Chapter 3: Homogeneous Markov Processes on Discrete State Spaces

L. Breuer
University of Kent, UK

November 7, 2013

From now on we shall convene on the technical assumption

$$\check{\lambda} := \inf\{\lambda_i : i \in E\} > 0$$

Transience, Recurrence etc.

From now on we shall convene on the technical assumption

$$\check{\lambda} := \inf\{\lambda_i : i \in E\} > 0$$

which holds for all applications that we will examine.

Transience, Recurrence etc.

From now on we shall convene on the technical assumption

$$\check{\lambda} := \inf\{\lambda_i : i \in E\} > 0$$

which holds for all applications that we will examine. Then a Markov process $\mathcal{Y}$ is called **irreducible**, **transient**, **recurrent** or **positive recurrent**

Transience, Recurrence etc.

From now on we shall convene on the technical assumption

$$\check{\lambda} := \inf\{\lambda_i : i \in E\} > 0$$

which holds for all applications that we will examine. Then a Markov process $\mathcal{Y}$ is called **irreducible**, **transient**, **recurrent** or **positive recurrent** if its embedded Markov chain $\mathcal{X}$ is.

An initial distribution π is called **stationary**

Stationary Distributions

An initial distribution π is called **stationary** if the process $\mathcal{Y}^\pi$ is stationary,

Stationary Distributions

An initial distribution π is called **stationary** if the process $\mathcal{Y}^\pi$ is stationary, i.e. if

$$\mathbb{P}(Y^\pi_{t_1} = j_1, \ldots, Y^\pi_{t_n} = j_n) = \mathbb{P}(Y^\pi_{t_1+s} = j_1, \ldots, Y^\pi_{t_n+s} = j_n)$$

for all $n \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_n$, and states $j_1, \ldots, j_n \in E$, and $s \geq 0$.

Theorem 3.9

A distribution π on E is stationary if and only if $\pi G = 0$ holds.

Theorem 3.9

A distribution π on E is stationary if and only if $\pi G = 0$ holds.

Proof:
First we obtain

$$\pi P(t) = \pi e^{G \cdot t} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \pi G^n = \pi I + \sum_{n=1}^{\infty} \frac{t^n}{n!} \pi G^n = \pi + \mathbf{0} = \pi$$

for all $t \geq 0$, with $\mathbf{0}$ denoting the zero measure on E.

Proof of theorem 3.9 (contd.)

With this, theorem 3.8 yields

$$\mathbb{P}(Y^\pi_{t_1} = j_1, \dots, Y^\pi_{t_n} = j_n)$$

$$= \sum_{i \in E} \pi_i P_{i,j_1}(t_1) P_{j_1,j_2}(t_2 - t_1) \dots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

Proof of theorem 3.9 (contd.)

With this, theorem 3.8 yields

$$\mathbb{P}(Y_{t_1}^{\pi} = j_1, \ldots, Y_{t_n}^{\pi} = j_n)$$

$$= \sum_{i \in E} \pi_i P_{i,j_1}(t_1) P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

$$= \pi_{j_1} P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

Proof of theorem 3.9 (contd.)

With this, theorem 3.8 yields

$$\mathbb{P}(Y_{t_1}^{\pi} = j_1, \ldots, Y_{t_n}^{\pi} = j_n)$$

$$= \sum_{i \in E} \pi_i P_{i,j_1}(t_1) P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

$$= \pi_{j_1} P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

$$= \sum_{i \in E} \pi_i P_{i,j_1}(t_1 + s) P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

Proof of theorem 3.9 (contd.)

With this, theorem 3.8 yields

$$\mathbb{P}(Y_{t_1}^{\pi} = j_1, \ldots, Y_{t_n}^{\pi} = j_n)$$

$$= \sum_{i \in E} \pi_i P_{i,j_1}(t_1) P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

$$= \pi_{j_1} P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

$$= \sum_{i \in E} \pi_i P_{i,j_1}(t_1 + s) P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

$$= \mathbb{P}(Y_{t_1+s}^{\pi} = j_1, \ldots, Y_{t_n+s}^{\pi} = j_n)$$

for all times $t_1 < \ldots < t_n$ with $n \in \mathbb{N}$, and states $j_1, \ldots, j_n \in E$.

