New Keynesian Roots

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Abstract

The stability properties of a macromodel are determined by its characteristic roots. Conventional price level determinacy is achieved by imposing restrictions on policy rules so that the characteristic roots satisfy a saddlepath criterion. This paper shows how to choose arbitrary roots for the canonical New Keynesian macromodel using a simple forward-looking Taylor-type policy rule. We are able to show the implications of discretionary and bubble-free policies for the characteristic roots of the model. Our approach sheds considerable light on the so-called Cochrane Critique, where the saddlepath criterion is insufficient to rule out indeterminacy.

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1 Feedback rules, determinacy and characteristic roots

How inflation is determined when monetary policy is set using an interest rate rule has recently become a topic for debate again after a long period of consensus. Textbook New Keynesian analysis (for example Woodford (2003)) would suggest that some simple conditions ensure that policy is well conducted and determinate. These conditions are encapsulated in the rule-of-thumb that is the Taylor Principle, where real interest rates are assumed to rise in response to inflationary pressures. Policy rules that reflect this principle impose saddlepath stability in common policy models, ruling out the multiplicity of equilibria associated with insufficiently active monetary policy. This approach assumes that the characteristic roots of the closed model are such that the resulting dynamic instabilities are cancelled by agents choosing appropriate initial conditions for free variables to yield a unique non-explosive expectational equilibrium. The conditions under which this holds have been extensively explored for variations on both the basic model and the assumed policy rule.

However, particularly since Cochrane (2011) (see also Benhabib, Schmitt-Grohé, and Uribe (2001); McCallum (2009b,a); Cochrane (2009)) it has been argued that such these conditions are insufficient to determine inflation in New Keynesian models, simply because there is nothing intrinsically wrong with an explosive path for prices. The Cochrane Critique is that there exists an infinity of admissible albeit unstable paths for prices; inflation therefore remains undetermined in these models. On the same theme, Atkeson, Chari, and Kehoe (2010) argue for something better than ‘unsophisticated’ monetary policy rules such as these which do no more than ensure determinacy by implying the non-existence of any other equilibrium through this explosiveness. Suddenly common practice looks to be on shaky foundations.

But at the same time policy rules that are argued to guarantee uniqueness with no other admissible solutions have been proposed independently by both Loisel (2009) and Adão, Correia, and Teles (2011). These solutions technically imply infinitely explosive behavior (demonstrated concisely by Loisel (2009)) and do so by adopting what we might describe as structural policy feedback rules. These are rules that for some vector of endogenous variables $Y_t$

\[
\ldots \text{manage to disconnect } Y_t \text{ from } E_t \{Y_{t+k}\} \ldots \text{by partly “mimicking” the structural equations.} \quad \ldots \text{[T]he expectation at date } t \text{ of one of these rules } \ldots \text{has the same forward-looking part as the } \ldots \text{structural equation, so that subtracting one from the other leads to a backward-looking equation } [...]. \text{As a consequence, this system pins down } Y_t \text{ uniquely.}
\]

(Loisel, 2009, p. 1533)
Because these rules are constructed using knowledge of the models structure rather than by using \textit{ad hoc} coefficients, it turns out that these rules can both behave remarkably closely to more conventional alternatives and contain neither the infinitesimal nor near infinite coefficients that might be thought to go along with infinite roots. We might have in mind here a rule for interest rates reacting to inflation such as $i_t = (1 + \theta_\pi) \pi_t$ where $\theta_\pi \rightarrow \infty$. However the structural rules so far investigated have unusual characteristics and appear to be quite different to Taylor-type policies. It would be interesting to establish how far away they really are from their more conventional brethren. It turns out that is also interesting to see how they relate to \textit{targeting rules}.

We will have something to say about each of the issues just raised and more. To do so we will use two different approaches. First, we explicitly focus on the implied characteristic roots of the New Keynesian model. We derive an alternative and more general structural interest rate feedback rule using a quite different approach to either Adão et al. (2011) and Loisel (2009). It can be used to set the characteristic roots of the model to any arbitrary desired values (both stable and unstable) which become arguments of the rule. We illustrate it by applying it to the canonical New Keynesian model, and deriving analytically a forward-looking structural feedback rule to impose the desired values for the characteristic roots. Globally unique policies are a special case of these where we impose particular values for the characteristic roots: this is why Loisel (2009) describes his rules as \textit{bubble free}. Second, we interpret all of these policies as targeting rules. This cuts to the heart of the policy proposals and shows how all of the instrument rules imply essentially the same targeting rule.

There are a number of advantages to this joint approach. Having derived an expression for the policy rule that assigns roots arbitrarily we can use it to analyze the implications of a rich menu of policy prescriptions by appropriate structural re-parameterizations. As a start, we can analyze conventional determinacy conditions for policy rules and show how they relate to each other. Trivially, we can confirm existing determinacy conditions derived by other means but at the same time provide additional insight into the coefficient restrictions that policy rules must satisfy for determinacy to hold and, importantly, where the Cochrane Critique is manifest. We can analyze bubble-free (and globally unique) policies in a common framework to all other feedback rules, remembering that bubble-free policies by definition require particular characteristic roots. We can also show what is implied by various targeting rules, and show how they relate to more conventional instrument rules.

To anticipate the results, it turns out that some of the frequently studied Taylor-type policy rules impose very specific values on the model’s characteristic roots. Thus if we choose to react to, say, expected inflation alone or to react to it with a unit coefficient but allow output feedback, this im-
poses certain model-related values on the characteristic roots. We also show that targeting rules imply particular characteristic roots and must therefore be associated with specific feedback rules. A bubble-free interest rate rule implies a very simple targeting rule in our example model, additionally exposing the underlying preferences of the policymaker. Optimal discretionary policy also implies certain characteristic roots, and it turns out that we can describe the Cochrane Critique in the same way. One surprising conclusion is that discretionary behavior by policymakers is guaranteed to be close to a region of multiple explosive equilibria. We also have something to say about implementing time inconsistent policies; this is a further illustration of the weakness of the saddlepath uniqueness concept hitherto employed.

The paper is organized as follows. In Section 2 we outline the model used and demonstrate how to arbitrarily choose its characteristic roots. To do so we derive a particular policy rule for a New Keynesian model that is a function of the desired characteristic roots and feeds back on forecast values. This enables us to investigate the properties of various policy rules and analyze determinacy regions, further examined in an Appendix. In Section 3 we consider the impact of rules that specifically impose either one or two zero roots in our model. These relate to discretionary and bubble-free behavior respectively. We then reinterpret the policy rules as targeting rules. In Section 4 we discuss what this means for how we should think about policy rules and equilibrium selection. A final section concludes.

## 2 Choosing New Keynesian characteristic roots

In this section we outline the canonical New Keynesian model and then the method suggested by Ackermann (1972) to assign arbitrary characteristic roots. This yields a structural forward-looking interest rate feedback rule. We then give some illustrative arbitrary choices and discuss conventional determinacy analysis.

