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EXIT PROBLEMS FOR REFLECTED MARKOV-ADDITIVE PROCESSES WITH PHASE–TYPE JUMPS

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Abstract

Let $(\mathcal{X}, \mathcal{J})$ denote a Markov-additive process with phase–type jumps (PH-MAP) and denote its supremum process by $\mathcal{S}$. For some $a > 0$, let $\tau(a)$ denote the time when the reflected process $\mathcal{Y} := \mathcal{S} - \mathcal{X}$ first surpasses the level a. Further, let $\tau_-(a)$ denote the last time before $\tau(a)$ when $\mathcal{X}$ attains its current supremum. In this paper we shall derive the joint distribution of $S_{\tau(a)}$, $\tau_-(a)$ and $\tau(a)$, where the latter two shall be given in terms of their Laplace transforms. Furthermore, we define scale functions for PH-MAPs and remark on some of their properties. This extends recent results for spectrally negative Lévy processes to the (dense) class of PH-MAPs. The result is of interest to applications such as the dividend problem in insurance mathematics and the buffer overflow problem in queueing theory. Examples will be given for the former.

Keywords: Markov-additive process; Lévy process; reflection; supremum; exit problem; dividends; buffer overflow

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1. Introduction

Markov–additive processes are a powerful generalisation of Lévy processes, becoming more and more popular in stochastic modelling. Based on an underlying finite state Markov process $\mathcal{J}$, called the phase process, the level process $\mathcal{X}$ evolves like a Lévy process for which the parameters change in time according to the phase process $\mathcal{J}$. Furthermore, phase changes (i.e. jumps of $\mathcal{J}$) may induce jumps of the level process $\mathcal{X}$. The joint process $(\mathcal{X}, \mathcal{J})$ is called a Markov–additive process (MAP).

A textbook introduction to MAPs is given in [1], chapter XI. First passage times (or the one-sided exit problem) are derived via martingales in [4] and solved iteratively in [5]. The two-sided exit problem is solved in [11] for MAPs with phase-type jumps (PH-MAPs). The

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class of PH-MAPs is dense within the class of all MAPs, see [2], proposition 1. The Gerber-Shiu function (which is quite popular in insurance mathematics) has been derived in [6] for the same class of MAPs.

It is this class of MAPs for which we wish to solve the following problem. Denote the supremum process of $\mathcal{X}$ by $\mathcal{S}$. For some $a > 0$, let $\tau(a)$ denote the time when the reflected process $\hat{\mathcal{X}} := \mathcal{S} - \mathcal{X}$ first surpasses the level a. Further, let $\tau_{-}(a)$ denote the last time before $\tau(a)$ when $\mathcal{X}$ attains its current supremum. We shall derive the joint distribution of $S_{\tau(a)}$, $\tau_{-}(a)$ and $\tau(a)$, where the latter two shall be given in terms of their Laplace transforms. This extends recent results for spectrally negative Lévy processes [13] to PH-MAPs. Even for the common subset of Lévy processes with phase-type jumps the approach in this paper may be advantageous, since the scale functions (which are the main ingredients in the formulas) are given explicitly.

The result is of interest to applications such as the dividend problem in insurance mathematics and the buffer overflow problem in queueing theory. An algorithmic solution for the time to buffer overflow in a Markov-additive framework is given in [4], section 6, see also [3]. A partial result of the present paper in the context of queueing theory is contained in [7]. An algorithmic solution for the expectation of the total dividend payments before ruin is presented in [8]. Moments of dividend payments in a Markov-additive risk model without Brownian component are derived in [12].

The analysis is performed mainly by matrix-analytic methods using probabilistic arguments wherever possible. This naturally results in formulas containing matrices which are to be computed via fixed point iterations. We shall present examples for the simple cases allowing explicit scalar solutions. This restriction is due to the circumstance that only for these there are solutions in the literature which can be compared with results in the present paper.

The paper is structured as follows. Section 2 contains an exact definition of the problem to be analysed. In section 3, preparatory results from recent literature are presented and scale functions for MAPs with phase-type jumps are introduced. Section 4 finally contains the main result. Examples will be developed throughout the paper in subsequent stages.

2. The exit problem for reflected MAPs

Let $\tilde{\mathcal{J}} = (\tilde{J}_t : t \geq 0)$ be an irreducible Markov process with finite state space $\tilde{E}$ and infinitesimal generator matrix $\tilde{Q} = (\tilde{q}_{ij})_{i,j\in\tilde{E}}$. We call $\tilde{J}_t$ the phase at time $t \geq 0$ (another common name is regime). Define the real–valued process $\tilde{\mathcal{X}} = (\tilde{X}_t : t \geq 0)$ as evolving like a Lévy process $\tilde{\mathcal{X}}^{(i)}$ with parameters $\tilde{\mu}_i$ (drift), $\tilde{\sigma}_i^2$ (variation), and $\tilde{\nu}_i$ (Lévy measure) during intervals when the phase equals $i \in \tilde{E}$. For the sake of a more concise presentation we exclude the case of $\tilde{\mu}_i = \tilde{\sigma}_i^2 = 0$, i.e. a pure jump process or the constant zero process, for any phase $i \in \tilde{E}$. Whenever $\tilde{\mathcal{J}}$ jumps from a state $i \in \tilde{E}$ to another state $j \in \tilde{E}$, this may be accompanied by a jump of $\tilde{\mathcal{X}}$ with some distribution function F_{ij}. Then the two–dimensional process $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ is called a Markov–additive process (or shortly MAP). In short, a MAP is a Markov-modulated Lévy process with possible jumps at phase changes. For a textbook introduction to MAPs see [1], chapter XI.