Proof of theorem 3.9 (contd.)

With this, theorem 3.8 yields

$$\mathbb{P}(Y^\pi_{t_1} = j_1, \dots, Y^\pi_{t_n} = j_n)$$

$$= \sum_{i \in E} \pi_i P_{i,j_1}(t_1) P_{j_1,j_2}(t_2 - t_1) \dots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

$$= \pi_{j_1} P_{j_1,j_2}(t_2 - t_1) \dots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

$$= \sum_{i \in E} \pi_i P_{i,j_1}(t_1 + s) P_{j_1,j_2}(t_2 - t_1) \dots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

$$= \mathbb{P}(Y^\pi_{t_1+s} = j_1, \dots, Y^\pi_{t_n+s} = j_n)$$

for all times $t_1 < \dots < t_n$ with $n \in \mathbb{N}$, and states $j_1, \dots, j_n \in E$. Hence the process $\mathcal{Y}^\pi$ is stationary.

Proof of theorem 3.9 (contd.)

On the other hand, if π is a stationary distribution,

Proof of theorem 3.9 (contd.)

On the other hand, if π is a stationary distribution, then we necessarily obtain

$$\pi P(t) = \pi e^{G \cdot t} = \pi$$

for all $t \geq 0$.

Proof of theorem 3.9 (contd.)

On the other hand, if π is a stationary distribution, then we necessarily obtain

$$\pi P(t) = \pi e^{G \cdot t} = \pi$$

for all $t \geq 0$.As above, this means

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \pi G^n = \mathbf{0}$$

for all $t \geq 0$,

Proof of theorem 3.9 (contd.)

On the other hand, if π is a stationary distribution, then we necessarily obtain

$$\pi P(t) = \pi e^{G \cdot t} = \pi$$

for all $t \geq 0$.As above, this means

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \pi G^n = \mathbf{0}$$

for all $t \geq 0$, which yields

$$\pi G = \mathbf{0}$$

Proof of theorem 3.9 (contd.)

On the other hand, if π is a stationary distribution, then we necessarily obtain

$$\pi P(t) = \pi e^{G \cdot t} = \pi$$

for all $t \geq 0$.As above, this means

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \pi G^n = \mathbf{0}$$

for all $t \geq 0$, which yields

$$\pi G = \mathbf{0}$$

because of the uniqueness of the zero power series.

Balance equations

The equation $\pi G = 0$ is equivalent to an equation system

$$\sum_{i \neq j} \pi_i g_{ij} = -\pi_j g_{jj} \iff \sum_{i \neq j} \pi_i g_{ij} = \pi_j \sum_{i \neq j} g_{ji}$$

for all $j \in E$.

Balance equations

The equation $\pi G = 0$ is equivalent to an equation system

$$\sum_{i \neq j} \pi_i g_{ij} = -\pi_j g_{jj} \quad \Longleftrightarrow \quad \sum_{i \neq j} \pi_i g_{ij} = \pi_j \sum_{i \neq j} g_{ji}$$

for all $j \in E$. We call the value $\pi_i g_{ij}$ **stochastic flow** from state i to state j in equilibrium.

Balance equations

The equation $\pi G = 0$ is equivalent to an equation system

$$\sum_{i \neq j} \pi_i g_{ij} = -\pi_j g_{jj} \quad \Longleftrightarrow \quad \sum_{i \neq j} \pi_i g_{ij} = \pi_j \sum_{i \neq j} g_{ji}$$

for all $j \in E$. We call the value $\pi_i g_{ij}$ **stochastic flow** from state i to state j in equilibrium. Then the above equations mean that the accrued stochastic flow into any state j equals the flow out of this state.

