1} & 0 & -\frac{1}{s_{2,2}} - \frac{(1-s_{2,2})}{s_{2,2}^2} & 0 \\ 0 & \frac{1}{s_{1,1}} & 0 & 0 \end{bmatrix}$$

has full rank 4. A modified PLUR decomposition of $\partial \boldsymbol{\kappa}_{uvw}(\mathbf{s},\mathbf{t})/\partial \boldsymbol{\kappa}_{st}$ shows this the model remains full rank for any value of $s_{1,1}, s_{1,2}, s_{2,2}$ or $t_{1,1}$. Therefore $\boldsymbol{\kappa}_{st}$ is an exhaustive summary when $n_1 = n_2 = 2$. If we extend the model to add another year of recapture, the exhaustive summary becomes

$$\boldsymbol{\kappa}_{uvw}(\mathbf{s},\mathbf{t}) = \begin{bmatrix} u_1 \\ u_2 \\ v_{1,1} \\ v_{1,2} \\ v_{2,2} \\ w_{1,1} \\ w_{1,2} \\ w_{2,2} \end{bmatrix} = \begin{bmatrix} s_{1,1}s_{2,2}s_{3,3} \\ s_{1,2}s_{2,3} \\ t_{1,1}/s_{1,1} \\ t_{1,2}/s_{1,2} \\ t_{2,2}/s_{2,2} \\ (1-s_{3,3})/s_{3,3} \\ (1-s_{2,3})/s_{2,3} \\ \{(1-s_{2,2}-t_{2,2}) + t_{2,2}(1-s_{3,3})\}/s_{2,2}s_{3,3} \end{bmatrix},$$

with parameters $\boldsymbol{\kappa}_{st} = [s_{1,1}, s_{1,2}, s_{2,2}, s_{2,3}, s_{3,3}, t_{1,1}, t_{1,2}, t_{2,2}]^T$. Note that the terms $u_2, v_{2,2}$ and $w_{1,2}$ are identical to $u_1, v_{1,1}$ and $w_{1,1}$ respectively in (1), if $s_{1,1}$ is re-labelled as $s_{1,2}$, $s_{2,2}$ as $s_{2,3}$, and $t_{1,1}$ as $t_{1,2}$. This can then form the first stage of the two-stage extension theorem with

$$\boldsymbol{\kappa}_1 = \begin{bmatrix} u_2 \\ v_{1,2} \\ w_{1,2} \end{bmatrix} = \begin{bmatrix} s_{1,2}s_{2,3} \\ t_{1,2}/s_{1,2} \\ (1-s_{2,3})/s_{2,3} \end{bmatrix},$$

with parameters $\boldsymbol{\theta}_1 = [s_{1,2}, s_{2,3}, t_{1,2}]$. The derivative matrix

$$\left[\frac{\partial \boldsymbol{\kappa}_1}{\partial \boldsymbol{\theta}_1}\right] = \begin{bmatrix} s_{2,3} & -\frac{t_{1,2}}{s_{1,2}^2} & 0 \\ s_{1,2} & 0 & -\frac{1}{s_{2,3}} - \frac{(1-s_{2,3})}{s_{2,3}^2} \\ 0 & \frac{1}{s_{1,2}} & 0 \end{bmatrix},$$

has full rank 3. The second stage involves the terms

$$\boldsymbol{\kappa}_2 = \begin{bmatrix} u_1 \\ v_{1,1} \\ v_{2,2} \\ w_{1,1} \\ w_{2,2} \end{bmatrix} = \begin{bmatrix} s_{1,1}s_{2,2}s_{3,3} \\ t_{1,1}/s_{1,1} \\ t_{2,2}/s_{2,2} \\ (1-s_{3,3})/s_{3,3} \\ \left\{(1-s_{2,2}-t_{2,2}) + t_{2,2}\left(1-s_{3,3}\right)\right\}/s_{2,2}s_{3,3} \end{bmatrix},$$

with the parameter set $\boldsymbol{\theta}_2 = [s_{1,1}, s_{2,2}, s_{3,3}, t_{1,1}, t_{2,2}]^T$. The derivative matrix

$$\left[\frac{\partial \boldsymbol{\kappa}_2}{\partial \boldsymbol{\theta}_2}\right] = \begin{bmatrix} s_{2,2}s_{3,3} & -\frac{t_{1,1}}{s_{1,1}^2} & 0 & 0 & 0 \\ s_{1,1}s_{3,3} & 0 & -\frac{t_{2,2}}{s_{2,2}^2} & 0 & -\frac{1}{s_{2,2}s_{3,3}} - \frac{\{(1-s_{2,2}-t_{2,2})+t_{2,2}(1-s_{3,3})\}}{s_{2,2}^2 s_{3,3}} \\ s_{1,1}s_{2,2} & 0 & 0 & -\frac{1}{s_{2,2}} - \frac{(1-s_{3,3})}{s_{2,2}^2} & -\frac{t_{2,2}}{s_{2,2}s_{3,3}} - \frac{\{(1-s_{2,2}-t_{2,2})+t_{2,2}(1-s_{3,3})\}}{s_{2,2}s_{3,3}^2} \\ 0 & \frac{1}{s_{1,1}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s_{2,2}} & 0 & -\frac{1}{s_{2,2}} \end{bmatrix},$$

has full rank 5. Therefore by the two-stage extension theorem, the model can be extended in terms of years of recapture. Adding a year of marking so $n_1 = 3$ while $n_2 = 3$ adds only the exhaustive summary term $u_3 = s_{1,3}$. As there is only one additional parameter, this extension is trivially full rank and means that the original model can be extended for a greater number years of marking. Therefore $\mathbf{s}$ and $\mathbf{t}$ form an exhaustive summary for any dimension. □