Define the supremum process $\tilde{\mathcal{S}} = (\tilde{S}_t : t \geq 0)$ by $\tilde{S}_t := \sup_{s\leq t} \tilde{X}_s \vee 0$ for all $t \geq 0$, and the reflected process $\tilde{\mathcal{Y}} := \tilde{\mathcal{S}} - \tilde{\mathcal{X}}$. For a fixed level $a > 0$ let

$$\tilde{\tau}(a) := \inf\{t \geq 0 : \tilde{Y}_t \geq a\} \quad \text{and} \quad \tilde{\tau}_-(a) := \sup\{t \leq \tilde{\tau}(a) : \tilde{Y}_t = 0\}$$

We shall seek to determine the joint distribution of $\tilde{\tau}(a)$, $\tilde{\tau}_-(a)$, and $\tilde{S}_{\tilde{\tau}(a)}$ in the form of the measure

$$\mathbb{E}\left(e^{-\alpha(\tilde{\tau}(a)-\tilde{\tau}_-(a))}e^{-\gamma\tilde{\tau}_-(a)}; \tilde{S}_{\tilde{\tau}(a)} \in dx\right)$$

where $\alpha, \gamma, x \geq 0$.

Example 1. We consider the classical compound Poisson model. Inter–claim times and claim sizes are iid exponential with parameter $\lambda > 0$ and $\beta > 0$, respectively. The rate of premium income is $c > 0$. Denote the initial risk reserve by $u \geq 0$. This model has been examined in [9]. The risk reserve at time $t \geq 0$ is given by

$$\tilde{X}_t = u + ct - \sum_{n=0}^{N_t} C_n \tag{1}$$

where $(N_t : t \geq 0)$ is a Poisson process with intensity λ and the C_n, $n \in \mathbb{N}$, are iid random variables with exponential distribution of parameter β.

The risk reserve process can be analysed as a MAP with exponential (and hence phase–type) negative jumps with parameter β. For this, we would need only one phase, i.e. $|\tilde{E}| = 1$.

This phase governs a Lévy process with parameters $\tilde{\sigma} = 0$, $\tilde{\mu} = c$, and Lévy measure $\tilde{\nu}(dx) = \lambda e^{-\beta\cdot(-x)}\beta dx$ for all $x < 0$.

If any risk reserve above u is paid out as dividends immediately (i.e. a constant barrier strategy), then $\tilde{\tau}(u)$ is the time of ruin under this strategy, $S_{\tilde{\tau}(u)}$ is the total amount of dividends paid before ruin, and $\tilde{\tau}_-(u)$ is the last time before ruin that dividends are paid.

3. Preliminaries

3.1. Markov–additive Processes with phase–type Jumps

In this section we introduce the restriction that all jumps have a phase-type distribution. Then we construct a new MAP $(\mathcal{X}, \mathcal{J})$ from the given MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ without losing any information. This new MAP will have continuous paths which simplifies the one- and two-sided exit problems (cf. sections 3.2 and 3.3) considerably.

Denote the indicator function of a set A by $\mathbb{I}_A$. We assume that the Lévy measures $\tilde{\nu}_i$ have the form

$$\begin{aligned} \tilde{\nu}_i(dx) = \lambda_i^+ \mathbb{I}_{\{x>0\}}\, \alpha^{(ii)+} \exp(T^{(ii)+}x)\eta^{(ii)+} dx \\ + \lambda_i^- \mathbb{I}_{\{x<0\}}\, \alpha^{(ii)-} \exp(-T^{(ii)-}x)\eta^{(ii)-} dx \end{aligned} \tag{2}$$

for all $i \in \tilde{E}$, where $\lambda_i^\pm \geq 0$ and $(\alpha^{(ii)\pm}, T^{(ii)\pm})$ are representations of phase–type distributions without an atom at 0. The $\eta^{(ii)\pm} := -T^{(ii)\pm}\mathbf{1}$ are called the exit vectors, where $\mathbf{1}$ denotes a column vector of appropriate dimension with all entries being 1. This means that the jump process induced by the Lévy measure ν_i is compound Poisson with jump sizes of a doubly phase–type distribution. Denote the order of $PH(\alpha^{(ii)\pm}, T^{(ii)\pm})$ by $m_{ii}^\pm$. Further write $\lambda_i := \lambda_i^+ + \lambda_i^-$.

Likewise, let p_{ij}^+ (resp. p_{ij}^-) denote the probability that a positive (resp. negative) jump is induced by a phase change from $i \in \tilde{E}$ to $j \in \tilde{E}$, and assume that these jumps have a $PH(\alpha^{(ij)\pm}, T^{(ij)\pm})$ distribution without an atom at 0. Note that $p_{ij}^+ + p_{ij}^- \leq 1$ for all $i, j \in \tilde{E}$. Let $m_{ij}^\pm$ denote the order of $PH(\alpha^{(ij)\pm}, T^{(ij)\pm})$ and define the exit vectors $\eta^{(ij)\pm} := -T^{(ij)\pm}\mathbf{1}$.

The main advantage of the phase–type restriction on the jump distributions is the possibility of transforming the jumps into a succession of linear pieces of exponential duration (each with slope 1 or -1), which yields a modified MAP with continuous paths. We can then retrieve the original process via a simple time change. This is explained in detail in sections 2.1 and 2.2 of [6]. Here we shall present only the pertinent information to make the present paper self-contained.