Balance equations

The equation $\pi G = 0$ is equivalent to an equation system

$$\sum_{i \neq j} \pi_i g_{ij} = -\pi_j g_{jj} \quad \Longleftrightarrow \quad \sum_{i \neq j} \pi_i g_{ij} = \pi_j \sum_{i \neq j} g_{ji}$$

for all $j \in E$. We call the value $\pi_i g_{ij}$ **stochastic flow** from state i to state j in equilibrium. Then the above equations mean that the accrued stochastic flow into any state j equals the flow out of this state. The above equations are called the (global) **balance equations**.

Example: Poisson process

The generator of the Poisson process with parameter λ is given by

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \ddots \\ 0 & 0 & -\lambda & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Example: Poisson process

The generator of the Poisson process with parameter λ is given by

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \ddots \\ 0 & 0 & -\lambda & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

This process has no stationary distribution, which can be seen as follows.

Example: Poisson process

The generator of the Poisson process with parameter λ is given by

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \ddots \\ 0 & 0 & -\lambda & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

This process has no stationary distribution, which can be seen as follows. The balance equations for the Poisson process are given by

$$\pi_0 \lambda = 0 \qquad \text{and} \qquad \pi_i \lambda = \pi_{i-1} \lambda$$

for all $i \geq 1$.

Example: Poisson process

The generator of the Poisson process with parameter λ is given by

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \ddots \\ 0 & 0 & -\lambda & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

This process has no stationary distribution, which can be seen as follows. The balance equations for the Poisson process are given by

$$\pi_0\lambda = 0 \qquad \text{and} \qquad \pi_i\lambda = \pi_{i-1}\lambda$$

for all $i \geq 1$. These are solvable only by $\pi_i = 0$ for all $i \in E$,

Example: Poisson process

The generator of the Poisson process with parameter λ is given by

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \ddots \\ 0 & 0 & -\lambda & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

This process has no stationary distribution, which can be seen as follows. The balance equations for the Poisson process are given by

$$\pi_0 \lambda = 0 \qquad \text{and} \qquad \pi_i \lambda = \pi_{i-1} \lambda$$

for all $i \geq 1$. These are solvable only by $\pi_i = 0$ for all $i \in E$, which means that there is no stationary distribution π.

Theorem 3.11

Let $\mathcal{Y}$ be a Markov process with embedded Markov chain $\mathcal{X}$.

Theorem 3.11

Let $\mathcal{Y}$ be a Markov process with embedded Markov chain $\mathcal{X}$. Let $\mathcal{X}$ be irreducible and positive recurrent.

Theorem 3.11

Let $\mathcal{Y}$ be a Markov process with embedded Markov chain $\mathcal{X}$. Let $\mathcal{X}$ be irreducible and positive recurrent. Further assume that $\check{\lambda} := \inf\{\lambda_i : i \in E\} > 0$.

Theorem 3.11

Let $\mathcal{Y}$ be a Markov process with embedded Markov chain $\mathcal{X}$. Let $\mathcal{X}$ be irreducible and positive recurrent. Further assume that $\check{\lambda} := \inf\{\lambda_i : i \in E\} > 0$. Then there is a unique stationary distribution for $\mathcal{Y}$.

Theorem 3.11

Let $\mathcal{Y}$ be a Markov process with embedded Markov chain $\mathcal{X}$. Let $\mathcal{X}$ be irreducible and positive recurrent. Further assume that $\check{\lambda} := \inf\{\lambda_i : i \in E\} > 0$. Then there is a unique stationary distribution for $\mathcal{Y}$.

Proof:
According to theorems 2.25 and 2.18, the transition matrix P of $\mathcal{X}$ admits a unique stationary distribution ν with $\nu P = \nu$.

Theorem 3.11

Let $\mathcal{Y}$ be a Markov process with embedded Markov chain $\mathcal{X}$. Let $\mathcal{X}$ be irreducible and positive recurrent. Further assume that $\check{\lambda} := \inf\{\lambda_i : i \in E\} > 0$. Then there is a unique stationary distribution for $\mathcal{Y}$.

