The Periodic $BMAP/PH/c$ Queue

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Abstract In queueing theory, most models are based on time-homogeneous arrival processes and service time distributions. However, in communication networks arrival rates and/or the service capacity usually vary periodically in time. In order to reflect this property accurately, one needs to examine periodic rather than homogeneous queues. In the present paper, the periodic $BMAP/PH/c$ queue is analyzed. This queue has a periodic BMAP arrival process, which is defined in this paper, and phase–type service time distributions. As a Markovian queue, it can be analysed like an (inhomogeneous) Markov jump process. The transient distribution is derived by solving the Kolmogorov forward equations. Furthermore, a stability condition in terms of arrival and service rates is proven and for the case of stability, the asymptotic distribution is given explicitly. This turns out to be a periodic family of probability distributions. It is sketched how to analyze the periodic $BMAP/M_t/c$ queue with periodically varying service rates by the same method.

Keywords: Periodic Queues, BMAP, Markov Jump Processes

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1. Introduction

In queueing theory, most models are based on time-homogeneous arrival processes and service time distributions. One of the most important features to be exploited is the Markov property which often appears after the construction of embedded Markov chains. The search for Markovian but versatile arrival processes has led to the concept of batch Markovian arrival processes (BMAPs, see Neuts [15] and Lucantoni [13]) which allow for a phase process controlling the arrival rates. This arrival process is often used for modelling communication networks.

A typical property of communication traffic is the dependence of its arrival rates on time. This aspect incites the use of time-inhomogeneous processes and queues for modelling communication networks. Typically, a periodic dependence of the arrival rates and/or the service time distribution can be assumed with period lengths of a day or a week.

While queues with periodic input naturally reflect the time-dependent amount of traffic that arrives in communication networks, the analysis of queues with inhomogeneous arrival rates is far less developed than the one for homogeneous queues. Some of the existing results in the literature are given in Asmussen and Thorisson [1], Bambos and Walrand [2], Falin [5], Harrison and Lemoine [8], Hasofer [9], Heyman and Whitt [10], Lemoine [12], [11], Massey [14], Rolski [17], [18], and Willie [19]. Although many types of stability conditions could be established, explicit formulae for asymptotic behaviour have not been derived yet.

The present paper is organized as follows. In the remainder of this section, the periodic $BMAP/PH/c$ queue as well as the basic notations are defined. The next section contains the transient distributions at arbitrary times. In section 3, a stability condition is given and the asymptotic distributions at any "day time" of one period length are derived explicitly in the case of stability.

Like all Markov arrival processes in queueing theory, BMAPs are Markov jump processes. As main reference for the theory of Markov jump processes, the book by Gikhman, Skorokhod [7] shall be referred to. Analogous to the definition of a BMAP (see Lucantoni [13]), an inhomogeneous BMAP shall be defined by its time-dependent transition rate matrices $(D_n(t) : n \in \mathbb{N}_0, t \in \mathbb{R}_0^+)$, assuming that the continuity conditions for the transition rates of inhomogeneous Markov jump processes are satisfied (see Gikhman, Skorokhod [7], p.362). Let the number of phases, which is the dimension of the square matrices $D_n(t)$, be denoted by $m \in \mathbb{N}$. Then $D_n(t)(i,j)$ is the infinitesimal transition rate of observing n arrivals at time t while changing from phase i to phase j. A periodic BMAP with period T is an inhomogeneous BMAP with the property $D_n(s+T) = D_n(s)$ for all $n \in \mathbb{N}_0$ and $s \in [0, T[$.

Let $Q = (Q_t : t \in \mathbb{R}_0^+)$ denote a periodic $BMAP/PH/c$ queue with a periodic BMAP arrival process and a phase–type service time distribution which is identical and independent for all c servers. Define the arrival process by its transition rate matrices $(D_n(t) : n \in \mathbb{N}_0, t \in \mathbb{R}_0^+)$ having dimension m and period T.

Every server shall be equal, and the service time distribution be phase–type with representation (α, S) and dimension r (for a description of phase–type distributions, see Neuts [16]). The absorbing state shall be denoted by 0 and the transient states by $\{1,\ldots,r\}$. For any $d \in I\!N$, let 1_d denote the d–dimensional column vector with all entries being 1 and denote the ith canonical column base vector of $I\!R^d$ by e_i if the dimension d is clear from the context. Denote the transpose of a matrix A or a vector v by A^T or v^T, respectively. Finally, define the exit vector $\eta := -S1_r$, the vectors $\alpha' := (0,\alpha)$ and $\eta' := (0,\eta^T)^T$ as well as the matrices

$$S' := \begin{pmatrix} 0\ldots 0 \\ \vdots \quad S \\ 0 \end{pmatrix}, \qquad B_0 := \begin{pmatrix} \alpha' \\ 0\ldots 0 \\ \vdots \\ 0\ldots 0 \end{pmatrix} \qquad \text{and} \qquad B_1 := \begin{pmatrix} 0\;\alpha \\ \vdots\;\; I \\ 0 \end{pmatrix} \tag{0}$$