Without the jumps, the Lévy process $\tilde{\mathcal{X}}^{(i)}$ during a phase $i \in \tilde{E}$ is either a linear drift (i.e. $\tilde{\sigma}_i = 0$) or a Brownian motion (with parameters $\tilde{\sigma}_i > 0$ and $\tilde{\mu}_i \in \mathbb{R}$). Considering this MAP (without the jumps) we can partition its phase space $\tilde{E}$ into the subspaces E_p (for positive drifts), E_σ (for Brownian motions), and E_n (for negative drifts). We thus define

$$\begin{aligned} & E_p := \{i \in \tilde{E} : \tilde{\mu}_i > 0, \tilde{\sigma}_i = 0\}, \quad E_n := \{i \in \tilde{E} : \tilde{\mu}_i < 0, \tilde{\sigma}_i = 0\} \\ \text{and} \qquad & E_\sigma := \{i \in \tilde{E} : \tilde{\sigma}_i > 0\} \end{aligned} \tag{3}$$

Note that $\tilde{E} = E_p \cup E_\sigma \cup E_n$, since we have excluded the case of $\tilde{\mu}_i = \tilde{\sigma}_i^2 = 0$ for any phase $i \in \tilde{E}$. Then we introduce two new phase spaces

$$E_\pm := \{(i, j, k, \pm) : i, j \in E_p \cup E_\sigma \cup E_n, 1 \le k \le m_{ij}^\pm\} \tag{4}$$

to model the jumps. Define now the enlarged phase space $E = E_+ \cup \tilde{E} \cup E_-$. We define the modified MAP $(\mathcal{X}, \mathcal{J})$ over the phase space E as follows. Set the parameters $(\mu_i, \sigma_i^2, \nu_i)$ for $i \in E$ as

$$(\mu_i, \sigma_i^2, \nu_i) := \begin{cases} (\pm 1, 0, \mathbf{0}), & i \in E_\pm \\ (\tilde{\mu}_i, \tilde{\sigma}_i, \mathbf{0}), & i \in \tilde{E} = E_p \cup E_\sigma \cup E_n \end{cases} \tag{5}$$

Let $E_c := E_p \cup E_\sigma \cup E_n$ denote the subspace of E that contains all phases under which the real time movements are continuous. The modified phase process $\mathcal{J}$ is determined by its generator matrix $Q = (q_{ij})_{i,j \in E}$. For this the construction above yields

$$q_{ih} = \begin{cases} \tilde{q}_{ii} - \lambda_i, & h = i \in E_c \\ \tilde{q}_{ih} \cdot (1 - p_{ih}^+ - p_{ih}^-), & h \in E_c, h \neq i \\ \lambda_i^\pm \alpha_k^{(ii)\pm}, & h = (i, i, k, \pm) \\ \tilde{q}_{ij} \cdot p_{ij}^\pm \cdot \alpha_k^{(ij)\pm}, & h = (i, j, k, \pm) \end{cases} \tag{6}$$

for $i \in E_c$ as well as

$$q_{(i,j,k,\pm),(i,j,l,\pm)} = T_{kl}^{(ij)\pm} \qquad \text{and} \qquad q_{(i,j,k,\pm),j} = \eta_k^{(ij)\pm} \tag{7}$$

for $i, j \in E_c$ and $1 \le k, l \le m_{ij}^\pm$.

The original level process $\tilde{\mathcal{X}}$ is retrieved via the time change

$$c(t) := \int_0^t \mathbb{I}_{\{J_s \in E_c\}}\, ds \quad \text{and} \quad \tilde{X}_{c(t)} = X_t \tag{8}$$

for all $t \geq 0$. Thus we obtain

$$S_t := \sup_{s \leq t} X_s \vee 0 = \sup_{s \leq c(t)} \tilde{X}_s \vee 0 = \tilde{S}_{c(t)}$$

as well as

$$\tilde{\tau}(a) = c(\tau(a)) \qquad \text{and} \qquad \tilde{\tau}_-(a) = c(\tau_-(a)) \tag{9}$$

for $Y_t := S_t - X_t$ and

$$\tau(a) := \inf\{t \geq 0 : Y_t \geq a\} \quad \text{and} \quad \tau_-(a) := \sup\{t \leq \tau(a) : Y_t = 0\}$$

In particular,

$$\tilde{S}_{\tilde{\tau}(a)} = S_{\tau(a)} \tag{10}$$

Equations (9) and (10) imply that we can perform an analysis of the MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ in terms of the modified MAP $(\mathcal{X}, \mathcal{J})$ alone.

Example 2. Continuing example 1, we obtain the MAP $(\mathcal{X}, \mathcal{J})$ as follows. Let the phase space be given by $E_p = \{1\}$, $E_- = \{2\}$, and $E_\sigma = \emptyset$. The parameters are given by $\sigma_1 = \sigma_2 = 0$, $\mu_1 = c$, $\mu_2 = -1$, $\nu_1 = \nu_2 = \mathbf{0}$, and

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \beta & -\beta \end{pmatrix}$$

The initial state is $(X_0, J_0) = (u, 1)$.

3.2. First Passage Times

Of central use in the present paper will be the recent derivation of the Laplace transforms for the first passage times of MAPs as given in [5]. Define the first passage times $\tilde{\sigma}(x) := \inf\{t \geq 0 : \tilde{X}_t > x\}$ and

$$\sigma(x) := \inf\{t \geq 0 : X_t > x\} = \inf\{t \geq 0 : X_t = x\}$$

for all $x \geq 0$ and assume that $\tilde{X}_0 = X_0 = 0$. Note that $\tilde{\sigma}(x)$ is the first passage time over the level x for the original MAP $\tilde{\mathcal{X}}$, meaning that we do not count the time spent in jump phases