Proof:
According to theorems 2.25 and 2.18, the transition matrix P of $\mathcal{X}$ admits a unique stationary distribution ν with $\nu P = \nu$. The generator G is defined by

$$G = \Lambda(P - I)$$

Theorem 3.11

Let $\mathcal{Y}$ be a Markov process with embedded Markov chain $\mathcal{X}$. Let $\mathcal{X}$ be irreducible and positive recurrent. Further assume that $\check{\lambda} := \inf\{\lambda_i : i \in E\} > 0$. Then there is a unique stationary distribution for $\mathcal{Y}$.

Proof:
According to theorems 2.25 and 2.18, the transition matrix P of $\mathcal{X}$ admits a unique stationary distribution ν with $\nu P = \nu$. The generator G is defined by

$$G = \Lambda(P - I)$$

with

$$\Lambda = diag(\lambda_i : i \in E)$$

Proof of theorem 3.11 (contd.)

Hence the measure

$$\mu := \nu\Lambda^{-1}$$

Proof of theorem 3.11 (contd.)

Hence the measure

$$\mu := \nu\Lambda^{-1}$$

is stationary for $\mathcal{Y}$.

Proof of theorem 3.11 (contd.)

Hence the measure

$$\mu := \nu\Lambda^{-1}$$

is stationary for $\mathcal{Y}$. Since $\check{\lambda} > 0$, the measure μ is finite,

Proof of theorem 3.11 (contd.)

Hence the measure

$$\mu := \nu \Lambda^{-1}$$

is stationary for $\mathcal{Y}$. Since $\check{\lambda} > 0$, the measure μ is finite, with total mass bounded by $\check{\lambda}^{-1} < \infty$.

Proof of theorem 3.11 (contd.)

Hence the measure

$$\mu := \nu \Lambda^{-1}$$

is stationary for $\mathcal{Y}$. Since $\check{\lambda} > 0$, the measure μ is finite, with total mass bounded by $\check{\lambda}^{-1} < \infty$. Now the normalization

$$\pi_j := \frac{\mu_j}{\sum_{i \in E} \mu_i} = \frac{\nu_j / \lambda_j}{\sum_{i \in E} \nu_i / \lambda_i}$$

for all $j \in E$

Proof of theorem 3.11 (contd.)

Hence the measure

$$\mu := \nu\Lambda^{-1}$$

is stationary for $\mathcal{Y}$. Since $\check{\lambda} > 0$, the measure μ is finite, with total mass bounded by $\check{\lambda}^{-1} < \infty$. Now the normalization

$$\pi_j := \frac{\mu_j}{\sum_{i \in E} \mu_i} = \frac{\nu_j/\lambda_j}{\sum_{i \in E} \nu_i/\lambda_i}$$

for all $j \in E$ yields a stationary distribution for $\mathcal{Y}$.

Proof of theorem 3.11 (contd.)

Hence the measure

$$\mu := \nu \Lambda^{-1}$$

is stationary for $\mathcal{Y}$. Since $\check{\lambda} > 0$, the measure μ is finite, with total mass bounded by $\check{\lambda}^{-1} < \infty$. Now the normalization

$$\pi_j := \frac{\mu_j}{\sum_{i \in E} \mu_i} = \frac{\nu_j / \lambda_j}{\sum_{i \in E} \nu_i / \lambda_i}$$

for all $j \in E$ yields a stationary distribution for $\mathcal{Y}$. This is unique because ν is unique.

Skip-free Markov processes

We define a **skip–free Markov process** by

$$g_{ij} = 0 \quad \text{for all states} \quad i, j \in E \subset \mathbb{N}_0 \quad \text{with} \quad |i - j| > 1$$

Skip-free Markov processes

We define a **skip–free Markov process** by

$$g_{ij} = 0 \quad \text{for all states} \quad i, j \in E \subset \mathbb{N}_0 \quad \text{with} \quad |i - j| > 1$$

Denote the remaining infinitesimal transition rates by

$$\lambda_i := g_{i,i+1} \qquad \text{and} \qquad \mu_i := g_{i,i-1}$$

for all possible values of i.