1.3. Proof of Theorem 1b.

In this section we provide a proof for Theorem 1b. The proof is similar to the proof for Theorem 1b. Theorem 1b states that a simpler exhaustive summary for the capture-recapture-recovery model consists of the terms $s_{i,j} = \phi_{i,j}p_{i+1,j+1}$ (for $i = 1, \ldots, n_2$ and $j = i, \ldots, \min(n_1+i-1, n_2)$), $t_{i,j} = \phi_{i,j}(1-p_{i+1,j+1})$ (for

$i = 1, \ldots, n_2 - 1$ and $j = i, \ldots, \min(n_1 + i - 1, n_2 - 1)$), and $r_{i,j} = (1 - \phi_{i,j})\lambda_{i,j}$ (for all $i = 1, \ldots, n_2$ and $j = i, \ldots, \min(n_1 + i - 1, n_2)$). The proof of Theorem 1b. is also split into three parts:

- In part one, we show that the original exhaustive summary consisting of the capture-histories can be reparameterised in terms of $[\mathbf{s}^T, \mathbf{t}^T, \mathbf{r}^T]$, utilising Theorem 8 of [2].
- In part two, we create a new exhaustive summary, denoted as $\boldsymbol{\kappa}_{uvwx}$. This is created so that the extension theorem of [1, 2] can be applied in order for results to be extended to any dimension.
- In part three, we show that the reparametrisation $\boldsymbol{\kappa}_{uvwx}(\mathbf{s}, \mathbf{t}, \mathbf{r})$ is an exhaustive summary, utilising Theorem 8 of [2].

We assume that none of the parameters are on boundary values, so that our parameter space is restricted to $0 < \phi_{i,j} < 1$, $0 < p_{i,j} < 1$ and $0 < \lambda_{i,j} < 1$ for all values of i and j for this theorem to apply.

Part one:

We show that any capture/recovery-history can be expressed as parameters $s_{i,j}$, $t_{i,j}$ and $r_{i,j}$. The probability of a particular capture/recovery-history, h, is

$$Pr(h) = \begin{cases} \displaystyle\prod_{k=a+1}^{b} \phi_{k-a,k-1} \left(\delta_k p_{k-a+1,k} + \bar{\delta}_k \bar{p}_{k-a+1,k}\right) \chi_{b-a+1,b} & \text{if } \delta_b = 1 \\ \displaystyle\prod_{k=a+1}^{b-1} \phi_{k-a,k-1} \left(\delta_k p_{k-a+1,k} + \bar{\delta}_k \bar{p}_{k-a+1,k}\right) \bar{\phi}_{b-a,b-1} \lambda_{b-a,b-1} & \text{if } \delta_b = 2, \end{cases}$$

where an animal is first recaptured at time a and last recaptured or recovered at time b, with individual history entry δ_k at time k, and $\bar{x} = 1 - x$. These probabilities can be reparameterised in terms of $s_{i,j} = \phi_{i,j} p_{i+1,j+1}$,

$t_{i,j} = \phi_{i,j}(1 - p_{i+1,j+1})$ and $r_{i,j} = (1 - \phi_{i,j})\lambda_{i,j}$ to give

$$Pr(h) = \begin{cases} \prod_{k=a+1}^{b} \left(\delta_k s_{k-a+1,k} + \bar{\delta}_k t_{k-a+1,k}\right) \chi_{b-a+1,b} & \text{if } \delta_b = 1 \\ \prod_{k=a+1}^{b-1} \left(\delta_k s_{k-a+1,k} + \bar{\delta}_k t_{k-a+1,k}\right) r_{b-a+1,b} & \text{if } \delta_b = 2 \end{cases}$$

The probability of never being seen again, $\chi_{i,j} = (1 - \phi_{i,j})(1 - \lambda_{i,j}) + \phi_{i,j}(1 - p_{i+1,j+1})\chi_{i+1,j+1}$, with $\chi_{i,n_2+1} = 1$, can be shown to be a function of $s_{i,j}$, $t_{i,j}$ and $r_{i,j}$, by first expanding $\chi_{i,j}$. This gives:

$$\begin{aligned} \chi_{i,j} &= (1 - \phi_{i,j})(1 - \lambda_{i,j}) + \phi_{i,j}(1 - p_{i+1,j+1}) \cdot \\ &\quad [(1 - \phi_{i+1,j+1})(1 - \lambda_{i+1,j+1}) + \phi_{i+1,j+1}(1 - p_{i+2,j+2}) \cdot \\ &\quad [(1 - \phi_{i+2,j+2})(1 - \lambda_{i+2,j+2}) + \phi_{i+2,j+2}(1 - p_{i+3,j+3}) \cdot \\ &\quad [\cdots[((1 - \phi_{n_2,n_2})(1 - \lambda_{n_2,n_2}) + \phi_{n_2,n_2}(1 - p_{n_2+1,n_2+1})]\cdots]]]. \end{aligned}$$

By noting that $(1 - \phi_{i,j})(1 - \lambda_{i,j}) = (1 - s_{i,j} - t_{i,j} - r_{i,j})$

$$\begin{aligned} \chi_{i,j} &= (1 - s_{i,j} - t_{i,j} - r_{i,j}) + t_{i,j} \cdot \\ &\quad [(1 - s_{i+1,j+1} - t_{i+1,j+1} - r_{i+1,j+1}) + t_{i+1,j+1} \cdot \\ &\quad [(1 - s_{i+2,j+2} - t_{i+2,j+2} - r_{i+2,j+2}) + t_{i+2,j+2} \cdot \\ &\quad [\cdots[(1 - s_{n_2,n_2} - r_{n_2,n_2})]\cdots]]]. \end{aligned}$$

Therefore all capture/recovery-histories can be parameterised in terms of **s**, **t** and **r** only.