### 2.1 The canonical New Keynesian model

The model we will use is the canonical deterministic\(^1\) New Keynesian model which comprises three equations

\[
y_t = y_{t+1} - \sigma^{-1} r_t \\
\pi_t = \beta \pi_{t+1}^e + \kappa y_t \\
r_t = i_t - \pi_{t+1}^e
\]

\(^1\)The characteristic roots are unaffected by whether the model is stochastic or not, and neither will the persistence of any shocks matter materially. We omit them with no loss of generality for analytic purposes although there are some implications for the path of, say, implied inflation.
where \( \pi_t \) is the inflation rate, \( y_t \) the output gap, \( i_t \) the nominal interest rate and \( r_t \) the real interest rate, all at time \( t \). All coefficients are defined to be positive, and in what follows we assume \( \beta < 1 \). This is a version of the model derived in, for example, Woodford (2003, ch. 4) or Galí (2008, ch. 3) consisting of a dynamic IS curve (1) and a Phillip's curve (2) together with the trivial Fisher identity (3). All coefficients can be satisfactorily microfounded, and this has become the workhorse specification for policy models.

After eliminating the real interest rate, the model can be written in matrix form as

\[
\begin{bmatrix}
  y_t \\
  \pi_t 
\end{bmatrix} = \begin{bmatrix}
  1 & \frac{1}{\sigma} \\
  \kappa & (\beta + \frac{1}{\sigma})
\end{bmatrix} \begin{bmatrix}
  y_{t+1}^e \\
  \pi_{t+1}^e 
\end{bmatrix} + \begin{bmatrix}
  -\frac{1}{\sigma} \\
  -\frac{\beta}{\sigma}
\end{bmatrix} i_t \\
= A \begin{bmatrix}
  y_{t+1}^e \\
  \pi_{t+1}^e 
\end{bmatrix} + Bi_t. 
\]

(4)

This model is widely used as an illustrative device and is particularly tractable for what follows.

### 2.2 Ackermann’s formula

The problem is then to find some feedback rule \( i_t = K \begin{bmatrix}
  y_{t+1}^e \\
  \pi_{t+1}^e 
\end{bmatrix} \) such that the characteristic roots of \( A + BK \) can be arbitrarily chosen. This is an entirely forward-looking Taylor-type rule, which as we see below is a frequently used variation of the conventional rule. It is also not restrictive, and appropriately re-parameterized can be used to investigate the implications for current dated rules, for example. Notwithstanding the Cochrane Critique, for the model to have a conventionally unique saddlepath solution we would need the characteristic roots of the model under control to be less than unity in absolute value.\(^2\)

There are several well-known formulae for the solution of this problem (see Datta (2003), for example), but we will use one that gives a convenient expression for the New Keynesian model. This is an approach widely used in control system design due to Ackermann (1972), appropriate for single-instrument problems like ours and yielding a unique solution. We implement the formula for the case of two dynamic variables as in the above model, but the analysis generalizes to more.

\(^2\)The more familiar criterion for determinacy when solving the model using the Blanchard and Kahn (1980) method is that both eigenvalues should be greater than unity in absolute value. However, as the model is written with the expectations on the right hand side (essentially inverted) the eigenvalues are now required to be less than unity in absolute value. This formulation, used by Galí (2008) for example, gives more convenient expressions in what follows particularly as we will be interested in infinite roots the other way around, which are conveniently expressed here as zeros.
Ackermann’s formula makes use of a basic result in linear algebra. The Cayley-Hamilton theorem (Lütkepohl, 1996, p. 67) tells us that every $n$-dimensional square matrix satisfies its own characteristic equation, so

$$\prod_{i}(A + BK - \lambda_i I) = 0$$

(5)

where $\lambda_i$ are the $n$ characteristic roots of $A + BK$. Multiplying (5) out and collecting terms we obtain for the $2 \times 2$ case

$$A^2 - (\lambda_1 + \lambda_2) A + \lambda_1 \lambda_2 I + ABK + BK (A + BK) - (\lambda_1 + \lambda_2) BK = 0$$

which can be further rearranged as

$$A^2 - (\lambda_1 + \lambda_2) A + \lambda_1 \lambda_2 I = -\begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} K(A + BK) - (\lambda_1 + \lambda_2) K \end{bmatrix}.$$.

This implies

$$K = -\begin{bmatrix} 0 & 1 \end{bmatrix} \left( [B \ AB]^{-1} (A^2 - (\lambda_1 + \lambda_2) A + \lambda_1 \lambda_2 I) \right)$$

(6)

where the selector matrix $\begin{bmatrix} 0 & 1 \end{bmatrix}$ picks out the last row of the bracketed expression. The formula (6) isolates $K$ given the model and the specified ‘closed loop’ response. Thus $K$ is a function both of the desired characteristic roots and potentially all the parameters of the model.$^3$ This formula can be used as long as the matrix $[B \ AB]$ is non-singular.$^4$

### 2.3 A structural forward-looking Taylor rule

We can apply (6) to the algebraic model (4) which yields a convenient expression with easy interpretation. It is simple to establish that

$$\begin{bmatrix} B & AB \end{bmatrix} = \frac{1}{\sigma} \begin{bmatrix} -1 & -\left(\frac{1}{1 + \beta} + \frac{1}{\sigma} \right) \\ -\kappa & -\kappa \left(1 + \beta + \frac{1}{\sigma} \right) \end{bmatrix}$$

These can be expanded to tractable formulae for higher order systems. The importance of zero roots will become apparent below.

$^3$There are useful special cases. If both characteristic roots are required to be zero then

$$K_0 = -\begin{bmatrix} 0 & 1 \end{bmatrix} [B \ AB]^{-1} A^2$$

which we will in effect use later. If one root is required to be zero and the other some single $\lambda$, then obviously

$$K_\lambda = -\begin{bmatrix} 0 & 1 \end{bmatrix} [B \ AB]^{-1} A(A - \lambda I).$$

These can be expanded to tractable formulae for higher order systems. The importance of zero roots will become apparent below.

