# Chapter 3: Markov Processes
# First hitting times

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If upon an arrival the system is filled, i.e. all servers are busy, this arriving user is not admitted into the system. In this case we say that the arriving user is lost. Queueing systems with the possibility of such an event are thus called **loss systems**.

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The queue described above is a skip-free Markov process $\mathcal{Y}$ with state space $E = \{0, \dots, c\}$ and generator matrix

$$Q = \begin{pmatrix} -\lambda & \lambda & & & & \\ \mu & -\lambda-\mu & \lambda & & & \\ & 2\mu & -\lambda-2\mu & \lambda & & \\ & & \ddots & \ddots & \ddots & \\ & & & (c-1)\mu & -\lambda-(c-1)\mu & \lambda \\ & & & & c\mu & -c\mu \end{pmatrix}$$

up to the first loss.

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Now

$$Z := \inf\{t \geq 0 : Y'_t = c+1\}$$

is the random variable of the time until the first loss.

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# Theorem 9.2

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for all $t > 0$. Here, the function

$$e^{T \cdot t} := \exp(T \cdot t) := \sum_{n=0}^{\infty} \frac{t^n}{n!} T^n$$

denotes a **matrix exponential function**.

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$$= \alpha_{m+1} + \alpha\mathbf{1} - \alpha e^{T \cdot t}\mathbf{1} = 1 - \alpha e^{T \cdot t}\mathbf{1}$$

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$$= -\alpha \frac{d}{dt} e^{T \cdot t}\mathbf{1} = -\alpha T e^{T \cdot t}\mathbf{1} = \alpha e^{T \cdot t}(-T\mathbf{1}) = \alpha e^{T \cdot t}\eta$$

which was to be proven.