Part two:

We derive a new exhaustive summary consisting of following terms:

- $u_j = \prod_{k=1}^{n_2-j+1} \phi_{k,k+j-1} p_{k+1,k+j}$,
  for all $j = 1, \ldots, n_2$;
- $v_{i,j} = \dfrac{(1 - p_{i+1,j+1})}{p_{i+1,j+1}}$,
  for all $i = 1, \ldots, n_2 - 1$ and $j = i, \ldots, \min(i + n_1 - 1, n_2 - 1)$;

- $w_{i,j} = \dfrac{\chi_{i+1,j+1}}{\left(\displaystyle\prod_{k=1}^{n_2-j} \phi_{k+i,k+j} p_{k+i+1,k+j+1}\right)}$,
  for all $i = 1, \ldots, n_2 - 1$ and $j = i, \ldots, \min(i + n_1 - 1, n_2 - 1)$;
- and $x_{i,j} = \dfrac{r_{i,j}}{\left(\displaystyle\prod_{k=0}^{n_2-j} \phi_{k+i,k+j} p_{k+i+1,k+j+1}\right)}$,
  for all $i = 1, \ldots, n_2$ and $j = i, \ldots, \min(i + n_1 - 1, n_2)$.

We can reparameterise the original exhaustive summary consisting of the capture-histories, when there are $n_1 = n_2 = 2$ years of marking and recapture/recovery, as

$$\boldsymbol{\kappa} = \begin{bmatrix} Pr(111) \\ Pr(101) \\ Pr(011) \\ Pr(110) \\ Pr(112) \\ Pr(120) \\ Pr(102) \\ Pr(012) \\ Pr(100) \\ Pr(010) \end{bmatrix} = \begin{bmatrix} \phi_{1,1} p_{2,2} \phi_{2,2} p_{3,3} \\ \phi_{1,1} \bar{p}_{2,2} \phi_{2,2} p_{3,3} \\ \phi_{1,2} p_{2,3} \\ \phi_{1,1} p_{2,2} \chi_{2,2} \\ \phi_{1,1} p_{2,2} \bar{\phi}_{2,2} \lambda_{2,2} \\ \bar{\phi}_{1,1} \lambda_{1,1} \\ \phi_{1,1} \bar{p}_{2,2} \bar{\phi}_{2,2} \lambda_{2,2} \\ \bar{\phi}_{1,2} \lambda_{1,2} \\ \chi_{1,1} \\ \bar{\phi}_{1,2} \bar{\lambda}_{1,2} + \phi_{1,2} \bar{p}_{2,3} \end{bmatrix} = \begin{bmatrix} u_1 \\ u_1 v_{1,1} \\ u_2 \\ u_1 w_{1,1} \\ u_1 x_{2,2} \\ u_1 x_{1,1} \\ u_1 v_{1,1} x_{2,2} \\ u_2 x_{1,2} \\ k_9 \\ 1 - u_2 - u_2 x_{1,2} \end{bmatrix},$$

where $k_9 = 1 - u_1 - u_1 v_{1,1} - u_1 w_{1,1} - u_1 x_{2,2} - u_1 x_{1,1} - u_1 v_{1,1} x_{2,2}$. The reparameterisation is $\boldsymbol{\kappa}_{uvwx} = [u_1, u_2, v_{1,1}, w_{1,1}, x_{1,1}, x_{1,2}, x_{2,2}]^T$. The derivative matrix,

$$\left[\frac{\partial \boldsymbol{\kappa}}{\partial \boldsymbol{\kappa}_{uvwx}}\right] = \begin{bmatrix} 1 & v_{1,1} & 0 & w_{1,1} & x_{2,2} & x_{1,1} & v_{1,1} x_{2,2} & 0 & D_{1,9} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_{1,2} & 0 & -1 - x_{1,2} \\ 0 & u_1 & 0 & 0 & 0 & 0 & u_1 x_{2,2} & 0 & -u_1(1 + x_{2,2}) & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 0 & -u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 & -u_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_2 & 0 & -u_2 \\ 0 & 0 & 0 & 0 & u_1 & 0 & u_1 v_{1,1} & 0 & -u_1(1 + v_{1,1}) & 0 \end{bmatrix},$$

has full rank 7 where $D_{1,9} = -1 - v_{1,1} + v_{1,1}x_{2,2} - x_{2,2} - x_{1,1} - w_{1,1}$. A modified PLUR decomposition of $\partial\boldsymbol{\kappa}/\partial\boldsymbol{\kappa}_{uvwx}$ shows this is valid for all values of $u_1, u_2, v_{1,1}$ and $w_{1,1}$ as long as $u_1 = \phi_{1,1}p_{2,2}\phi_{2,2}p_{3,3}$ and $u_2 = \phi_{1,2}p_{2,3}$ are non-zero. This only occurs at boundary values, which have been excluded from the parameter space. Therefore by the Theorem 8 of [2], when $n_1 = n_2 = 2$, $\boldsymbol{\kappa}_{uvwx}$ is an alternative exhaustive summary for the model. Now consider extending the model firstly by adding another year of recapture so that $n_2 = 3$, while keeping $n_1 = 2$. The original exhaustive summary is then