$^4$We can weaken any number of conditions at the expense of computational complexity (see Hsu and Chang (1996)). Nonsingularity of $[B \ AB]$ is sufficient for a solution to exist. Other approaches are needed if this condition fails, when it may be that only certain characteristic roots can be arbitrarily chosen. It holds for our model.
with inverse
\[
[B \ AB]^{-1} = \frac{1}{\beta} \begin{bmatrix}
-((1 + \beta)\sigma + \kappa) & (1 + \frac{\sigma}{\kappa}) \\
\sigma & -\frac{\sigma}{\kappa}
\end{bmatrix}.
\] (7)

It is also straightforward to establish that \( A^2 - (\lambda_1 + \lambda_2) A + \lambda_1 \lambda_2 I \) is
\[
\begin{bmatrix}
1 + \frac{\kappa}{\sigma} - \lambda_1 - \lambda_2 + \lambda_1 \lambda_2 & \frac{1}{\sigma} (1 + \beta + \frac{\kappa}{\sigma} - \lambda_1 - \lambda_2) \\
\kappa (1 + \beta + \frac{\kappa}{\sigma} - \lambda_1 - \lambda_2) & \frac{\kappa}{\sigma} + (\beta + \frac{\kappa}{\sigma} - \lambda_1 - \lambda_2) (\beta + \frac{\kappa}{\sigma}) + \lambda_1 \lambda_2
\end{bmatrix}. \quad (8)
\]

Using (7) and (8) in (6) gives the following expression for \( K \)
\[
K = \left[ \sigma \left( 1 - \frac{\lambda_1 \lambda_2}{\beta} \right) + 1 + \frac{\sigma}{\kappa} \left( \beta - \lambda_1 - \lambda_2 + \frac{\lambda_1 \lambda_2}{\beta} \right) \right]
\]
or in terms of an explicit policy rule
\[
i_t = \theta_{yf} y_{t+1}^c + \left( 1 + \theta_{\pi f} \right) \pi_{t+1}^c \quad (9)
\]
where
\[
\theta_{yf} = \sigma \left( 1 - \frac{\lambda_1 \lambda_2}{\beta} \right) \\
\theta_{\pi f} = \frac{\sigma}{\kappa} \left( \beta - \lambda_1 - \lambda_2 + \frac{\lambda_1 \lambda_2}{\beta} \right).
\]

This is an entirely forward-looking Taylor rule of the type that is extensively discussed in Bullard and Mitra (2002), Sec. 3.3. It might seem surprising that it can be used to assign arbitrary characteristic roots, but the simplicity of this formulation is a consequence of the controllability of the New Keynesian model, as \([B \ AB]^{-1}\) exists. It is a rather familiar-looking rule with uncontroversial, structural coefficients. In particular \(\theta_{yf}\) and \(\theta_{\pi f}\) are clearly finite for any choice of characteristic roots such that \(-1 < \lambda_1, \lambda_2 < 1\).

### 2.4 Some arbitrary roots

As an illustration we apply (9) to achieve particular pairs of characteristic roots. In general, a good policy rule should avoid areas that are close to either indeterminacy or instability without the need for any additional criteria. To illustrate this we choose three pairs well away from regions of conventional indeterminacy: these are \((\frac{1}{2}, \frac{1}{2})\), \((\frac{1}{2}, -\frac{1}{2})\) and \((-\frac{1}{2}, -\frac{1}{2})\). As we discuss later, these roots are also well away from the values implied by more conventional policy rules.

**Characteristic roots of \((\frac{1}{2}, \frac{1}{2})\)** Applying (9) with \(\lambda_1 = \lambda_2 = \frac{1}{2}\) implies
\[
i_t = \sigma \left( 1 - \frac{1}{4\beta} \right) y_{t+1}^c + \left( 1 + \frac{\sigma}{\kappa} \left( 1 - \frac{2\beta}{4\beta} \right)^2 \right) \pi_{t+1}^c. \quad (10)
\]

7
Characteristic roots \( (\frac{1}{2}, -\frac{1}{2}) \) Now (9) gives

\[
i_t = \sigma \left( 1 + \frac{1}{4\beta} \right) y_{t+1}^c + \left( 1 + \frac{\sigma}{\kappa} \left( \beta - \frac{1}{4\beta} \right) \right) \pi_{t+1}^c
\]

Characteristic roots of \( (\frac{1}{2}, -\frac{i}{2}) \) As nothing so far rules out complex characteristic roots we can choose them to be \( \pm \frac{i}{2} \) where \( i = \sqrt{-1} \). For the complex conjugate pair \( \lambda_1, \lambda_2 = 0 \) and \( \lambda_1 \lambda_2 = \mu_0^2 + \mu_1^2 = \frac{1}{4} \). Again from (9) the real rule

\[
i_t = \sigma \left( 1 - \frac{1}{4\beta} \right) y_{t+1}^c + \left( 1 + \frac{\sigma}{\kappa} \left( \beta + \frac{1}{4\beta} \right) \right) \pi_{t+1}^c
\]

gives the required complex pair.

In Table 1 we give some representative numbers for rules (10), (11) and (12). These desired roots all imply relatively large feedback coefficients for standard parameter choices. This reflects the need for strong feedback to place the closed loop characteristic roots at their design points. What is not so far obvious is what are reasonable choices for the roots? Good choices will reflect the desired dynamic behavior of the model but also particular design criteria. We discuss these issues below, particularly in the context of discretion and bubble free rules. Next we investigate conventional determinacy regions before discussing some implications of choosing particular algebraic values for characteristic roots.

### 2.5 Conventional determinacy analysis

The model (4) is not usually closed with (9) but often with the contemporaneous Taylor rule

\[
i_t = \theta_y y_t + (1 + \theta_\pi) \pi_t.
\]

Conventional reasoning is that real interest rates should rise to choke off inflation, so it is a feature of Taylor rules that \( \theta_\pi \) is chosen to be strictly positive for the model to have a unique equilibrium. This is often known as the Taylor Principle. Determinacy regions for models closed using (13)

<table>
<thead>
<tr>
<th>( (\lambda_1, \lambda_2) )</th>
<th>( (\theta_y, \theta_\pi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\frac{1}{2}, \frac{1}{2}) )</td>
<td>( (1.494, 5.851) )</td>
</tr>
<tr>
<td>( (\frac{1}{2}, -\frac{1}{2}) )</td>
<td>( (2.505, 15.750) )</td>
</tr>
<tr>
<td>( (\frac{1}{2}, -\frac{i}{2}) )</td>
<td>( (1.494, 25.851) )</td>
</tr>
</tbody>
</table>

Table 1: Selected numerical values for roots, \( \beta = 0.99 \)
and similar rules have been widely studied. Woodford (2003) shows that the determinacy condition is actually slightly weaker than the Taylor Principle might seem to imply, and for the rule (9) there is perhaps surprisingly an upper bound on $\theta_{\pi_f}$. A by-product of our analysis shows that this result obviously obtains directly from (9).

Here we show determinacy analysis for the rule

$$i_t = \left(1 + \theta_{\pi_f}\right) \pi_{t+1}^e$$

leaving richer policy specifications to an Appendix. Comparing (9) and (14) it is clear that the consequence of only feeding back on expected inflation is that the necessary zero feedback on expected output requires $\lambda_1 \lambda_2 = 0$. This follows directly from knowing the assignment rule: any feedback on expected inflation alone imposes the restriction that the product of the characteristic roots is $0$. A further consequence is that they must be of the same sign.