$i \in E_\pm$. This means that

$$\tilde{\sigma}(x) = c(\sigma(x)) = \int_0^{\sigma(x)} \mathbb{I}_{\{J_s \in E_c\}} ds$$

according to (8). In particular, we may compute expectations over $\tilde{\sigma}(x)$ using the distribution of the modified MAP $(\mathcal{X}, \mathcal{J})$ only and without needing to recur to the original MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$. For $\gamma \geq 0$ denote

$$\mathbb{E}_{ij}(e^{-\gamma\tilde{\sigma}(x)}) := \mathbb{E}(e^{-\gamma\tilde{\sigma}(x)}; J_{\tau(x)} = j | J_0 = i, X_0 = 0)$$

for all $i, j \in E$. Let $\mathbb{E}(e^{-\gamma\tilde{\sigma}(x)})$ denote the matrix with these entries and write

$$\mathbb{E}(e^{-\gamma\tilde{\sigma}(x)}) = \begin{pmatrix} \mathbb{E}_{(a,a)}(e^{-\gamma\tilde{\sigma}(x)}) & \mathbb{E}_{(a,d)}(e^{-\gamma\tilde{\sigma}(x)}) \\ \mathbb{E}_{(d,a)}(e^{-\gamma\tilde{\sigma}(x)}) & \mathbb{E}_{(d,d)}(e^{-\gamma\tilde{\sigma}(x)}) \end{pmatrix}$$

in obvious block notation with respect to the subspaces $E_a := E_+ \cup E_p \cup E_\sigma$ (ascending phases) and $E_d := E_n \cup E_-$ (descending phases).

According to section 3 in [5] we can write

$$\mathbb{E}(e^{-\gamma\tilde{\sigma}(x)}) = \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} \begin{pmatrix} e^{U(\gamma)x} & \mathbf{0} \end{pmatrix} \tag{11}$$

where I_a denotes the identity matrix of dimension $E_a \times E_a$, $\mathbf{0}$ the zero matrix of dimension $E_a \times E_d$, $U(\gamma)$ is a sub–generator matrix of dimension $E_a \times E_a$, and $A(\gamma)$ is a sub–transition matrix of dimension $E_d \times E_a$. An iteration to determine $A(\gamma)$ and $U(\gamma)$ is derived in [5] and further specified to the case of phase-type jumps in [6].

Example 3. Continuing example 2, first note that phase 2 represents the downward jumps and we will not discount the time during sojourns in it. According to the formulas above, the Laplace transform of the first passage time $\tilde{\sigma}(x)$ to a level $x > u$ is given by $\mathbb{E}(e^{-\gamma\tilde{\sigma}(x)}) = e^{U(\gamma)\cdot(x-u)}$, where

$$U(\gamma) = -\frac{\lambda + \gamma}{c} + \frac{\lambda}{c} A(\gamma) \qquad \text{and} \qquad A(\gamma) = \frac{\beta}{\beta - U(\gamma)}$$

Noting that $U(\gamma)$ must be negative, this resolves as

$$U(\gamma) = \frac{1}{2c}\left(c\beta - \gamma - \lambda - \sqrt{(c\beta - \gamma - \lambda)^2 + 4c\beta\gamma}\right)$$

cf. equation (3.12) in [9], noting that γ is denoted as δ there.

3.3. The two-sided Exit Problem

Define the stopping times $\sigma(0,b) := \inf\{t \geq 0 : X_t < 0 \quad \text{or} \quad X_t > b\}$ and

$$\tilde{\sigma}(0,b) := \int_0^{\sigma(0,b)} \mathbb{I}_{\{J_s \in E_c\}} ds = \inf\{t \geq 0 : \tilde{X}_t < 0 \quad \text{or} \quad \tilde{X}_t > b\} \tag{12}$$

which are the exit times of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ from the interval $[0,b]$, respectively. For the main result we need an expression for

$$\Psi_{ij}^+(b|x) := \mathbb{E}\left(e^{-\gamma\tilde{\sigma}(0,b)}; X_{\sigma(0,b)} = b, J_{\sigma(0,b)} = j | J_0 = i, X_0 = x\right)$$

where $x \in [0,b]$ and $i,j \in E$. Clearly $\Psi_{ij}^+(b|x) = 0$ for $j \in E_d$ since an exit over the upper boundary can occur only in an ascending phase. Define the matrix $\Psi_\gamma^+(b|x) := (\Psi_{ij}^+(b|x))_{i\in E, j\in E_a}$. A formula for $\Psi_\gamma^+(b|x)$ has been derived in [11]. In order to state it we need some additional notation.

Let $(\mathcal{X}^+, \mathcal{J})$ denote the MAP as constructed in section 3.1 and define the process $\mathcal{X}^- = (X_t^- : t \geq 0)$ by $X_t^- := -X_t^+$ for all $t > 0$ given that $X_0^+ = X_0^- = 0$. Thus $(\mathcal{X}^-, \mathcal{J})$ is the negative of $(\mathcal{X}^+, \mathcal{J})$. The two processes have the same generator matrix Q for $\mathcal{J}$, but the drift parameters are different. Denoting variation and drift parameters for $\mathcal{X}^\pm$ by $\sigma_i^\pm$ and $\mu_i^\pm$, respectively, this means $\sigma_i^+ = \sigma_i^-$ and $\mu_i^- = -\mu_i^+$ for all $i \in E$. This of course implies that phases $i \in E_+ \cup E_p$ (resp. $i \in E_- \cup E_n$) are descending (resp. ascending) phases for $\mathcal{X}^-$.

Let $A^\pm(\gamma)$ and $U^\pm(\gamma)$ denote the matrices that determine the first passage times in (11). We shall write $A^\pm = A^\pm(\gamma)$ and $U^\pm = U^\pm(\gamma)$ except in cases when we wish to underline the dependence on γ.