$$\boldsymbol{\kappa} = \begin{bmatrix} Pr(1111) \\ Pr(1011) \\ Pr(0111) \\ Pr(1100) \\ Pr(1120) \\ Pr(1200) \\ Pr(1020) \\ Pr(0120) \\ Pr(1000) \\ Pr(0101) \\ Pr(1110) \\ Pr(1010) \\ Pr(0110) \\ Pr(1101) \\ Pr(1001) \\ Pr(0100) \\ Pr(1112) \\ Pr(1102) \\ Pr(1012) \\ Pr(1002) \\ Pr(0112) \\ Pr(0102) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_1v_{1,1} \\ u_2 \\ u_1w_{1,1} \\ u_1x_{2,2} \\ u_1x_{1,1} \\ u_1v_{1,1}x_{2,2} \\ u_2x_{1,2} \\ k_9 \\ u_2v_{1,2} \\ u_1w_{2,2} \\ u_1v_{1,1}w_{2,2} \\ u_2w_{1,2} \\ u_1v_{2,2} \\ u_1v_{1,1}v_{2,2} \\ k_{16} \\ u_1x_{3,3} \\ u_1v_{2,2}x_{3,3} \\ u_1v_{1,1}x_{3,3} \\ u_1v_{1,1}v_{2,2}x_{3,3} \\ u_2x_{2,3} \\ u_2v_{1,2}x_{2,3} \end{bmatrix},$$

where $k_9 = 1 - u_1 - u_1v_{1,1} - u_1w_{1,1} - u_1x_{2,2} - u_1x_{1,1} - u_1v_{1,1}x_{2,2} - u_1w_{2,2} - u_1v_{1,1}w_{2,2} - u_1v_{2,2} - u_1v_{1,1}v_{2,2} - u_1x_{3,3} - u_1v_{2,2}x_{3,3} - u_1v_{1,1}x_{3,3} - u_1v_{1,1}v_{2,2}x_{3,3}$ and $k_{16} = 1 - u_2 - u_2x_{1,2} - u_2v_{1,2} - u_2w_{1,2} - u_2x_{2,3} - u_2v_{1,2}x_{2,3}$. We write $\boldsymbol{\kappa}_{uvwx} = [u_1, u_2, v_{1,1}, v_{1,2}, v_{2,2}, w_{1,1}, w_{1,2}, w_{2,2}, x_{1,1}, x_{1,2}, x_{2,2}, x_{2,3}, x_{3,3}]^T$. The two-stage extension theorem of Section 1.1 is applied next. The first stage involves the exhaustive summary terms

$$\boldsymbol{\kappa}_1 = \begin{bmatrix} u_1 \\ u_1v_{1,1} \\ u_2 \\ u_1w_{1,1} \\ u_1x_{2,2} \\ u_1x_{1,1} \\ u_1v_{1,1}x_{2,2} \\ u_2x_{1,2} \end{bmatrix},$$

with parameters $\boldsymbol{\theta}_1 = [u_1, u_2, v_{1,1}, w_{1,1}, x_{1,1}, x_{1,2}, x_{2,2}]$. The derivative matrix

$$\left[\frac{\partial \boldsymbol{\kappa}_1}{\partial \boldsymbol{\theta}_1}\right] = \begin{bmatrix} 1 & v_{1,1} & 0 & w_{1,1} & x_{2,2} & x_{1,1} & v_{1,1}x_{2,2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_{1,2} \\ 0 & u_1 & 0 & 0 & 0 & 0 & u_1x_{2,2} & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 & u_1 & 0 & u_1v_{1,1} & 0 \end{bmatrix},$$

has full rank 7. The second stage examines the remaining exhaustive summary terms

$$\boldsymbol{\kappa}_2 = \begin{bmatrix} k_9 \\ u_2 v_{1,2} \\ u_1 w_{2,2} \\ u_1 v_{1,1} w_{2,2} \\ u_2 w_{1,2} \\ u_1 v_{2,2} \\ u_1 v_{1,1} v_{2,2} \\ k_{16} \\ u_1 x_{3,3} \\ u_1 v_{2,2} x_{3,3} \\ u_1 v_{1,1} x_{3,3} \\ u_1 v_{1,1} v_{2,2} x_{3,3} \\ u_2 x_{2,3} \\ u_2 v_{1,2} x_{2,3} \end{bmatrix},$$

with parameters $\boldsymbol{\theta}_2 = [v_{1,2}, v_{2,2}, w_{1,2}, w_{2,2}, x_{2,3}, x_{3,3}]^T$. The derivative matrix $\partial \boldsymbol{\kappa}_2 / \partial \boldsymbol{\theta}_2$ has full rank 6. Therefore by the two-stage extension theorem, the model can be extended in terms of years of recapture. Adding a year of marking so $n_1 = 3$ while $n_2 = 3$ adds the following exhaustive summary terms

$$\boldsymbol{\kappa}_3 = \begin{bmatrix} Pr(0011) \\ Pr(0010) \\ Pr(0012) \end{bmatrix} = \begin{bmatrix} \phi_{1,3} p_{2,4} \\ 1 - \phi_{1,3} p_{2,4} \\ \bar{\phi}_{1,3} \lambda_{1,3} \end{bmatrix} = \begin{bmatrix} u_3 \\ 1 - u_3 - u_3 x_{1,3} \\ u_1 x_{1,3} \end{bmatrix},$$

with parameters $\boldsymbol{\theta}_1 = [u_3, x_{1,3}]$. The derivative matrix

$$\left[\frac{\partial \boldsymbol{\kappa}_3}{\partial \boldsymbol{\theta}_3}\right] = \begin{bmatrix} 1 & -1 - x_{1,3} & x_{1,3} \\ 0 & -u_3 & u_3 \end{bmatrix},$$

has full rank 2. Therefore $\boldsymbol{\kappa}_{uvwx}$ is an exhaustive summary for any dimension.