It must then be that the feedback rule collapses to

$$i_t = \left(1 + \sigma (1 + \beta - \lambda_1 - \lambda_2) / \kappa\right) \pi_{t+1}^e$$

so $\theta_{\pi_f} = \varpi (1 + \beta - \lambda_1 - \lambda_2)$ in (14). With this restriction in place the maximum determinate value of $\lambda_1 + \lambda_2$ is less than $1 + \beta$ and the minimum is greater than zero. This is easy to prove as follows. Write $\lambda_1$ as $1 - \varepsilon$ where $\varepsilon \in [0, 1 - \beta]$. By just choosing $\varepsilon$ we set both characteristic roots, as $\lambda_2 = \beta / (1 - \varepsilon)$ by the product restriction. Then

$$\lambda_1 + \lambda_2 - 1 = 1 - \varepsilon + \frac{\beta}{1 - \varepsilon} - 1 - \beta = \frac{(\varepsilon + \beta - 1)\varepsilon}{1 - \varepsilon}.$$

As $(1 - \varepsilon), \varepsilon > 0$ the sign of this expression depends on $(\varepsilon + \beta - 1)$ which is negative for allowable values of $\varepsilon$ which bounds the sum above by $1 + \beta$. Reversing this we get the lower limit. This means the model is determinate for the rule $i_t = \left(1 + \theta_{\pi_f}\right) \pi_{t+1}^e$ on the interval $\theta_{\pi_f} \in (0, \frac{2\beta}{\kappa} (1 + \beta))$. The same determinacy conditions are derived in Woodford (2003), Appendix C or Bilbiie (2008), Appendix B, by different methods. All of the conditions implied by examining the policy rule coefficients in (9) are implied by their analysis, including $\lambda_1 \lambda_2 = 0$, but not exploited.\footnote{This is reflected by the determinant of $A + BK$, which is $|A + B [0 \ 1 + \theta_{\pi_f}]| = \beta$ and therefore is independent of both $\lambda_1$ and $\lambda_2$ and is the product of the characteristic roots.}

3 \ Less conventional analysis: zero roots

In any model with forward expectations solved using the normalization adopted here, zero roots have special significance. A zero root implies that

\footnote{For, say, the parameters $\beta = 0.99$, $\sigma = 1$ and $\kappa = 0.2$ this gives a maximum feedback coefficient $1 + \theta_{\pi_f} < 20.9$.}.
one of the variables has a trivial rational expectation, perhaps as a function of another expectation, but nonetheless one that can be substituted out. Solving the model then becomes much simpler, and for our model is analytic. Two zero roots with two forward expectational variables means that there is no meaningful forward-looking behavior in that any expectation can be trivially predicted from a backward-looking set of equations.

This doesn’t mean that expectations do not feature in the policy rule; agent behavior is still forward looking, it is just that those expectations are trivially formed. This turns out to be the characteristic required of bubble-free policies. One zero root most decidedly does not imply bubble free behavior. What is interesting is that the one-zero root case corresponds to the type of targeting rule associated with discretionary policymaking. It also embodies the kind of private-sector behavior that has been used to analyze the Cochrane Critique in simpler models. We discuss all of these issues further below.

3.1 One zero root

Let us arbitrarily select λ₂ = 0. We would therefore simplify (9) to

\[ i_t = \sigma y_{t+1}^e + \left(1 + \frac{\sigma}{\kappa} (\beta - \lambda_1)\right) \pi_{t+1}^e. \]  

(16)

Substituting out for \( y_{t+1}^e \) and \( \pi_{t+1}^e \) using (1) and (2) and collecting terms implies the targeting rule (see, e.g., Svensson (2003))

\[ 0 = y_t + \frac{\beta - \lambda_1}{\kappa \lambda_1} \pi_t. \]  

(17)

Conventional determinacy implies easy-to-establish limits on \( \frac{\beta - \lambda_1}{\kappa \lambda_1} \) showing that negative values are permissible. Below we will use the obvious result that for \( \frac{\beta - \lambda_1}{\kappa \lambda_1} \) to be positive it must be that \( \beta > \lambda_1 > 0 \), which is within these limits.

What does this mean for solving the model? We know that

\[ y_t = \frac{\lambda_1 - \beta}{\kappa \lambda_1} \pi_t \]  

(18)

so

\[ y_{t+1}^e = \frac{\lambda_1 - \beta}{\kappa \lambda_1} \pi_{t+1}^e. \]

Hence the expected value \( y_{t+1}^e \) can be trivially calculated once \( \pi_{t+1}^e \) is known.

Rearranging the IS curve and using (16) gives

\[ \pi_{t+1}^e - \sigma y_t = i_t - \sigma y_{t+1}^e \]

\[ = \left(1 + \frac{\sigma}{\kappa} (\beta - \lambda_1)\right) \pi_{t+1}^e \]
so that
\[ y_t = \frac{\lambda_1 - \beta}{\kappa} \pi_t^{e} - 1. \]

Together with the Phillips curve this implies
\[
\pi_t = \beta \pi_{t+1}^{e} + (\lambda_1 - \beta) \pi_{t+1}^{e}
\]
\[= \lambda_1 \pi_{t+1}^{e}. \]

This is as we should expect, given that there is only one dynamic variable and one non-zero root.

A specific candidate for the remaining root that yields very convenient expressions is \( \lambda_1 = \beta \). This immediately implies
\[
i_t = \sigma y_t^{e} + \pi_t^{e} \]
(19)
where feedback on expected inflation is unity.\(^7\) Substituting out for \( y_t^{e} \) using the IS curve this simplifies to the targeting rule
\[ 0 = \sigma y_t \Rightarrow 0 = y_t \]
and, of course,
\[ \pi_t = \beta \pi_{t+1}^{e}. \]

This shows that the ‘extreme Keynesian’ targeting assumption that \( y_t = 0 \) makes the model determinate on its own and implies characteristic roots of zero (reflecting the predictability of \( y_t^{e} \)) and \( \beta \). Again, the latter should not be a surprise, as we can establish this by inspection by simply appending this targeting rule to the model. However, this way around it is made obvious that we are imposing a zero root as well.

The ‘one zero root’ case also illustrates that with specific expected output gap and expected inflation feedback we can achieve an output gap target continuously in this model but only if inflation is stabilized. If in addition \( \lambda_1 \) is chosen to be \( \beta \) there is a complete disconnect of output and inflation. So inflation stabilization does not necessarily imply two zero roots: as we shall see later, inflation could be at zero with just a single stable root.

It is a simple matter to show that for one zero and one arbitrary non-zero root \( \lambda_1 \) the dynamics of the solved model must be
\[
\pi_{t+1}^{e} = \frac{1}{\lambda_1 - \beta} \pi_t \]
(20a)
\[ y_t = \frac{\lambda_1 - \beta}{\kappa \lambda_1} \pi_t \]
(20b)
\[ i_t = \left( \lambda_1 + \frac{\sigma}{\kappa} (1 - \lambda_1) (\lambda_1 - \beta) \right) \pi_t. \]
(20c)

\(^7\)Properties of other rules with unit feedback on expected inflation are discussed in the Appendix.
Thus the single root $\lambda_1$ is all that is needed to solve the model, and we can determine that root by choice of policy rule.