Skip-free Markov processes

We define a **skip–free Markov process** by

$$g_{ij} = 0 \quad \text{for all states} \quad i, j \in E \subset \mathbb{N}_0 \quad \text{with} \quad |i - j| > 1$$

Denote the remaining infinitesimal transition rates by

$$\lambda_i := g_{i,i+1} \qquad \text{and} \qquad \mu_i := g_{i,i-1}$$

for all possible values of i. The rates λ_i and μ_i are called **arrival rates** and **departure rates**, respectively.

Skip-free Markov processes

We define a **skip–free Markov process** by

$$g_{ij} = 0 \quad \text{for all states} \quad i, j \in E \subset \mathbb{N}_0 \quad \text{with} \quad |i - j| > 1$$

Denote the remaining infinitesimal transition rates by

$$\lambda_i := g_{i,i+1} \qquad \text{and} \qquad \mu_i := g_{i,i-1}$$

for all possible values of i. The rates λ_i and μ_i are called **arrival rates** and **departure rates**, respectively. The state transition graph of such a process assumes the form

Skip-free Markov processes

Its balance equations are given by

$$\lambda_0 \pi_0 = \mu_1 \pi_1$$

Its balance equations are given by

$$\lambda_0 \pi_0 = \mu_1 \pi_1$$

and

$$(\lambda_i + \mu_i)\pi_i = \lambda_{i-1}\pi_{i-1} + \mu_{i+1}\pi_{i+1}$$

for all $i \in \mathbb{N}$.

Skip-free Markov processes

Its balance equations are given by

$$\lambda_0\pi_0 = \mu_1\pi_1$$

and

$$(\lambda_i + \mu_i)\pi_i = \lambda_{i-1}\pi_{i-1} + \mu_{i+1}\pi_{i+1}$$

for all $i \in \mathbb{N}$. By induction on i it is shown that these are equivalent to the equation system

$$\lambda_{i-1}\pi_{i-1} = \mu_i\pi_i$$

for all $i \in \mathbb{N}$.

Skip-free Markov processes

This system is solved by successive elimination

This system is solved by successive elimination with a solution of the form

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}} = \pi_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}$$

for all $i \geq 1$.

Skip-free Markov processes

This system is solved by successive elimination with a solution of the form

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}} = \pi_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}$$

for all $i \geq 1$. The solution π is a probability distribution if and only if it can be normalized,

Skip-free Markov processes

This system is solved by successive elimination with a solution of the form

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}} = \pi_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}$$

for all $i \geq 1$. The solution π is a probability distribution if and only if it can be normalized, i.e. if $\sum_{n \in E} \pi_n = 1$.

Skip-free Markov processes

This system is solved by successive elimination with a solution of the form

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}} = \pi_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}$$

for all $i \geq 1$. The solution π is a probability distribution if and only if it can be normalized, i.e. if $\sum_{n \in E} \pi_n = 1$. This condition implies

$$1 = \sum_{n \in E} \pi_0 \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}} = \pi_0 \sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}}$$

with the empty product being defined as one.

Skip-free Markov processes

This system is solved by successive elimination with a solution of the form

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}} = \pi_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}$$

for all $i \geq 1$. The solution π is a probability distribution if and only if it can be normalized, i.e. if $\sum_{n \in E} \pi_n = 1$. This condition implies

$$1 = \sum_{n \in E} \pi_0 \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}} = \pi_0 \sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}}$$

with the empty product being defined as one. This means that

$$\pi_0 = \left(\sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}} \right)^{-1}$$

Skip-free Markov processes

This system is solved by successive elimination with a solution of the form

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}} = \pi_0 \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}$$

for all $i \geq 1$. The solution π is a probability distribution if and only if it can be normalized, i.e. if $\sum_{n \in E} \pi_n = 1$. This condition implies

$$1 = \sum_{n \in E} \pi_0 \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}} = \pi_0 \sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}}$$

with the empty product being defined as one. This means that

$$\pi_0 = \left(\sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}} \right)^{-1}$$

and thus π is a probability distribution if and only if the series in the brackets converges.

Competing exponentials

Let $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$ denote two independent random variables.