Part three:

This part involves checking whether the derivative matrix $\partial\boldsymbol{\kappa}_{uvwx}(\mathbf{s},\mathbf{t},\mathbf{r})/\partial[\mathbf{s},\mathbf{t},\mathbf{r}]$ is full rank and then using the two-stage theorem to show it is always full rank for larger dimensions. Starting with $n_1 = n_2 = 2$ we can reparameterise $\boldsymbol{\kappa}_{uvwx}$ in terms of $\mathbf{s}$, $\mathbf{t}$ and $\mathbf{r}$ to get

$$\boldsymbol{\kappa}_{uvwx}(\mathbf{s},\mathbf{t},\mathbf{r}) = \begin{bmatrix} u_1 \\ v_{1,1} \\ w_{1,1} \\ u_2 \\ x_{1,1} \\ x_{1,2} \\ x_{2,2} \end{bmatrix} = \begin{bmatrix} s_{1,1}s_{2,2} \\ t_{1,1}/s_{1,1} \\ (1 - s_{2,2} - r_{2,2})/s_{2,2} \\ s_{1,2} \\ r_{1,1}/s_{1,1}s_{2,2} \\ r_{1,2}/s_{1,2} \\ r_{2,2}/s_{2,2} \end{bmatrix}, \tag{2}$$

with parameter set $\boldsymbol{\kappa}_{str} = [s_{1,1}, s_{1,2}, s_{2,2}, t_{1,1}, r_{1,1}, r_{1,2}, r_{2,2}]^T$. The derivative matrix

$$\left[\frac{\partial\boldsymbol{\kappa}_{uvwx}(\mathbf{s},\mathbf{t},\mathbf{r})}{\partial\boldsymbol{\kappa}_{str}}\right] = \begin{bmatrix} s_{2,2} & -\frac{t_{1,1}}{s_{1,1}^2} & 0 & 0 & -\frac{r_{1,1}}{s_{1,1}^2 s_{2,2}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{-r_{1,2}}{s_{1,2}} & 0 \\ s_{1,1} & 0 & \frac{-1}{s_{2,2}} - \frac{1-s_{2,2}-r_{2,2}}{s_{2,2}^2} & 0 & -\frac{r_{1,1}}{s_{1,1}s_{2,2}^2} & 0 & -\frac{r_{2,2}}{s_{2,2}^2} \\ 0 & \frac{1}{s_{1,1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{s_{2,2}s_{1,1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{s_{1,2}} & 0 \\ 0 & 0 & -\frac{1}{s_{2,2}} & 0 & 0 & 0 & \frac{1}{s_{2,2}} \end{bmatrix},$$

has full rank 7. A modified PLUR decomposition of $\partial\boldsymbol{\kappa}_{uvwx}(\mathbf{s},\mathbf{t},\mathbf{r})/\partial\boldsymbol{\kappa}_{str}$ shows this the model remains full rank for any value of $s_{1,1}, s_{1,2}, s_{2,2}, t_{1,1}, r_{1,1}, r_{1,2}$ or $r_{2,2}$. Therefore $\boldsymbol{\kappa}_{uvwx}(\mathbf{s},\mathbf{t},\mathbf{r})$ is an exhaustive summary when $n_1 = n_2 = 2$. If we extend the model to add another year of recapture, the parameter set is

$$\boldsymbol{\kappa}_{uvwx}(\mathbf{s},\mathbf{t},\mathbf{r}) = \begin{bmatrix} u_1 \\ u_2 \\ v_{1,1} \\ v_{1,2} \\ v_{2,2} \\ w_{1,1} \\ w_{1,2} \\ w_{2,2} \\ x_{1,1} \\ x_{1,2} \\ x_{2,2} \\ x_{2,3} \\ x_{3,3} \end{bmatrix} = \begin{bmatrix} s_{1,1}s_{2,2}s_{3,3} \\ s_{1,2}s_{2,3} \\ t_{1,1}/s_{1,1} \\ t_{1,2}/s_{1,2} \\ t_{2,2}/s_{2,2} \\ (1-s_{3,3}-s_{3,3})/s_{3,3} \\ (1-s_{2,3}-r_{2,3})/s_{2,3} \\ \left\{(1-s_{2,2}-t_{2,2}-r_{2,2})+t_{2,2}\,(1-s_{3,3})\right\}/s_{2,2}s_{3,3} \\ r_{1,1}/s_{1,1}s_{2,2}s_{3,3} \\ r_{1,2}/s_{1,2}s_{2,3} \\ r_{2,2}/s_{2,2}s_{3,3} \\ r_{2,3}/s_{2,3} \\ r_{3,3}/s_{3,3} \end{bmatrix},$$

with parameters $\boldsymbol{\kappa}_{str} = [s_{1,1}, s_{1,2}, s_{2,2}, s_{2,3}, s_{3,3}, t_{1,1}, t_{1,2}, t_{2,2}, r_{1,1}, r_{1,2}, r_{2,2}, r_{2,3}, r_{3,3}]^T$. Note that the terms $u_2, v_{2,2}w_{1,2}, x_{1,2}$ and $x_{2,3}$ are identical to $u_1, v_{1,1}, w_{1,1}, x_{1,1}$ and $x_{2,2}$ respectively in (2), if $s_{1,1}$ is re-labelled as $s_{1,2}$, $s_{2,2}$ as $s_{2,3}$, $t_{1,1}$ as $t_{1,2}$, $r_{1,1}$ as $r_{1,2}$, and $r_{2,2}$ as $r_{2,3}$. This can then form stage one of the two-stage extension theorem with