This policy regime can be used to illustrate Cochrane’s argument. What is it that rules out any slowly (or indeed rapidly) explosive solution in the nominal variables? The main mechanism is the promise of explosive nominal interest rates that are never observed. However, in the case of $\lambda_1 \neq \beta$ explosive interest rates (and inflation) would imply an explosive output gap, but for $\lambda_1 \approx \beta$ it wouldn’t, or at least not immediately. As a corollary, as $\lambda_1 \to 0$ expected inflation is anticipated to explode at an even faster rate, and to take output with it. We return later to discuss possible values for $\lambda_1$ and why $\lambda_1 \approx \beta$ for policy choices under discretion.

We can make this argument concrete quite easily. Consider a sunspot solution to the model, such that

$$\pi_t = \varphi \pi_{t-1} + \varsigma_t$$

(21)

where we assume that $\varphi$ is some as yet undetermined coefficient, and $\varsigma_t$ is a ‘sunspot’ that can take on any desired value that agents determine. This is essentially (Cochrane, 2011, eq. 4). Obviously from (20a) it must be that $\varphi = \frac{1}{\lambda_1}$. Thus if $|\lambda_1| \geq 1$ we have a indeterminate solution, in that for any value of $\varsigma_t$, $\pi_{t-1}$ there is a stable solution to (21).

If instead $|\lambda_1| < 1$ there is only one stable solution, where $\varsigma_t = \varphi \pi_{t-1}$ so that $\pi_t = 0$. All other solutions are explosive at the rate $\frac{1}{\lambda_1}$. Notice that there is also real explosiveness, despite there being only a single root because $y_t$ is linearly related to $\pi_t$ by (20b). As discussed above, if $\lambda_1 \approx \beta$ then this real explosiveness disappears. Determinacy depends then on the choice of stable eigenvalue $\lambda_1$, and, as Cochrane (2011) points out, quite counterintuitive values and give the same solution.

Atkeson et al. (2010) in a sense respond to this by introducing the concept of ‘sophisticated monetary policies’. These ensure the unique implementation of any desired competitive equilibrium, with the implementation defined both on and off the equilibrium path. We don’t pursue the discussion of this interesting idea here — although we think it invaluable — but instead note that policies defined by Taylor-type rules fall into their category of ‘unsophisticated’ rules. These Atkeson et al. (2010) think are inadequate, claiming that

‘... unsophisticated implementation is deficient because it does not describe how the economy will behave after a deviation by private agents from the desired outcome. This deficiency leaves open the possibility that the approach achieves implementation via nonexistence ... [as] no continuation equilibrium exists after private agent deviations.’

(Atkeson et al., 2010, p. 50)
All of the policy proposals we have just discussed clearly fall into this category: one zero characteristic root leaves the rest of the model dynamics completely determined by the other finite root. That finite stable root, be it positive or negative, is used to justify a single solution \( \zeta_t = \varphi \pi_{t-1} \), as all other dynamic paths are unstable. Indeed, it turns out that we can implement any desired competitive equilibrium by choice of an appropriate unsophisticated equilibria. We return to this further below.

3.2 Discretionary policy

Importantly, the imposition of one zero root and a specific second one is the form of policy rule associated with discretionary equilibrium in this model. To show this we first describe the discretionary equilibrium. Consider the discounted quadratic loss function

\[
W_0 = \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \omega y_t^2) \tag{22}
\]

where \( \omega \) is the relative weight on output gap to inflation stabilization, which can be micro-founded. The discretionary solution to this implies the targeting rule

\[
0 = y_t + \frac{\kappa}{\omega} \pi_t. \tag{23}
\]

Compare this with (17). Thus any choice of a particular \( \omega \) implies one zero root and one that satisfies \( \frac{\beta - \lambda_1}{\kappa \lambda_1} = \frac{\pi}{\omega} \), so \( \lambda_1 = \frac{\beta \omega}{\kappa^2 + \omega} \leq \beta \).

A useful result of this is that discretionary policy rule (or indeed any policy rule with \( \lambda_2 = 0 \) and \( \lambda_1 = \beta \omega/(\kappa^2 + \omega) \) for some \( \omega \)) can be written as

\[
i_t = \sigma y_{t+1}^e + \left(1 + \frac{\sigma}{\kappa} (\beta - \lambda_1)\right) \pi_{t+1}^e \tag{23}
\]

This is a Taylor-type rule that implements discretionary policy in conventional policy rule space.

We can now think about discretionary policy in terms of the implied characteristic roots. The extreme Keynesian and extreme monetarist assumptions of \( \omega \to \infty \) and \( \omega \to 0 \) respectively predictably imply one zero root and a second markedly different root. These are:

- **Extreme Keynesian**: \( \omega \to \infty, y_t \to 0, \theta_{yf} = \sigma, \theta_{\pi f} \to 0, \lambda_1 \to \beta, \lambda_2 = 0 \)
- **Extreme monetarist**: \( \omega \to 0, \pi_t \to 0, \theta_{yf} = \sigma, \theta_{\pi f} \to \sigma \beta/\kappa, \lambda_1 \to 0, \lambda_2 = 0 \)
Discretion still automatically implies one zero and one non-zero root when away from these extremes, and consequently has the characteristics discussed in the last section. The non-zero root imposed by (23) is \( \beta \left( \frac{\omega}{\pi + \omega} \right) \). This implies that although it must be conventionally determinate, the zero root is not exclusively associated with the dynamics of \( y_t \), as \( \frac{\omega}{\kappa + \omega} \neq 1 \). As \( \frac{\omega}{\kappa + \omega} \to 1 \) then \( y_t \to 0 \) for all \( t \) so discretion does not necessarily rule out explosive inflation, depending on the value of \( \omega \). In the limit, extreme monetarism can be interpreted as a targeting rule which implies both characteristic roots are zero; extreme Keynesianism implies only one.

This all implies that the discretionary solution for any value of \( \omega \) must satisfy

\[
\begin{align*}
\pi_{t+1} & = \beta^{-1} \frac{\kappa \omega + \omega}{\omega} \pi_t \quad \text{(24a)} \\
y_t & = -\frac{\kappa}{\omega} \pi_t \quad \text{(24b)}
\end{align*}
\]

where the last equation follows directly from the targeting rule. The equivalent to (21) is therefore

\[
\pi_t = \beta^{-1} \frac{\kappa \omega + \omega}{\omega} \pi_{t-1} + \varsigma_t
\]

which is clearly explosive and implies the same sunspot solution as before. However, as \( \omega \to 0 \) then output volatility becomes very large from (24b) until it suddenly becomes zero at the limit.