2. Transient Distributions

As a Markovian queue, the system can be analysed like an (inhomogeneous) Markov jump process. The process Q has state space $I\!N \times \{1,\ldots,m\} \times \{0,\ldots,r\}^c$ and hence a finite phase space of dimension $1+c$ with $d := m \cdot c \cdot (r+1)$ possible phases. The first dimension indicates the arrival phase and the last c dimensions shall describe the phases of the respective servers. If a server is in phase 0 at time t, it means that this server is idle at that time.

Let 0 and I denote the zero and identity matrix, respectively. Denote the Kronecker product of two quadratic matrices A and B by $A \otimes B$ (cf. Bellman [3]). The neutral element with respect to the Kronecker product is the scalar $1 \in I\!R$, interpreted as a one–dimensional matrix. Define the iteration of Kronecker products by $A^{\otimes 0} := 1$ and $A^{\otimes n+1} := A^{\otimes n} \otimes A$ for all quadratic matrices A.

Now the infinitesimal generator $G(t)$ of the queue process Q can be written as an $I\!N_0 \times I\!N_0$ block matrix with entries being the $d \times d$ matrices

$$G_{kn}(t) = \begin{cases} 0 & \text{for } k > n+1 \\ \sum_{i=1}^{c} I^{\otimes i} \otimes \eta' e_0^T \otimes I^{\otimes c-i} & \text{for } k = n+1 \leq c \\ \sum_{i=1}^{c} I^{\otimes i} \otimes \eta' \alpha' \otimes I^{\otimes c-i} & \text{for } k = n+1 > c \\ D_0(t) \otimes I^{\otimes c} + \sum_{i=1}^{c} I^{\otimes i} \otimes S' \otimes I^{\otimes c-i} & \text{for } k = n \\ D_{n-k}(t) \otimes I^{\otimes c} & \text{for } c \leq k < n \\ D_{n-k}(t) \otimes B_1^{\otimes c} & \text{for } k < c \leq n \end{cases}$$

as well as

$$G_{kn}(t) = \binom{c-k}{c-n}^{-1} \sum_{1 \leq i_1 < \ldots < i_{n-k} \leq c} D_{n-k}(t) \otimes M_1 \otimes \ldots \otimes M_c$$

for $k < n < c$, with $M_i = B_0$ for $i \in \{i_1, \ldots, i_{n-k}\}$ and $M_i = I$ else. The interpretation of the last equation is that for several possibilities of filling idle servers, every possibility shall be equally probable. This convention allows us to write the above generator in block matrix form without needing to define the elements of the blocks separately, which notationally would be much more inconvenient. Denote the (i,j)th entry of the matrix $G_{kn}(t)$ by $G_{kn}(t)(i,j)$ for $i,j \in \{1, \ldots, d\}$.

Remark 1. Instead of the periodic $BMAP/PH/c$ queue as described above, one can analyze the periodic $BMAP/M_t/c$ queue with periodically varying service rates by the same method. In this case, every server is equal, and the service time distribution function B_s for a user arriving at time $s \in \mathbb{R}_0^+$ is defined by

$$B_s(t-s) := 1 - e^{-\int_s^t \mu_u du}$$

for all $t > s$. This means that the service process without idle periods would be an inhomogeneous Poisson process with rates $(\mu_t : t \in \mathbb{R}_0^+)$. Periodicity of the service rates means $\mu_{s+T} = \mu_s$ for all $s \in [0, T[$.

Here, we would have an infinitesimal generator $G(t)$ of the queue process which can be written as an $\mathbb{N}_0 \times \mathbb{N}_0$ block matrix with entries being the $m \times m$ matrices

$$G_{kn}(t) = \begin{cases} 0 & \text{for} \quad k > n+1 \\ k\mu_t \cdot I & \text{for} \quad k = n+1 \leq c \\ c\mu_t \cdot I & \text{for} \quad k = n+1 > c \\ D_0(t) - k\mu_t \cdot I & \text{for} \quad k = n \leq c \\ D_0(t) - c\mu_t \cdot I & \text{for} \quad k = n > c \\ D_{n-k}(t) & \text{for} \quad k < n \end{cases}$$

for $k, n \in \mathbb{N}_0$. All the statements in the following apply to this queue with periodic service rates, too.