$$\boldsymbol{\kappa}_c = \begin{bmatrix} u_2 \\ v_{1,2} \\ w_{1,2} \\ x_{1,2} \\ x_{2,2} \end{bmatrix} = \begin{bmatrix} s_{1,2}s_{2,3} \\ t_{1,2}/s_{1,2} \\ (1-s_{2,3}-r_{2,3})/s_{2,3} \\ r_{1,2}/s_{1,2}s_{2,3} \\ r_{2,3}/s_{2,3} \end{bmatrix},$$

with parameters $\boldsymbol{\theta}_1 = [s_{1,2}, s_{2,3}, t_{1,2}, r_{1,2}, r_{2,3}]$. The derivative matrix

$$\left[\frac{\partial \boldsymbol{\kappa}_1}{\partial \boldsymbol{\theta}_1}\right] = \begin{bmatrix} s_{2,3} & -\frac{t_{1,2}}{s_{1,2}^2} & 0 & -\frac{r_{1,2}}{s_{1,2}^2 s_{2,3}} & 0 \\ s_{1,2} & 0 & -\frac{1}{s_{2,3}} - \frac{1-s_{2,3}-r_{2,3}}{s_{2,3}^2} & -\frac{r_{1,2}}{s_{1,2}s_{2,3}^2} & \frac{r_{2,3}}{s_{2,3}^2} \\ 0 & \frac{1}{s_{1,2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s_{1,2}s_{2,3}} & 0 \\ 0 & 0 & \frac{-1}{s_{2,3}} & 0 & \frac{1}{s_{2,3}} \end{bmatrix},$$

has full rank 5. The second stage involves the exhaustive summary terms

$$\boldsymbol{\kappa}_2 = \begin{bmatrix} u_1 \\ v_{1,1} \\ v_{2,2} \\ w_{1,1} \\ w_{2,2} \\ x_{1,1} \\ x_{2,2} \\ x_{3,3} \end{bmatrix} = \begin{bmatrix} s_{1,1}s_{2,2}s_{3,3} \\ t_{1,1}/s_{1,1} \\ t_{2,2}/s_{2,2} \\ (1-s_{3,3})/s_{3,3} \\ \left\{(1-s_{2,2}-t_{2,2}-r_{2,2})+t_{2,2}\,(1-s_{3,3})\right\}/s_{2,2}s_{3,3} \\ r_{1,1}/s_{1,1}s_{2,2}s_{3,3} \\ r_{2,2}/s_{2,2}s_{3,3} \\ r_{3,3}/s_{3,3} \end{bmatrix},$$

with the parameter set $\boldsymbol{\theta}_2 = [s_{1,1}s_{2,2}, s_{3,3}, t_{1,1}, t_{2,2}, r_{1,1}, r_{2,2}, r_{3,3}]^T$. The derivative matrix $\partial\boldsymbol{\kappa}_2/\partial\boldsymbol{\theta}_2$ has full rank 5. Therefore by the two-stage extension theorem, the model can be extended in terms of years of recapture. Adding a year of marking so $n_1 = 3$ while $n_2 = 3$ adds the following exhaustive summary terms

$$\boldsymbol{\kappa}_3 = \begin{bmatrix} u_3 \\ x_{1,3} \end{bmatrix} = \begin{bmatrix} s_{1,3} \\ r_{1,3}/s_{1,3} \end{bmatrix},$$

with parameters $\boldsymbol{\theta}_1 = [s_{1,3}, x_{1,3}]$. The derivative matrix

$$\left[\frac{\partial\boldsymbol{\kappa}_3}{\partial\boldsymbol{\theta}_3}\right] = \begin{bmatrix} 1 & \frac{-r_{1,3}}{s_{1,3}^2} \\ 0 & \frac{1}{s_{1,3}} \end{bmatrix},$$

has full rank 2. Therefore **s**, **t** and **r** form an exhaustive summary for any dimension. □

1.4. Proof of Theorem 2 (Full Rank Theorem)

In this section we provide a proof of Theorem 2, which states that if the capture-recapture y/z_1 model is full rank, then the capture-recapture-recovery $y/(z_1; z_2)$ model with the same y and z_1, but any z_2 is also full rank.

Consider the exhaustive summary for the capture-recapture-recovery $y/(z_1; z_2)$ model as consisting of two parts. The first part, $\boldsymbol{\kappa}_1$ consists of the terms $s_{i,j} = \phi_{i,j}p_{i+1,j+1}$ for all $i = 1, \ldots, n_2$ and $j = i, \ldots, n_1 + i - 1 \le n_2$ and

$t_{i,j} = \phi_{i,j}(1-p_{i+1,j+1})$ for all $i = 1, \ldots, n_2-1$ and $j = i, \ldots, n_1+i-1 \leq n_2-1$. The second part, $\boldsymbol{\kappa}_2$, consists of the terms $r_{i,j} = (1-\phi_{i,j})\lambda_{i,j}$ for all $i = 1, \ldots, n_2$ and $j = i, \ldots, \min(n_1 + i - 1, n_2)$. Let the parameter vector $\boldsymbol{\theta}_1$ consist of the parameters $\phi_{i,j}$ and $p_{i,j}$, and the parameter vector $\boldsymbol{\theta}_2$ consist of the parameters $\lambda_{i,j}$.