### 3.3 Two zero roots

Now we turn to bubble-free policies, which we have already touched on in the last section in the guise of extreme Monetarism. Above we quoted Loisel (2009) who characterized bubble-free feedback rules as ones where forward behavior is substituted out. We do the same, but with the express intent of engendering infinitely unstable dynamic responses. The argument is that if model dynamics are infinitely unstable, then they cannot be sustained for even an infinitesimal period of time.

It is almost too trivial to use (9) to find a rule that gives two zero roots. Setting \( \lambda_1 = \lambda_2 = 0 \) in (9) yields

\[
i_t = \sigma y_{t+1}^e + \left( 1 + \sigma \frac{\beta}{\kappa} \right) \pi_{t+1}^e.
\]

This very simple forward-looking two parameter feedback rule implies exactly the same behavior as the rule suggested by Loisel. Indeed, substituting out for \( i_t \) in (1) gives

\[
y_t = \frac{\beta}{\kappa} \pi_{t+1}^e
\]
and (2) as the reduced-form model.

For this model Loisel (2009) suggests the following as an example of a bubble-free rule

$$i_t = \sigma (y_{t+1}^e - y_t) + \pi_{t+1}^e + \psi \pi_t. \quad (27)$$

It is easy to see why (25) and (27) are equivalent. Immediately the implications of Loisel’s rule (27) are clear; substituting in the IS curve in yields

$$0 = \psi \pi_t \Rightarrow \pi_t = 0. \quad (28)$$

If instead we use (26) to substitute out for $\pi_{t+1}^e$ in (2) we obtain

$$0 = \frac{\beta}{\kappa} \pi_{t+1}^e + \frac{1}{\kappa} \pi_t - y_t = y_t + \frac{1}{\kappa} \pi_t - y_t$$

which implies the targeting rule

$$0 = \frac{1}{\kappa} \pi_t \Rightarrow \pi_t = 0. \quad (29)$$

This implies that the standard Taylor-type rule (25) is functionally equivalent to (27). Both rules must also imply a zero output gap, for the same logic expressed in Loisel (2009, sec. 2.3); zero inflation implies zero output through the IS curve (1). An important consequence is that future inflation is now perfectly predictable.

Separately Adão et al. (2011) suggested a seemingly-different approach, designed instead to guarantee uniqueness rather than bubble-free behavior. These turn out to be equivalent. They suggest a specific policy rule based on the consumption Euler equation. For the time being we interpret their proposed interest rate policy rule (equation (2.5) in their paper) as

$$i_t = r_t + p_t \quad (30)$$

where $r_t$ is the real interest rate and $p_t$ is the price level, defined as

$$p_t = \pi_t + p_{t-1}. \quad (31)$$

This implies

$$i_t = r_t + \pi_{t+1}^e + p_t$$

which is the Fisher identity (3) with the current price level added. This therefore imposes

$$p_t = 0$$

or

$$\pi_t = -p_{t-1} \quad (32)$$

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which superficially differs from the Loisel targeting rule (29) derived above. However, in effect they will behave in exactly the same way. Potentially in the first period inflation jumps to clear any past deviations from target prices, but beginning from equilibrium, nothing else is different. The material effect is that \( \pi_{t+1}^e \) is essentially exogenous at time \( t \).

We can emphasize this by noting that instead of using (31) we could use the rule

\[
i_t = r_t + \pi_{t+1}^e + \psi \pi_t
\]

which uses the Fisher identity instead of the IS curve to imply (28). In fact Adão et al. (2011) effectively suggest

\[
i_t = r_t + \psi p_t^{e_{t+1}}
\]

\[
= r_t + \sum_{j=1}^{\infty} \psi_j \pi_{t-j}^e
\]

which requires \( \psi = 1 \) or equivalently \( \psi_j = 1 \) for all \( j \) for the rule to work otherwise the Fisher identity need not be satisfied simultaneously with \( p_t = 0 \). From (33) it is clearly only necessary for \( \psi_{-1} = 1, \psi_0 \neq 0 \).

Having established this, another price-level based equivalent possibility is, of course,

\[
i_t = \sigma (y_{t+1}^e - y_t) + p_t^{e_{t+1}}.
\]

(34)

Again, it should be easy to see why this gives the same answer as (30). This is effectively the targeting rule \( p_t = 0 \) as before, where we have simply used the IS curve instead of the Fisher identity and substituted \( p_t^{e_{t+1}} \) for \( \pi_{t+1}^e \).

At this point we can usefully recap on the various suggestions.

1. Adão et al. (2011) suggest an Euler-equation based rule that leaves the price level after substituting out for expectations using the structural equations of the model, with no other possible solution as in equilibrium expectations are structurally irrelevant. As we have seen, the key to bubble free behavior is making nominal variables perfectly predictable.

2. Relatedly, Loisel (2009) outlines an approach where expectations are substituted out in the model by using an interest rate rule that mimics the structural impact of expectations. For the model we have discussed, a policy rule of the IS curve with an additional contemporaneous inflation rate is one such rule. With a trivial role for expectations, model equilibria are unique and (equivalently) bubble free.

3. Using Ackermann’s (1972) formula we can derive a minimum-dimension feedback rule that arbitrarily assigns characteristic roots, in particular to the zero values implied by both of the previous approaches. This rule is of a familiar forward-looking Taylor type.
4. All of these rules are actually instrument-rule representations of one of two targeting rules: either $\pi_t = 0$ or $\pi_t = -p_{t-1}$. The difference between these two targeting rules is immaterial, so we can treat them as identical.

In none of these rules is it the case that expectations do not matter. It is only that these expectations are trivial to calculate, as all of the roots associated with them are zero.

### 4 Implications for policy rules

The two formulations (33) and (27) are exactly equivalent, and of course (33) would be a perfectly acceptable bubble-free rule. Indeed, the apparent difference that the Adão et al. (2011) approach implies price level targeting (or that the level of prices must enter into interest rate determination) whilst the Loisel (2009) rule does not is easily revealed to be unimportant. We have derived a price level version of the Loisel (2009) rule and an inflation version of the Adão et al. (2011) one. It is the given restriction on the *allowable value of current inflation through a targeting rule* that does the work of generating both of the zero characteristic roots.

We can therefore interpret this targeting rule as one which imposes

$$
\pi_t = \xi_t
$$

(35)

where $\xi_t$ is some appropriate predetermined sequence. An exogenous sequence can have no implication for the characteristic roots. To turn this on its head, we can then see that *any* desired exogenous sequence of inflation rates can be implemented via a targeting rule and that for this model any of the suggested bubble-free rules could be a formulation of targeting rule. Above we used $\xi_t = p_{t-1}$, $\xi_t = 0$ and so on.