The multiplication of two $\mathbb{N}_0 \times \mathbb{N}_0$ block matrices A and B is defined by

$$(AB)_{kn}(i,j) := \sum_{l=0}^{\infty} \sum_{h=1}^{d} A_{kl}(i,h) B_{ln}(h,j)$$

for every $k, n \in I\!N_0$ and $i, j \in \{1, \ldots, m\} \times \{0, \ldots, r\}^c = \{1, \ldots, d\}$.

Define $P_{kn}(s,t)(i,j)$ as the probability of having $n \in I\!N_0$ users in the queue and being in phase j at time $t > s$ under the condition of having $k \in I\!N_0$ users in the queue and being in phase i at time s. Further define $P_{kn}(s,t)$ as the $d \times d$ matrix with entries $P_{kn}(s,t)(i,j)$ and $P(s,t) = (P_{kn}(s,t))_{k,n \in I\!N_0}$ as the $I\!N_0 \times I\!N_0$ block matrix with entries $P_{kn}(s,t)$.

Solving Kolmogorov's forward equations via the iteration method by Picard and Lindelöf (cf. Gikhman, Skorokhod [7], p.317), it can be shown that the transition probability matrices of the queue can be written as

$$P(s,t) = \sum_{k=0}^{\infty} P^{(k)}(s,t)$$

with

$$P^{(k)}(s,t) = \underbrace{\int_s^t \int_s^{u_k} \ldots \int_s^{u_2}}_{k \; integrals} G(u_1) \ldots G(u_k) du_1 \ldots du_k$$

In the periodic case, this formula can be simplified as follows. The periodicity of the generator yields

$$P(0, nT) = P(0, (n-1)T)P((n-1)T, nT) = P(0, (n-1)T)P(0,T) = P(0,T)^n$$

Let μ denote the initial distribution of the queue process Q. Define

$$\lfloor t/T \rfloor := \max\{n \in I\!N_0 : nT \leq t\}$$

as the number of period lengths that have passed until time $t \in I\!R^+$. Now the transient distribution of Q is given by

$$Q_t = \int d\mu P(0, \lfloor t/T \rfloor T) P(\lfloor t/T \rfloor T, t) = \int d\mu P(0,T)^{\lfloor t/T \rfloor} P(0, t - \lfloor t/T \rfloor T)$$

This expression allows a computation of the transient distribution at any time $t \in I\!R^+$ without needing to integrate over ranges larger than the period T. For computing the remaining terms $P(0,s)$ with $s \leq T$, one can use the following iteration as given in Bellman [3], p.168: Starting with $I_0(u) := Id$ for all $u \leq s$, the iteration

$$I_{n+1}(u) := \int_0^u I_n(v) Q(v) dv + Id$$

leads to the limit

$$P(0,s) = \lim_{n \to \infty} I_n(s)$$

for all $s \leq T$.

3. Stability and Asymptotic Distributions

The asymptotic analysis of the $BMAP/PH/c$ queue first requires the derivation of an ergodicity condition, before asymptotic distributions can be given. In Fayolle, Malyshev, Menshikov [6], a version of Foster's criterion for state spaces $\mathbb{N}_0 \times \{1, \ldots, d\}$, hence for Markovian BMAP queues with phase–type service, can be found. This shall be used in this section in order to prove ergodicity criteria for the periodic $BMAP/PH/c$ queue. After that, the concept of an asymptotic distribution (which is valid for homogeneous queues) is adapted to periodic queues by the definition of a periodic family of asymptotic distributions. This is given explicitly for the case of ergodicity at the end of this section.

Define $Y = (Y_n : n \in \mathbb{N}_0)$ as the homogeneous Markov chain with transition probability matrix $P(0,T)$ and let $Y^\mu = (Y_n^\mu : n \in \mathbb{N})$ denote the version of Y with initial distribution μ. Assume in the following that the arrival phase process as well as the service phase processes have stationary distribution π_A and π_B, respectively. This means, the equations $\pi_A \sum_{n=0}^\infty D_n = 0$ and $\pi_B(S + \eta\alpha) = 0$ are satisfied. For the latter as well as for all following statements on PH–renewal processes, see Neuts [16], p.231ff.

Theorem 2. The Markov chain Y is ergodic if and only if the stability condition

$$\frac{1}{T} \int_0^T \pi_A \sum_{n=1}^{\infty} n D_n(t) 1_m \, dt < c \cdot \pi_B \eta \tag{1}$$

holds.