Example 4. If $(\mathcal{X}^+, \mathcal{J})$ is the MAP as constructed in example 2, then $(\mathcal{X}^-, \mathcal{J})$ would be the net claim process for the compound Poisson model. As shown in [5], example 5, the Laplace transform of the first passage time $\tilde{\sigma}^-(x) := \inf\{t \geq 0 : \tilde{X}_t^- > x\}$ to a level $x > 0$ is given by

$$\mathbb{E}(e^{-\gamma\tilde{\sigma}^-(x)}) = A^- e^{U^- x} \quad \text{where} \quad A^- = \frac{\beta - R}{\beta}, \quad U^- = -R$$

and

$$-R = \frac{1}{2c}\left(\lambda + \gamma - c\beta - \sqrt{(c\beta - \gamma - \lambda)^2 + 4c\beta\gamma}\right)$$

This coincides with equation (4.24) in [9], noting that γ is denoted as δ there.

Define the matrices

$$C^+ := \begin{pmatrix} \mathbf{0} & I_\sigma \\ A^+ & \end{pmatrix} \qquad \text{and} \qquad C^- := \begin{pmatrix} A^- & \\ I_\sigma & \mathbf{0} \end{pmatrix} \tag{13}$$

of dimensions $(E_\sigma \cup E_d) \times E_a$ and $E_a \times (E_\sigma \cup E_d)$, respectively, where I_σ denotes the identity matrix of dimension $E_\sigma \times E_\sigma$. Further define

$$W^+ := \begin{pmatrix} I_a \\ A^+ \end{pmatrix} \qquad \text{and} \qquad W^- := \begin{pmatrix} A^- \\ I_{\sigma\cup d} \end{pmatrix} \tag{14}$$

which are matrices of dimensions $E \times E_a$ and $E \times (E_\sigma \cup E_d)$. Here, $I_{\sigma\cup d}$ denotes the identity matrix of dimension $(E_\sigma \cup E_d) \times (E_\sigma \cup E_d)$. Finally, let $Z^\pm := C^\pm e^{U^\pm \cdot b}$. Then equation (23) in [11] states that

$$\Psi^+_\gamma(b|x) = \left(W^+ e^{U^+\cdot(b-x)} - W^- e^{U^-\cdot x} Z^+ \right) \cdot \left(I - Z^- Z^+ \right)^{-1} \tag{15}$$

for $0 \le x \le b$. This matrix has dimension $E \times E_a$, due to the fact that exit from below can only happen in an ascending phase.

By reflection at the initial level x, we obtain

$$\begin{aligned} \Psi^-_\gamma(b|x) &:= \mathbb{E}\left(e^{-\gamma\tilde{\sigma}(0,b)}; X_{\sigma(0,b)} = 0 | X_0 = x \right) \\ &= \left(W^- e^{U^- x} - W^+ e^{U^+\cdot(b-x)} Z^- \right) \cdot \left(I - Z^+ Z^- \right)^{-1} \end{aligned} \tag{16}$$

for $0 \le x \le b$. This matrix has dimension $E \times (E_\sigma \cup E_d)$. Note that the expressions on the right-hand sides of (15) and (16) depend on a choice of $\gamma \ge 0$.

Remark 1. Noting that $\left(I - Z^- Z^+\right)^{-1} = \sum_{n=0}^\infty \left(Z^- Z^+\right)^n$ and $Z^- Z^+$ represents a crossing of the interval $[0, b]$ from b to 0 and back, formula (15) has a clear probabilistic interpretation. The term $W^+ e^{U^+\cdot(b-x)}$ simply yields the event that $\mathcal{X}$ exits from b (before an exponential time of parameter γ). The correction term $W^- e^{U^-\cdot x} Z^+$ refers to the event that $\mathcal{X}$ descends below 0 before exiting from b. Multiplication by $\left(I - Z^- Z^+\right)^{-1}$ yields all possible combinations with any number of subsequent (down and up) crossings over the complete interval $[0, b]$.

Remark 2. Since $Z^+ = C^+ e^{U^+\cdot b}$ we can write $\Psi^+_\gamma(b|x)$ in the form

$$\Psi^+_\gamma(b|x) = \left(W^+ e^{-U^+\cdot x} - W^- e^{U^-\cdot x} C^+ \right) \left(e^{-U^+\cdot b} - C^- e^{U^-\cdot b} C^+ \right)^{-1}$$

This comes closer to the usual expression of the exit time distribution in terms of scale functions. For instance, let $\mathcal{X}$ be a Brownian motion with variation $\sigma > 0$ and drift $\mu \in \mathbb{R}$. We then obtain

$$U^{\pm} = \frac{\pm\mu - \sqrt{\mu^2 + 2\gamma\sigma^2}}{\sigma^2}$$

Denote $-r := U^+$ and $s := U^-$. Then

$$\Psi_\gamma^+(b|x) = \frac{e^{rx} - e^{sx}}{e^{rb} - e^{sb}}$$

cf. [10], (2.12 - 2.15), where the γ-scale function is given as $g(x) = e^{rx} - e^{sx}$.

Example 5. Continuing example 3, we obtain

$$U^{\pm} = \frac{1}{2c}\left(\pm(c\beta - \gamma - \lambda) - \sqrt{(c\beta - \gamma - \lambda)^2 + 4c\beta\gamma}\right)$$

cf. [9], equations (3.12) and (4.24) with $\delta = \gamma$. Denote $-\rho := U^+$ and $-R := U^-$. Section 3.2 further yields $A^+ = \beta/(\beta+\rho)$ and $A^- = (\beta - R)/\beta$. Thus

$$\begin{aligned}
\Psi_\gamma^+(b|x) &= \left(e^{-U^+\cdot x} - A^- e^{U^-\cdot x} A^+\right) \cdot \left(e^{-U^+\cdot b} - A^- e^{U^-\cdot b} A^+\right)^{-1} \\
&= \frac{e^{\rho x} - \frac{\beta-R}{\beta} e^{-Rx} \frac{\beta}{\beta+\rho}}{e^{\rho a} - \frac{\beta-R}{\beta} e^{-Ra} \frac{\beta}{\beta+\rho}} = \frac{e^{\rho x} - \psi(x)}{e^{\rho b} - \psi(b)}
\end{aligned}$$

if we write $\psi(x) := e^{-Rx} \cdot (\beta - R)/(\beta + \rho)$, cf. equation (6.37) in [9]. This coincides with formula (6.25) in [9], where $\Psi_\gamma^+(b|x)$ is denoted by $B(0, b|x)$.