As the capture-recapture y/z_1 model is full rank $\mathbf{D}_1 = \partial\boldsymbol{\kappa}_1/\partial\boldsymbol{\theta}_1$ is full rank. The derivative matrix $\mathbf{D}_2 = \partial\boldsymbol{\kappa}_2/\partial\boldsymbol{\theta}_2$ consists of the terms $-\phi_{i,j}$ on the diagonal and 0 elsewhere. As long as all $\phi_{i,j}$ are non-zero this will always be full rank. Then as $\mathbf{D}_1$ and $\mathbf{D}_2$, are full rank by the extension theorem of [1] the capture-recapture-recovery $y/(z_1; z_2)$ model is full rank. □

2. Web Appendix B: Models with juvenile survival probabilities

This section deals with models which have a different juvenile survival probability compared with adult survival probability. For example you may assume that juvenile survival depends on the time of marking in its first year of life, but then has only a constant yearly survival probability after its first year of life. The notation of the main paper is extended. The capture-recapture model has the notation $x^J/y/z_1$, where x^J denotes juvenile survival probability for the first J years of life (where $1 \leq J < n_2 - 1$), y denotes adult survival probability and z_1 denotes recapture probability. This is extended in the capture-recapture-recovery model to being of the form $x^J/y/(z_1; z_2)$, where z_2 denotes recovery probability. y, z_1 and z_2 can be either constant (C), time-dependent (T), age-dependent (A), or age- and time-dependent (A,T), with x only being either constant (C) or time dependent (T). This changes the juvenile survival probability as the $\mathrm{C}^1/\mathrm{C}/z_1$ model has survival probabilities ϕ_1 and ϕ_a, while the $\mathrm{T}^1/\mathrm{C}/z_1$ model has survival probabilities $\phi_{1,j}$ for $j = 1, \ldots, n_1$ and ϕ_a. We show three tables here: Table 1 is for the capture-recapture model where $J = 1$, as different first-year survival is a commonly used model in ecology (for example [8]), Table 2 is for a general J value in the capture-recapture model, and Table 3 is for $J = 1$ in the capture-recapture-recovery model. It is assumed that there are at least two years of marking and two years of recapture/recovery with

$n_2 \geq n_1$. The second column in the three tables refers to the rank of the model, which is the number of estimable parameters in the model. The third column in Table 1 refers to the parameter deficiency, d, of the model. The third column in Table 2 refers to the parameter deficiency when $(n_2 - n_1) < J$ and the fourth column refers to the parameter deficiency when $(n_2 - n_1) \geq J$. There are models that are excluded from the tables as they are identical to simpler models:

- $C^J/A/z_1$ is identical to A/z_1,
- $T^J/A, T/z_1$ is identical to $A, T/z_1$,
- $C^J/A/(z_1; z_2)$ is identical to $A/(z_1; z_2)$,
- $T^J/A, T/(z_1; z_2)$ is identical to $A, T/(z_1; z_2)$.

Results for these models are given in the main paper. The Full Rank Theorem also reduces the number of rows in Table 3, as if the $x^J/y/z_1$ model is full rank, then the $x^J/y/(z_1; z_2)$ model must also be full rank. The models where this occurs are

- $C^J/C/(z_1; z_2)$,
- $C^J/T/(C; z_2)$,
- $C^J/T/(A; z_2)$,
- $C^J/A, T/(C; z_2)$,
- $C^J/A, T/(T; z_2)$ (for $J = 1$),
- $T^J/C/(C; z_2)$,
- $T^J/C/(T; z_2)$,
- $T^J/C/(A; z_2)$,
- $T^J/T/(C; z_2)$,
- $T^J/T/(A; z_2)$,

- $T^J/A/(C;z_2)$,
- $T^J/A/(T;z_2)$.

Table 1: Table of parameter deficiencies for capture-recapture $x^1/y/z$ models

Model	Rank	Deficiency
C^1/C/C	3	0
C^1/C/T	n_2+2	0
C^1/C/A	n_2+2	0
C^1/C/A,T	$E+2$	0
C^1/T/C	n_2+1	0
C^1/T/T †	$2n_2-1$	1
C^1/T/A	$2n_2$	0
C^1/T/A,T	$E+n_2-1$	1
C^1/A,T/C	$E-n_1+2$	0
C^1/A,T/T †	$E-n_1+n_2$	1
C^1/A,T/A	$E-n_1+n_2$	1
C^1/A,T/A,T †	$2E-2n_1+1$	n_1
T^1/C/C	n_1+2	0
T^1/C/T	n_1+n_2+1	0
T^1/C/A	n_1+n_2+1	0
T^1/C/A,T ‡	$E+n_1+1$	0
T^1/T/C	n_1+n_2	0
T^1/T/T	n_1+2n_2-2	1
T^1/T/A	n_1+2n_2-1	0
T^1/T/A,T ‡	$E+n_1+n_2-2$	1
T^1/A/C	n_1+n_2	0
T^1/A/T	n_1+2n_2-1	0
T^1/A/A	n_1+2n_2-2	1
T^1/A/A,T ‡	$E+n_1+n_2-2$	1

Key: $E = n_1n_2 - \frac{1}{2}n_1^2 + \frac{1}{2}n_1$;
†: when $n_1 = n_2$, then the rank increases by 1 and the parameter deficiency decreases by 1;
‡: when $n_1 = n_2$, then the rank decreases by 1 and the parameter deficiency increases by 1.