Proof Let $A = (A_t : t \in \mathbb{R}_0^+)$ and $B = (B_t : t \in \mathbb{R}_0^+)$ denote the BMAP arrival process into the queue and the c-fold superposition of the PH–renewal process with representation (α, S), respectively. That means, A_t is the random variable of all arrivals into the queue until time t. Define $Z := A - B$ as the difference of these independent processes. Then Z is a periodic Markov jump process. For any two–dimensional process X, denote the marginal process in the first dimension by X^1. The mean expectation of Z^1 over one period length in phase equilibrium equals

$$E(Z_T^1) = E(A_T^1) - E(B_T^1) = \int_0^T \pi \sum_{n=1}^{\infty} nD_n(t) 1_m \, dt - c \cdot \pi_B \eta \cdot T \tag{2}$$

In order to show the necessity of condition 1, assume that $E(Z_T^1) \geq 0$. Since the state space of Z is $\mathbb{Z} \times \{1, \ldots, d\}$, but the chain Y has a barrier at the zero level, we have

$$E(Y_1^1) > E(Z_T^1) \geq 0$$

for initial distributions with support $\{0\} \times \{1, \ldots, d\}$. Starting in phase equlibrium, the asymptotic expectation

$$\lim_{n\to\infty} E(Y_n^1) = \lim_{n\to\infty} \sum_{k=1}^{n} E(Y_1^1) = \infty$$

diverges to infinity. Hence, there is no asymptotic distribution for Y.

Now we show sufficiency. Denote the transition probability matrix of the homogeneous Markov chain $(Z_{nT} : n \in \mathbb{N}_0)$ by p^Z. Since Z is homogeneous in the first component, we can define

$$p_k^Z(i,j) := p_{(0,i),(k,j)}^Z = P(Z_T = (k,j) | Z_0 = (0,i))$$

for all $k \in \mathbb{Z}$ and $i, j \in \{1, \ldots, d\}$. Further define $\pi := \pi_A \otimes \pi_B^{\otimes c}$, using the obvious adaptation of the Kronecker product to vectors. The above observation 2 yields

$$\sum_{i=1}^{d} \pi_i \sum_{k\in\mathbb{Z}} \sum_{j=1}^{d} k \cdot p_k^Z(i,j) < 0$$

According to Fayolle, Malyshev, Menshikov [6], p.35, there is an $\varepsilon > 0$ and a positive function f such that

$$\sum_{n=0}^{\infty} \sum_{j=1}^{d} p_{(k,i),(n,j)}^Z \cdot f(n,j) - f(k,i) < -\varepsilon$$

for almost all states $(k,i) \in \mathbb{N}_0 \times \{1, \ldots, d\}$. Furthermore, there are numbers $a_1, \ldots, a_d$ such that $f(k,i) = k + a_i$ for almost all states (k,i).

Define the event

$$R(n) := \{\exists\, t \in [nT, (n+1)T[: Q_t = 0\}$$

for all $n \in I\!N_0$. Then the transition probabilities of the homogeneous chain Y can be decomposed in

$$\begin{aligned} p^Y_{(k,i),(l,j)} &:= P(Y_{n+1} = (l,j)|Y_n = (k,i)) \\ &= P(Y_{n+1} = (l,j)|Y_n = (k,i), R(n)) \cdot P(R(n)|Y_n = (k,i)) \\ &\quad + P(Y_{n+1} = (l,j)|Y_n = (k,i), R(n)^c) \cdot P(R(n)^c|Y_n = (k,i)) \\ &= P(Y_{n+1} = (l,j)|Y_n = (k,i), R(n)) \cdot P(R(n)|Y_n = (k,i)) \\ &\quad + p^Z_{(k,i)(l,j)} \cdot P(R(n)^c|Y_n = (k,i)) \end{aligned}$$

for all $n \in I\!N_0$. Since

$$\lim_{k\to\infty} P(R(n)|Y_n = (k,i)) = 0 \tag{2}$$

there is a $k_0 \in I\!N$ such that

$$P(R(n)|Y_n = (k,i)) < \frac{\varepsilon}{2 \cdot M'}$$

for all $k > k_0$, with $M' := M + \sum_{i=1}^d |a_i|$ and some $M \in I\!N$, for which condition 1 implies

$$\max_{i\in\{1,\dots,m\}} \int_0^T e_i \sum_{n=1}^{\infty} nD_n(t) 1_m \, dt < M < \infty$$