4. Main Result

Theorem 1. *The joint distribution of $\tilde\tau(a)$, $\tilde\tau_-(a)$, and $\tilde S_{\tilde\tau(a)}$ is given by*

$$\mathbb{E}\left(e^{-\alpha(\tilde\tau(a) - \tilde\tau_-(a))} e^{-\gamma\tilde\tau_-(a)}; \tilde S_{\tilde\tau(a)} \in dx\right) = \Psi_\gamma^+(a|a) e^{G^{(\gamma)}(a)\cdot x} H^{(\alpha)}(a)\, dx$$

for $\alpha, \gamma, x \geq 0$, where

$$\begin{aligned}
G^{(\gamma)}(a) = &\left(U^+(\gamma) e^{-U^+(\gamma)a} + C^-(\gamma) e^{U^-(\gamma)a} U^-(\gamma) C^+(\gamma)\right) \\
&\times \left(e^{-U^+(\gamma)a} - C^-(\gamma) e^{U^-(\gamma)a} C^+(\gamma)\right)^{-1}
\end{aligned}$$

and

$$\begin{aligned}
H^{(\alpha)}(a) = &\left(U^-(\alpha) + C^+(\alpha) U^+(\alpha) C^-(\alpha)\right) \\
&\times \left(C^+(\alpha) e^{U^+(\alpha)a} C^-(\alpha) - e^{-U^-(\alpha)a}\right)^{-1}
\end{aligned}$$

Proof: We consider the sequence $(\tilde{T}_n(\varepsilon) : n \in \mathbb{N}_0)$ of stopping times defined by

$$\tilde{T}_n(\varepsilon) := \inf\{t \geq 0 : \tilde{S}_t > n\varepsilon, t < \tilde{\tau}(a)\}$$

where $\inf \emptyset := \infty$. Assume that $\tilde{X}_0 = 0$, i.e. $\tilde{S}_0 = 0$. On the set $\{\tilde{T}_n(\varepsilon) < \infty\}$, we observe that $\tilde{T}_n(\varepsilon) = \tilde{\sigma}(-a, n\varepsilon) = c(\sigma(-a, n\varepsilon)) = c(T_n(\varepsilon))$, see (12) and (8). Here, of course,

$$T_n(\varepsilon) := \inf\{t \geq 0 : S_t > n\varepsilon, t < \tau(a)\}$$

Since $X_{T_n(\varepsilon)} = S_{T_n(\varepsilon)}$ for all $\{n \in \mathbb{N} : T_n(\varepsilon) < \infty\}$, the times $T_{n+1}(\varepsilon) - T_n(\varepsilon)$ and thus $\tilde{T}_{n+1}(\varepsilon) - \tilde{T}_n(\varepsilon)$ are conditionally independent given the phase process $\mathcal{J}$. Thus

$$\mathbb{E}\left(e^{-\gamma \tilde{T}_n(\varepsilon)}; \tilde{S}_{\tilde{\tau}} > n\varepsilon\right) = \Psi_\gamma^+(a+\varepsilon|a) \left(\Psi_\gamma^+(a+\varepsilon|a)_{(a,a)}\right)^{n-1}$$

for $n \in \mathbb{N}$ and $\gamma \geq 0$, where $\Psi_\gamma^+(a+\varepsilon|a)_{(a,a)}$ denotes the upper block of the matrix

$$\Psi_\gamma^+(a+\varepsilon|a) = \begin{pmatrix} \Psi_\gamma^+(a+\varepsilon|a)_{(a,a)} \\ \Psi_\gamma^+(a+\varepsilon|a)_{(d,a)} \end{pmatrix}$$

referring to ascending initial phases (i.e. those in E_a). Thus

$$\begin{aligned} \Psi_\gamma^+(a+\varepsilon|a)_{(a,a)} = & \left(e^{U^+(\gamma)\varepsilon} - C^-(\gamma)e^{U^-(\gamma)a}C^+(\gamma)e^{U^+(\gamma)\cdot(a+\varepsilon)}\right) \\ & \times \left(I_a - C^-(\gamma)e^{U^-(\gamma)\cdot(a+\varepsilon)}C^+(\gamma)e^{U^+(\gamma)\cdot(a+\varepsilon)}\right)^{-1} \end{aligned}$$

according to (15), (13) and (14). The probabilities of failure for this matrix-geometric distribution are the entries of the upper block of

$$\mathbb{E}\left(e^{-\alpha\tilde{\sigma}(0,a+\varepsilon)}; X_{\sigma(0,a+\varepsilon)} = 0 | X_0 = a\right) = \Psi_\alpha^-(a+\varepsilon|a) = \begin{pmatrix} \Psi_\alpha^-(a+\varepsilon|a)_{(a,\sigma\cup d)} \\ \Psi_\alpha^-(a+\varepsilon|a)_{(d,\sigma\cup d)} \end{pmatrix}$$