This can be taken further. For example, Adão et al. (2011) suggest that this implies that as any desired policy can be implemented uniquely using their rule, this includes the Pareto optimum. Consider the fully optimal sequence for $\pi_t$ associated with (22). First order conditions associated with (22) under commitment are

$$
0 = y_0 + \frac{\kappa}{\omega} \pi_{0}^t
$$

(36a)

$$
0 = \Delta y_{t>0} + \frac{\kappa}{\omega} \pi_{t>0}^t
$$

(36b)

which imply a particular path for $\pi_t$ for every $t$. See, e.g., Woodford (2003, ch. 7, sec. 5.1) or Galí (2008, sec. 5.1.2) for discussion and derivations. Having determined $\pi^*_t$ then clearly a path for $\pi_t = \pi_t^*$ could be implemented using the targeting rule, and this would uniquely impose that solution. Commitment in this sense to a unique (bubble free) rule allows us to implement
any desired sequence including one that is time inconsistent. Thus a so-called unique solution can be a time inconsistent one. This is a perfect example of how a simple rule is a commitment device. The sustainability of the path for $\pi^*_t$ is a direct consequence of commitment; we have simply re-located the commitment to a path to a commitment to a rule to implement that path.

This is at the heart of the Cochrane Critique. Cochrane disputes the usefulness of standard equilibria, as

‘[h]yperinflating away the entire monetary system ... [is] perhaps more effective than an inflation or deflation that slowly gains steam.

‘However, it is not clear that [this proposal] rule[s] out equilibria. A currency can be completely inflated away in finite time.’

(Cochrane, 2011, pp. 588-9)

The results for one and two zero roots show that for the canonical New Keynesian model the extreme Keynesian targeting rule ($y_t = 0$) under this interpretation is insufficient to determine inflation.\(^8\) It is not that this targeting rule is conventionally indeterminate; on the contrary as we showed above it gives a zero root and a $\beta$ root. What it implies is that inflation is unaffected by aggregate demand and so there is nothing to rule out an unstable path for inflation.

So this is a stable but non-zero root that doesn’t determine inflation. But the extreme monetarist assumption that $\pi_t = \xi_t$ at all times must be sufficient because we determine inflation directly in every period. This also implies two zero roots, consistent with bubble-free, unique behavior. But are bubble-free policies with only zero roots really viable policy rules that exclude possible explosive equilibria, as implied by, say, Adão et al. (2011)?

Is it really the case that forward expectations can be systematically neutralized by monetary policy alone? In Cochrane’s terms this would imply the ability, the credibility and the will to inflate away the currency instantaneously. Atkeson et al. (2010) would presumably argue that such bubble-free, unique-equilibrium policies imply an extreme form of implementation by non-existence, and as such assume monetary Armageddon in the face of out-of-equilibrium behavior. This also sits oddly with the time inconsistency argument. If the implied path of $\xi_t$ is time inconsistent, why is a rule that implements it by promising extreme (but unobserved) behavior feasible?

Even on a more prosaic level, all of this seems an odd implication of the policy rule

$$i_t = \sigma y^e_{t+1} + \left(1 + \sigma \frac{\beta}{\kappa}\right) \pi^e_{t+1}$$

\(^8\)See also, in particular, the discussion in McCallum (2009b), sec. 2.
but follows from our analysis. With \( \sigma = 1 \), \( \kappa = 0.1 \) and \( \beta = 0.99 \) this implies (we would argue) the perfectly uncontroversial although markedly anti-inflationary parametric rule of

\[
i_t = y_{t+1}^e + 10.9\pi_{t+1}^e.
\]

It would be a surprise if agents interpreted this rule as embodying such a strong reaction to behavioral anomalies. Indeed McCallum (2009a) finds it difficult to see the mechanism so emphasized by Cochrane (and indeed Atkeson et al.) of ‘hyperinflationary threats’ to ‘blow up the world’. As he rightly identifies, for the model under discussion a Taylor-type rule is assumed to be followed with no contingencies, so

‘...[i]f “threats” were relevant there would be a threatened departure from this rule, to be invoked in certain specified situations. ...[I]n the ... model under discussion there is nothing of that type. What the central bank’s rule promises is to make nominal interest rates higher than otherwise when inflation is above its target value. In the presence of some price-level stickiness ... the higher nominal interest rate brings about a reduced level of real aggregate demand.’

(McCallum, 2009a, p. 1115)

Our analysis should make this mechanism crystal clear, particularly in the context of bubble-free rules written in Ackermann form. In the ultimate example of implementation by non-existence, the policy rule looks like a fairly standard one, but only one equilibrium doesn’t involve all participants choosing infinities. So what we might describe as conventional equilibrium behavior is just as described; standard Taylor-type interest rate reactions go along with an equilibrium driven by the Taylor principle. With a bubble-free rule there is no other solution as output, inflation and interest rates would all need to be infinite. This is entirely driven by the policy rule; any other policy rule — even one that involves reacting to output and inflation in almost the same way — would not involve those infinities, but would involve explosive behavior. So the ‘threats’ are inherent to the policy rule. All that is needed is for agents to choose an initial value of, say, inflation off this equilibrium path (equivalently that \( \varsigma_0 \neq \varphi \pi_{-1} \)). Then the announced path of interest rates given inflation and output are not sufficient to stabilize them and instead they all follow unstable trajectories. Cochrane’s objection to this as a way of determining inflation in these models is that it is only on the single stable path that the logic of raising real interest rates to reduce inflation works, a justification he describes as old Keynesian thinking.

A particular problem is what this implies for discretionary equilibria. In Table 2 we indicate the values of \( \lambda_1 \) implied by particular model parameterizations. For small values of \( \kappa \) and larger values of \( \omega \) the remaining stable
Table 2: Characteristic roots, $\beta = 0.99$, $\kappa = 0.025$

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$\omega = 0.05$</th>
<th>$\omega = 0.1$</th>
<th>$\omega = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1 = \beta \omega / (\kappa^2 + \omega)$</td>
<td>0.9778</td>
<td>0.9839</td>
<td>0.9869</td>
</tr>
<tr>
<td>$\beta - \lambda_1$</td>
<td>0.0122</td>
<td>0.0061</td>
<td>0.0031</td>
</tr>
</tbody>
</table>

root is remarkably close to $\beta$. Discretion inevitably puts us somewhere near a region for the characteristic roots where the Cochrane Critique is most effective. This is important; even if a policymaker would like to avoid explosive indeterminacy the inability to commit to a rule makes it more likely. The oft-repeated argument for a less than socially optimal value of $\omega$ for our monetary policy makers is reinforced by a renewed concern for any concomitant excessive inflation.

Perhaps this explosive variety of nominal indeterminacy could be avoided by policymakers choosing rules that guarantee real as well as nominal explosiveness, as long as agents believe that such real explosions will be validated. This doesn’t feel very satisfactory as a policy prescription: perhaps Axel Leijonhufvud has a point, when he remarks that the existence of bubbles that subsequently burst is strong evidence that such transversality conditions are ‘utter superstition’ (Leijonhufvud, 2011, p. 6).

