Chapter 4: Renewal Processes

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for all $t \geq 0$ with the convention $\max \emptyset = 0$. Then the continuous time process $\mathcal{N} = (N_t : t \in \mathbb{R}_0^+)$ is called a **renewal process**.

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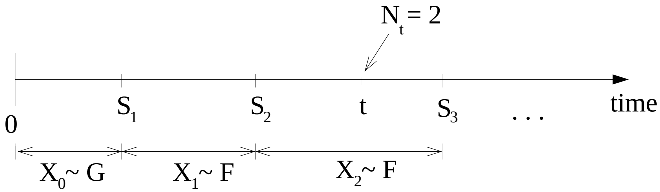

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$\mathcal{N}$ is a **counting process**, i.e. $N_t \in \mathbb{N}_0$ for all $t \geq 0$ and $N_s \leq N_t$ for all $s \leq t$.

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A Poisson process with intensity λ is an ordinary renewal process with $F(x) = G(x) = 1 - e^{-\lambda x}$, i.e. the renewal intervals have an exponential distribution. Thus a renewal process can be seen as a generalization of the Poisson process with respect to the distribution of the renewal intervals.

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$$X_n = \tau_j^{(n+1)} - \tau_j^{(n)}$$

for $n \in \mathbb{N}$.

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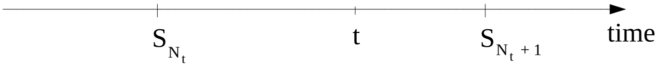
Proof of theorem 6.4

By definition of N_t, the inequalities $S_{N_t} \leq t < S_{N_t+1}$ hold with probability one for all times t.



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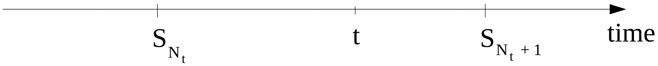

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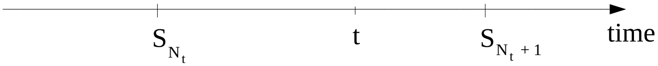

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Thus a Poisson process with intensity λ has at time t a Poisson distribution with parameter $\lambda \cdot t$. Moreover, the intensity λ is also the rate of the Poisson process, since a mean renewal interval has length $1/\lambda$. Now theorem 6.4 states that a consistent statistical estimator for the intensity λ is given by $\hat{\lambda} = N(t)/t$.

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which implies that the event $N_t \leq k$ depends only on $X_0, \ldots, X_k$.

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$$\mathbb{E}\left(\sum_{n=0}^{S} X_n\right) = \mathbb{E}(X_0) + \mathbb{E}(S) \cdot m$$

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$$\mathbb{P}(S \geq n|\mathcal{X}) = 1 - \mathbb{P}(S \leq n-1|\mathcal{X}) = 1 - \mathbb{P}(S \leq n-1|X_0, \ldots, X_{n-1})$$

for all $n \in \mathbb{N}$.

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Theorem 6.12: Elementary renewal theorem

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which trivially holds for the case $m = \infty$.

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$$\tilde{X}_n = \min(X_n, M)$$

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$$\sum_{n=0}^{\tilde{N}_t} \tilde{X}_n \leq t + M$$

holds almost certainly for all $t \geq 0$.

Proof of theorem 6.12 (contd.)

Taking expectations and applying Wald's lemma, we obtain

$$\mathbb{E}(X_0) + \mathbb{E}(\tilde{N}_t) \cdot \tilde{m} = \mathbb{E}\left(\sum_{n=0}^{\tilde{N}_t} \tilde{X}_n\right) \leq t + M$$

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for any constant M. Now the result follows for $M \to \infty$.

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Let U denote a random variable which is uniformly distributed on the interval $]0,1[$. Further define the random variables $(V_n : n \in \mathbb{N})$ by

$$V_n := \begin{cases} 0, & U > 1/n \\ n, & U \leq 1/n \end{cases}$$

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Since $U > 0$ with probability one, we obtain the limit

$$V_n \to 0, \qquad n \to \infty$$

with probability one.

Remark on the proof (contd.)

On the other hand, the expectation for V_n is given by

$$\mathbb{E}(V_n) = n \cdot \mathbb{P}(U \leq 1/n) = n \cdot \frac{1}{n} = 1$$

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for all $n \in \mathbb{N}$ and thus $\mathbb{E}(V_n) \to 1$ as $n \to \infty$.