According to (16), (13) and (14)

$$\begin{aligned} \Psi_\alpha^-(a+\varepsilon|a)_{(a,\sigma\cup d)} &= \left(C^-(\alpha)e^{U^-(\alpha)a} - e^{U^+(\alpha)\varepsilon}C^-(\alpha)e^{U^-(\alpha)(a+\varepsilon)}\right) \\ &\quad \times \left(I_{\sigma\cup d} - C^+(\alpha)e^{U^+(\alpha)(a+\varepsilon)}C^-(\alpha)e^{U^-(\alpha)(a+\varepsilon)}\right)^{-1} \\ &= \left(C^-(\alpha)e^{-U^-(\alpha)\varepsilon} - e^{U^+(\alpha)\varepsilon}C^-(\alpha)\right) \\ &\quad \times \left(e^{-U^-(\alpha)(a+\varepsilon)} - C^+(\alpha)e^{U^+(\alpha)(a+\varepsilon)}C^-(\alpha)\right)^{-1} \end{aligned}$$

Altogether we obtain

$$\mathbb{E}\left(e^{-\gamma \tilde{T}_n(\varepsilon)} e^{-\alpha(\tilde{\tau}(a)-\tilde{T}_n(\varepsilon))}; n\varepsilon < \tilde{S}_{\tilde{\tau}} < (n+1)\varepsilon\right)$$
$$= \Psi_\gamma^+(a+\varepsilon|a)\left(\Psi_\gamma^+(a+\varepsilon|a)_{(a,a)}\right)^{n-1} \Psi_\alpha^-(a+\varepsilon|\varepsilon)_{(a,\sigma\cup d)}$$

Now letting ε tend to 0 we obtain that $\mathbb{E}\left(e^{-\alpha(\tilde{\tau}(a)-\tilde{\tau}_-(a))} e^{-\gamma\tilde{\tau}_-(a)}; \tilde{S}_{\tilde{\tau}(a)} \in dx\right)$ has a defective matrix-exponential distribution with parameters

$$G^{(\gamma)}(a) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(\Psi_\gamma^+(a+\varepsilon|a)_{(a,a)} - I_a\right)$$

and

$$H^{(\alpha)}(a) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(\Psi_\alpha^-(a+\varepsilon|\varepsilon)_{(a,\sigma\cup d)} - \mathbf{0}\right)$$

For the first parameter we obtain

$$\begin{aligned} G^{(\gamma)}(a) &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(e^{U^+\varepsilon} - I_a + C^- e^{U^- a}\left(e^{U^-\varepsilon} - I_{\sigma\cup d}\right) C^+ e^{U^+\cdot(a+\varepsilon)}\right) \\ &\quad \times \left(I_a - C^- e^{U^-\cdot(a+\varepsilon)} C^+ e^{U^+\cdot(a+\varepsilon)}\right)^{-1} \\ &= \left(U^+(\gamma) + C^-(\gamma) e^{U^-(\gamma)a} U^-(\gamma) C^+(\gamma) e^{U^+(\gamma)a}\right) \\ &\quad \times \left(I_a - C^-(\gamma) e^{U^-(\gamma)a} C^+(\gamma) e^{U^+(\gamma)a}\right)^{-1} \\ &= \left(U^+(\gamma) e^{-U^+(\gamma)a} + C^-(\gamma) e^{U^-(\gamma)a} U^-(\gamma) C^+(\gamma)\right) \\ &\quad \times \left(e^{-U^+(\gamma)a} - C^-(\gamma) e^{U^-(\gamma)a} C^+(\gamma)\right)^{-1} \end{aligned}$$

where, for notational simplicity, the dependence on γ is omitted in the first equality. The second parameter is

$$\begin{aligned} H^{(\alpha)}(a) &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\left(C^-(\alpha) e^{-U^-(\alpha)\varepsilon} - e^{U^+(\alpha)\varepsilon} C^-(\alpha)\right) \\ &\quad \times \left(e^{-U^-(\alpha)a} - C^+(\alpha) e^{U^+(\alpha)a} C^-(\alpha)\right)^{-1} \\ &= \left(C^-(\alpha) U^-(\alpha) + U^+(\alpha) C^-(\alpha)\right) \\ &\quad \times \left(C^+(\alpha) e^{U^+(\alpha)a} C^-(\alpha) - e^{-U^-(\alpha)a}\right)^{-1} \end{aligned}$$

Altogether this yields the statement.

□

Example 6. The example in remark 2 of a Brownian motion fluid flow is explicated in section 6 of [13]. There it is stated that

$$\mathbb{E}\left(e^{-\alpha\tau(a)-\beta S_{\tau(a)}-\gamma\tau_-(a)}\right) = \frac{2e^{-2am}W^{(\alpha+\gamma)}(a)}{W^{(\alpha)}(a)(W^{(\alpha+\gamma)\prime}(a)+\beta W^{(\alpha+\gamma)}(a))} \tag{17}$$

for $\sigma^2 = 1$ and $\mu = m$, where the scale function is defined as

$$W^{(\alpha)}(a) = \frac{e^{-a(m-\sqrt{m^2+2\alpha})} - e^{-a(m+\sqrt{m^2+2\alpha})}}{\sqrt{m^2+2\alpha}} \tag{18}$$

We now wish to arrive at the same expression via theorem 1. We begin by observing that

$$\begin{aligned}\mathbb{E}\left(e^{-\alpha\tau(a)-\beta S_{\tau(a)}-\gamma\tau_-(a)}\right) &= \int_0^\infty e^{G^{(\alpha+\gamma)}(a)x}e^{-\beta x}\,dx\; H^{(\alpha)}(a)\\ &= \left(\beta I_a - G^{(\alpha+\gamma)}(a)\right)^{-1} H^{(\alpha)}(a)\end{aligned}$$