The bubble-free approach (in particular the appeal to global uniqueness) gives center-stage to such imaginary figments. It is only with them that no other equilibria exists. In a sense, the Taylor principle is essentially a statement about characteristic roots; uniqueness is a statement about how all agents respond to those roots. The policy rule is completely consistent with explosiveness if agents do not rule it out and completely consistent with stabilization if they do. In the case where the policymaker chooses roots of zero and $\beta$ and there are no output consequences why would agents rule out nominal explosiveness? Further, if only discretionary behavior by policymakers is feasible then all the other agents in the model impose roots of zero and $\beta \frac{\omega}{\kappa^2 + \omega}$. Then there may be almost no immediate consequences of explosive behavior if $\frac{\omega}{\kappa^2 + \omega}$ is sufficiently close to unity.

5 Conclusions

We began this paper by deriving a simple forward-looking Taylor rule to arbitrarily choose the characteristic roots of the canonical New Keynesian model. That this can be done is not a new result — we do after all make use of a forty-year-old formula — but is perhaps an unfamiliar one, and the rule itself is a radical reinterpretation of standard analysis. Just playing around with the implications of using a rule to select characteristic roots is an education in itself.
In this vein a natural application is to investigate issues such as determinacy, as characteristic roots can be chosen to ensure conventional saddlepath stability. Although in the paper we investigate one simple model, this approach can be extended to derive forward-looking rules for larger models, although there may be a need to resort to numerical methods for more than one extra state. All of this is implicit using a more standard approach; we emphasize that particular rules have exact implications for characteristic roots.

But focusing on the characteristic roots sheds light on several issues of current concern. One extremely useful by-product is showing how certain policies can be interpreted as targeting rules. An important contribution is to start thinking of discretionary policy rule in terms of the implied characteristic roots. This allows us to investigate the Cochrane Critique when the policymaker cannot commit to a particular rule or set of characteristic roots. Interestingly, discretion makes the Cochrane Critique more likely to bite in the limit simply because of the characteristic roots this implies. Commitment solutions — including bubble-free ones — don’t share this problem, but relocate it by making the alternative unthinkable. We would argue this also makes them untenable.

At one point, the economics literature used to make a distinction between real and nominal indeterminacy. Here we investigated rules that imply real and nominal explosiveness, and considered how this might impact on equilibrium selection. Purely nominal explosions rule nothing out; ‘sufficient’ real explosions may. But we would need to define sufficiency as it seems unlikely that policies that implied finite real effects when discounted would suffice. And real explosiveness needs to be credible to rule out admissible solutions; if it is, then in the extreme this makes any equilibrium supportable, including time inconsistent ones.
References


A Conventional determinacy analysis

No feedback on expected output In Section 2 we examined the determinacy region for the rule (14). The analysis yields an additional minor insight. It turns out that over most of the conventional determinacy interval there are complex conjugate pairs of stable characteristic roots. This follows because as $\lambda_1 + \lambda_2 = 2\mu_0$ and $\lambda_1\lambda_2 = \mu_0^2 + \mu_1^2 = \beta$ there exist pairs of roots on the interval $[-\sqrt{\beta}, \sqrt{\beta}]$ indexed by choice of $\mu_0$ which then have the product $\beta$ by choice of $\mu_1$. For example, if $\theta_{\pi_f} = \frac{\sigma}{\pi}(1 + \beta)$ this is only consistent with $\lambda_1, \lambda_2 = \pm \sqrt{\beta}$, so that they sum to zero with product $\beta$. The values where feedback produces both determinacy and real roots is where $0 < \theta_{\pi_f} < \frac{\sigma}{\pi}(1 + \sqrt{\beta})$ and $x_{\pi_f} (2 + 2\beta - \sqrt{\beta}) < \theta_{\pi_f} < \frac{\sigma}{\pi}(2 + 2\beta)$. Forecast inflation targeting whilst determinate over the given range is associated with mostly complex roots.

Unit feedback on expected inflation This is also easy to characterize from our rule. Let one characteristic root equal $\beta$, say $\lambda_2$, then (9) gives

$$i_t = \sigma (1 - \lambda_1) y_{t+1} + \pi_{t+1}$$

which implies that whatever the choice for $\lambda_1$, unit feedback on $\pi_{t+1}$ implies $\lambda_2 = \beta$. For determinacy we need only consider appropriate feedback to set $\lambda_1$ by choice of $\theta_{yf} = \sigma (1 - \lambda_1)$. It is easy to see determinacy requires that $\theta_{yf} \in (0, 2\sigma)$.

This shows that if $\theta_{\pi_f} = 0$ holds for models so that inflation forecast feedback is unity, determinacy is implied by any positive forecast output feedback up to $2\sigma$. Further, using (1) we can rewrite the rule as a feedback on $y_t$

$$i_t = \sigma \lambda_1^{-1} (1 - \lambda_1) y_{t+1} + \pi_{t+1}$$

It is easy to establish that this implies unit feedback on expected inflation and given positive $\sigma$ then any positive coefficient on contemporaneous $y_t$ makes the model conventionally determinate. This is a robust policy specification, requiring no knowledge of the underlying model parameters other than sign: $\theta_{\pi_f} = 0, \theta_{y} > 0$ is all that is required, rather like an ‘alternative’ Taylor principle. If we do know $\sigma$ we can work out the remaining characteristic root (the one that isn’t $\beta$) from (38).\(^9\)

Non-zero feedback on expected inflation and current output More generally we can allow for any non-zero feedback on expected inflation and

\(^9\)Bilbiie (2008) shows that for a class of models with some consumers facing limited asset market participation, the sign of $\sigma$ can switch. This inverts the Taylor principle result and requires less than unit absolute feedback on inflation. In (38) this is obviously manifested as requiring a negative coefficient on $y_t$. 

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mix it with contemporaneous feedback on the output gap. Take equation (9) and substitute out using (1) to give

\[ i_t = \sigma \left( \frac{\lambda_1 \lambda_2}{\beta} - 1 \right) y_t + \left( 1 + \frac{\sigma}{\kappa} + \frac{\sigma \beta}{\kappa \lambda_1 \lambda_2} (\beta - \lambda_1 - \lambda_2) \right) \pi_{t+1}^e. \] (39)

The intuition from this formulation is much more muddled. The important message from this parameterization is that the elimination of expected output by substituting out using (1) puts \( \lambda_1 \lambda_2 \) in the denominator of the expression for \( \theta_{\pi I} \). Thus smaller roots require ever-larger feedback on expected inflation but not the output gap. Trivially, imposing \( \lambda_1 \lambda_2 = \beta \) yields the same no output feedback result as before. As should be expected this removes the roots from the denominator in the term in \( \theta_{\pi I} \) so that it reduces back to (15). However, considering contemporaneous feedback for output means that two zero roots would require near-infinite feedback on expected inflation. That contemporaneous feedback requires large coefficients is compounded when the rule become a feedback on contemporaneous output and inflation. Substituting out for \( \pi_{t+1}^e \) in (39) using the Phillips curve shows why.