Thus the estimation

$$\begin{aligned} &\sum_{l=0}^{\infty}\sum_{j=1}^{d} P(Y_{n+1} = (l,j)|Y_n = (k,i), R(n)) \cdot f(l,j) \\ &\qquad \le \sum_{l=0}^{\infty}\sum_{j=1}^{d} P(Y_{n+1} = (l,j)|Y_n = (k,i), R(n)) \cdot \left(l + \sum_{j=1}^{d} |a_j|\right) \\ &\qquad < M + \sum_{i=1}^{d} |a_i| \end{aligned} \tag{3}$$

holds. Using the positive function f, we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{j=1}^{d} p^Y_{(k,i),(n,j)} \cdot f(n,j) - f(k,i) \\ &\qquad < \sum_{n=0}^{\infty}\sum_{j=1}^{d} p^Z_{(k,i),(n,j)} \cdot f(n,j) - f(k,i) + \frac{\varepsilon}{2 \cdot M'} \cdot \left(M + \sum_{i=1}^{d} |a_i|\right) \\ &\qquad < -\frac{\varepsilon}{2} \end{aligned} \tag{4}$$

for almost all states $(k,i) \in I\!N_0 \times \{1,\ldots,m\}$. Now Foster's criterion as stated in Fayolle, Malyshev, Menshikov [6], p.29, assures that Y is ergodic. □

Now the concept of an asymptotic distribution shall be adapted to periodic queues. After that, the main theorem of this paper gives an explicit formula for the asymptotic distribution of a stable periodic $BMAP/PH/c$ queue.

A family $(q_s : s \in [0,T[)$ of probability distributions shall be called a periodic family of asymptotic distributions for Q if

$$q_s = \lim_{n\to\infty} Q_{nT+s}$$

does exist for every $s \in [0,T[$. Here, the limit shall be defined in terms of weak convergence.

Theorem 3. Let Q denote a periodic $BMAP/PH/c$ queue with period T. Q has a periodic family of asymptotic distributions if and only if the stability condition 1 holds. In this case, the periodic family of asymptotic distributions is uniquely determined by

$$q_s = \int dq\ P(0,s)$$

for all $s \in [0,T[$, with q being the stationary distribution of the homogeneous Markov chain $Y = (Y_n : n \in I\!N_0)$ with transition probability matrix $P(0,T)$.

Proof Let μ denote the initial distribution of Q. First, assume that a periodic family $(q_s : s \in [0,T[)$ of asymptotic distributions does exist for Q. A necessary property of q_0 is

$$q_0 = \lim_{n\to\infty} \int d\mu P(0,nT) = \lim_{n\to\infty} \int d\mu P(0,(n-1)T)P(0,T)$$

$$= \left(\lim_{n\to\infty} \int d\mu P(0,(n-1)T)\right) P(0,T) = \int dq_0 P(0,T)$$

which means that q_0 is the stationary distribution of Y.

Now let q be the stationary distribution of Y. Then

$$q_0 = \lim_{n\to\infty} \int d\mu P(0,nT) = \lim_{n\to\infty} \int d\mu P(0,T)^n = \lim_{n\to\infty} Y_n^\mu = q$$

does exist.

Hence, the first statement follows from the above theorem 2. If the stationary distribution q of Y does exist, then the periodic family $(q_s : s \in [0,T])$ of asymptotic distributions for Q is given by

$$q_s = \lim_{n\to\infty} \int d\mu P(0, nT + s) = \lim_{n\to\infty} \int d\mu P(0, nT)P(0, s)$$

$$= \left(\lim_{n\to\infty} \int d\mu P(0, nT)\right) P(0, s) = \int dq P(0, s)$$

□

As intuitively plausible, this theorem shows that periodic Markovian arrival rates (and service capacity, in case of the $BMAP/M_t/c$ queue) yield a periodic asymptotic behaviour of the queue. The stability condition coincides with intuition, too, as it compares the accrued workload and service capacity over one period length. For the special case of $m = 1$ and constant service capacity, the results yield an analysis of the periodic $M_t/M/c$ queue. This extends the results of Heyman, Whitt [10].

4. Conclusion

In this paper, formulae for the transient as well as for asymptotic distributions for the periodic $BMAP/PH/c$ queue are given. The stability condition, which is derived, is easy to check, since it is written in terms of arrival and service rates and asymptotic distributions of the phase process. A structural result is that the family of asymptotic distributions of the examined periodic queue is again periodic.

The method applied in this paper seems to work only for Markovian queues. In order to model periodically varying service rates, one can apply the same method to the periodic $BMAP/M_t/c$ queue as described in remark 1. Furthermore, the same method can be applied towards the spatial Markovian $SMAP/M_t/c/c$ queue with a periodic spatial Markovian arrival process (SMAP) and a limited number of users in the queue. For an analysis of this queue see Breuer [4]. Spatial queues are used for modelling mobile communication networks and take the spatial location of their users in account.

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