Since there is only one phase and $E = E_\sigma$, we obtain $W^+ = W^- = C^+ = C^- = 1$ and $\Psi_\gamma^+(a,a) = 1$. Further

$$U^\pm(\alpha) = \pm m - \sqrt{m^2+2\alpha} \tag{19}$$

for $\sigma^2 = 1$ and $\mu = m$, i.e. $U^\pm(\alpha)$ are real numbers. This implies

$$G^{(\alpha+\gamma)}(a) = \frac{U^+(\alpha+\gamma)e^{-U^+(\alpha+\gamma)a} + e^{U^-(\alpha+\gamma)a}U^-(\alpha+\gamma)}{e^{-U^+(\alpha+\gamma)a} - e^{U^-(\alpha+\gamma)a}}$$

and

$$H^{(\alpha)}(a) = \frac{U^-(\alpha)+U^+(\alpha)}{e^{U^+(\alpha)a} - e^{-U^-(\alpha)a}}$$

Equations (19) and (18) yield

$$W^{(\alpha)}(a) = \frac{e^{-U^+(\alpha)a} - e^{U^-(\alpha)a}}{\sqrt{m^2+2\alpha}} \quad \text{and} \quad G^{(\alpha+\gamma)}(a) = -\frac{W^{(\alpha+\gamma)\prime}(a)}{W^{(\alpha+\gamma)}(a)}$$

as well as

$$H^{(\alpha)}(a) = \frac{2\sqrt{m^2+2\alpha}\,e^{-2ma}}{e^{-U^+(\alpha)a} - e^{U^-(\alpha)a}} = \frac{2\,e^{-2ma}}{W^{(\alpha)}(a)}$$

Hence we obtain

$$\mathbb{E}\left(e^{-\alpha\tau(a)-\beta S_{\tau(a)}-\gamma\tau_-(a)}\right) = \frac{W^{(\alpha+\gamma)}(a)}{W^{(\alpha+\gamma)\prime}(a)+\beta W^{(\alpha+\gamma)}(a)}\frac{2\,e^{-2ma}}{W^{(\alpha)}(a)}$$

which is the same expression as (17).

Remark 3. Defining a γ-scale function for MAPs with phase-type jumps by

$$W^{(\gamma)}(x) := e^{-U^+(\gamma)x} - C^-(\gamma)e^{U^-(\gamma)x}C^+(\gamma)$$

for $x > 0$, we see first that $G^{(\gamma)}(a) = -W^{(\gamma)\prime}(a)[W^{(\gamma)}(a)]^{-1}$ where $W^{(\gamma)\prime}(a)$ denotes the derivative of the function $W^{(\gamma)}(x)$ at $x = a$.

In applications to insurance risk, a popular question is the expected amount of γ-discounted dividends paid before ruin. If the initial risk reserve is $u \geq 0$ and dividends are paid above a constant barrier of $b \geq u$, then the mean discounted dividends paid out before ruin can be computed as

$$\begin{aligned}
V_\gamma(b|u) &:= \Psi_\gamma^+(b|u)\,\mathbb{E}\left(\tilde{S}_{\tilde{\tau}(b)}e^{-\gamma\tilde{\tau}_-(b)}\right) = \Psi_\gamma^+(b|u)\int_0^\infty e^{G^{(\gamma)}(b)x}\,dx \\
&= \Psi_\gamma^+(b|u)\,[-G^{(\gamma)}(b)]^{-1} \\
&= \left(W^+(\gamma)e^{-U^+(\gamma)u} - W^-(\gamma)e^{U^-(\gamma)u}C^+(\gamma)\right) \\
&\quad\times\left(-U^+(\gamma)e^{-U^+(\gamma)b} + C^-(\gamma)e^{U^-(\gamma)b}\left(-U^-(\gamma)\right)C^+(\gamma)\right)^{-1}
\end{aligned}$$

Example 7. We continue the example in remark 2 of a Brownian motion fluid flow. Since there is only one phase, we get $W^+ = W^- = C^+ = C^- = 1$ and hence

$$V_\gamma(b|u) = \frac{e^{ru} - e^{su}}{re^{rb} - se^{sb}}$$

which is equation (2.11) in [10]. Note that for $\gamma = 0$ we obtain

$$(s,r) = \begin{cases} \left(-2\frac{\mu}{\sigma^2}, 0\right), & \mu > 0 \\ \left(0, -2\frac{\mu}{\sigma^2}\right), & \mu < 0 \end{cases}$$

This implies

$$V_0(b|u) = \begin{cases} \dfrac{\sigma^2}{2\mu}\left(e^{2\mu b/\sigma^2} - e^{2\mu(b-u)/\sigma^2}\right), & \mu > 0 \\ -\dfrac{\sigma^2}{2\mu}\left(e^{2\mu(b-u)/\sigma^2} - e^{2\mu b/\sigma^2}\right), & \mu < 0 \end{cases}$$

cf. equation (2.22) in [10] for the case $\mu > 0$.

Example 8. Another example is the compound Poisson model, continued from example 5.

Starting in the ascending phase (collecting premiums), we obtain

$$\begin{aligned}
V_\gamma(b|u) &= \left(e^{-U^+u} - A^- e^{U^-u} A^+\right) \cdot \left(-U^+ e^{-U^+b} + A^- e^{U^-b}\left(-U^-\right) A^+\right)^{-1} \\
&= \frac{e^{\rho u} - \frac{\beta-R}{\beta+\rho} e^{-Ru}}{\rho e^{\rho b} + R\frac{\beta-R}{\beta+\rho} e^{-Rb}} \\
&= \frac{(\beta+\rho)e^{\rho u} - (\beta-R)e^{-Ru}}{\rho\cdot(\beta+\rho)e^{\rho b} + R\cdot(\beta-R)e^{-Rb}}
\end{aligned}$$

which is formula (7.8) in [9].

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