References

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Table 2: Table of parameter deficiencies for capture-recapture $x^J/y/z$ models

Model	Rank	Deficiency (1)	Deficiency (2)
C^J/C/C	$J+2$	0	0
C^J/C/T	n_2+J+1	0	0
C^J/C/A	n_2+J+1	0	0
C^J/C/A,T	$E+J+1$	0	0
C^J/T/C	n_2+1	0	0
C^J/T/T	$2n_2-1$	0	1
C^J/T/A	$2n_2$	0	0
C^J/T/A,T	$E+n_2-1$	1	1
C^J/A,T/C	$E-B-n_1+n_2+J+1+G$	0	0
C^J/A,T/T	$E-B-n_1+2n_2+J+G-d$	0	1
C^J/A,T/A	$E-B-n_1+2n_2+J-1+G$	1	1
C^J/A,T/A,T	$2E-B-n_1+n_2+J+G-d$	n_2-J	n_1
T^J/C/C	$B+n_1-n_2+2-G$	0	0
T^J/C/T	$B+n_1+1-G$	0	0
T^J/C/A	$B+n_1+1-G$	0	0
T^J/C/A,T	$E+B+n_1-n_2+1-G-d$	n_1-n_2+J	0
T^J/T/C	$B+n_1-J+1-G$	0	0
T^J/T/T	$B+n_1+n_2-J-1-G$	1	1
T^J/T/A	$B+n_1+n_2-J-G$	0	0
T^J/T/A,T	$E+B+n_1-J-G-d$	n_1-n_2+J+1	1
T^J/A/C	$B+n_1-J+1-G$	0	0
T^J/A/T	$B+n_1+n_2-J-G$	0	0
T^J/A/A	$B+n_1+n_2-J-1-G$	1	1
T^J/A/A,T	$E+B+n_1-J-G-d$	n_1-n_2+J+1	1

Deficiency (1) is when $(n_2-n_1)\leq J$ and Deficiency (2) is when $(n_2-n_1)\geq J$;
Key: $E=n_1n_2-\frac{1}{2}n_1^2+\frac{1}{2}n_1$; $B=\frac{1}{2}J(2n_2-J+1)$;
$G=\frac{1}{2}[(n_2-n_1-1)^2+(n_2-n_1-1)]$ when $(n_2-n_1)<J$ and
$G=\frac{1}{2}(J-1)(2n_2-J-2n_1)$ when $(n_2-n_1)\geq J$.

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Table 3: Table of parameter deficiencies for capture-recapture-recovery $x^1/y/(z_1;z_2)$ models

Model	Rank	Deficiency
C^1/T/(T;C)	$2n_2+1$	0
C^1/T/(T;T) †	$3n_2-1$	1
C^1/T/(T;A)	$3n_2$	0
C^1/T/(T;A,T) †	$E+2n_2-1$	1
C^1/T/(A,T;C)	$E+n_2+1$	0
C^1/T/(A,T;T) †	$E+2n_2-1$	1
C^1/T/(A,T;A)	$E+2n_2$	0
C^1/T/(A,T;A,T)	$2E+n_2-1$	1
C^1/A,T/(A;C)	$E-n_1+n_2+2$	0
C^1/A,T/(A;T)	$E-n_1+2n_2+1$	0
C^1/A,T/(A;A)	$E-n_1+2n_2$	1
C^1/A,T/(A;A,T)	$2E-n_1+n_2$	1
C^1/A,T/(A,T;C)	$2E-n_1+1$	0
C^1/A,T/(A,T;T) †	$2E-n_1+n_2$	1
C^1/A,T/(A,T;A)	$2E-n_1+n_2$	1
C^1/A,T/(A,T;A,T) †	$3E-2n_1+1$	n_1
T^1/C/(A,T;C)	$E+n_1+2$	0
T^1/C/(A,T;T)	$E+n_1+n_2+1$	0
T^1/C/(A,T;A)	$E+n_1+n_2+1$	0
T^1/C/(A,T;A,T) ‡	$2E+n_1+1$	0
T^1/T/(T;C)	n_1+2n_2	0
T^1/T/(T;T) †	n_1+3n_2-2	1
T^1/T/(T;A)	n_1+3n_2-1	0
T^1/T/(T;A,T)	$E+n_1+2n_2-2$	1
T^1/T/(A,T;C)	$E+n_1+n_2$	0
T^1/T/(A,T;T)	$E+n_1+2n_2-2$	1
T^1/T/(A,T;A)	$E+n_1+2n_2-1$	0
T^1/T/(A,T;A,T) ‡	$2E+n_1+n_2-2$	1
T^1/A/(A;C)	n_1+2n_2	0
T^1/A/(A;T)	n_1+3n_2-1	0
T^1/A/(A;A)	n_1+3n_2-2	1
T^1/A/(A;A,T)	$E+n_1+2n_2-2$	1
T^1/A/(A,T;C)	$E+n_1+n_2$	0
T^1/A/(A,T;T)	$E+n_1+2n_2-1$	0
T^1/A/(A,T;A)	$E+n_1+2n_2-2$	1
T^1/A/(A,T;A,T) ‡	$2E+n_1+n_2-2$	1

Key: $E=n_1n_2-\frac{1}{2}n_1^2+\frac{1}{2}n_1$;
†: when $n_1=n_2$, then the rank increases by 1 and the parameter deficiency decreases by 1;
‡: when $n_1=n_2$,then the rank decreases by 1 and the parameter deficiency increases by 1.