Competing exponentials

Let $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$ denote two independent random variables. Then

$$\min(X, Y) \sim Exp(\lambda + \mu)$$

Competing exponentials

Let $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$ denote two independent random variables. Then

$$\min(X, Y) \sim Exp(\lambda + \mu)$$

$$\mathbb{P}(X < Y) = \frac{\lambda}{\lambda + \mu}$$

Competing exponentials

Let $X \sim Exp(\lambda)$ and $Y \sim Exp(\mu)$ denote two independent random variables. Then

$$\min(X, Y) \sim Exp(\lambda + \mu)$$

$$\mathbb{P}(X < Y) = \frac{\lambda}{\lambda + \mu}$$

$$\mathbb{P}(X = Y) = 0$$

Proof - 1

Let $Z := \min(X, Y)$.

Proof - 1

Let $Z := \min(X, Y)$. Then

$$Z > t \iff X > t \quad \text{and} \quad Y > t$$

Proof - 1

Let $Z := \min(X, Y)$. Then

$$Z > t \iff X > t \quad \text{and} \quad Y > t$$

Independence of X and Y yields

$$\mathbb{P}(Z > t) = \mathbb{P}(X > t) \cdot \mathbb{P}(Y > t)$$

Proof - 1

Let $Z := \min(X, Y)$. Then

$$Z > t \iff X > t \quad \text{and} \quad Y > t$$

Independence of X and Y yields

$$\mathbb{P}(Z > t) = \mathbb{P}(X > t) \cdot \mathbb{P}(Y > t)$$

and thus

$$\mathbb{P}(\min(X, Y) > t) = e^{-\lambda t} e^{-\mu t} = e^{-(\lambda+\mu)t}$$

for all $t \geq 0$.

Proof - 2

Conditioning on $X \in dt$ yields

$$\mathbb{P}(X < Y) = \int_0^\infty \lambda e^{-\lambda t} \, \mathbb{P}(Y > t) \, dt$$

Proof - 2

Conditioning on $X \in dt$ yields

$$\mathbb{P}(X < Y) = \int_0^\infty \lambda e^{-\lambda t} \, \mathbb{P}(Y > t) \, dt$$

$$= \lambda \int_0^\infty e^{-\lambda t} e^{-\mu t} dt = \frac{\lambda}{\lambda + \mu}$$

Proof - 2

Conditioning on $X \in dt$ yields

$$\mathbb{P}(X < Y) = \int_0^\infty \lambda e^{-\lambda t}\, \mathbb{P}(Y > t)\, dt$$

$$= \lambda \int_0^\infty e^{-\lambda t} e^{-\mu t} dt = \frac{\lambda}{\lambda + \mu}$$

Finally,

$$\mathbb{P}(X = Y) = \lim_{h \to 0} \int_0^\infty \lambda e^{-\lambda t}\, \mathbb{P}(Y \in [t, t+h])\, dt$$

Proof - 2

Conditioning on $X \in dt$ yields

$$\mathbb{P}(X < Y) = \int_0^\infty \lambda e^{-\lambda t}\, \mathbb{P}(Y > t)\, dt$$

$$= \lambda \int_0^\infty e^{-\lambda t} e^{-\mu t} dt = \frac{\lambda}{\lambda + \mu}$$

Finally,

$$\mathbb{P}(X = Y) = \lim_{h \to 0} \int_0^\infty \lambda e^{-\lambda t}\, \mathbb{P}(Y \in [t, t+h])\, dt$$

$$= \int_0^\infty \lambda e^{-\lambda t} \lim_{h \to 0} \left(e^{-\mu t} - e^{-\mu(t+h)} \right) dt = 0$$

Example: M/M/1 queue

Arrivals: Poisson process with rate $\lambda > 0$

Example: M/M/1 queue

Arrivals: Poisson process with rate $\lambda > 0$

Service times: iid, exponentially distributed with rate $\mu > 0$

Example: M/M/1 queue

Arrivals: Poisson process with rate $\lambda > 0$

Service times: iid, exponentially distributed with rate $\mu > 0$

1 server

Example: M/M/1 queue

Arrivals: Poisson process with rate $\lambda > 0$

Service times: iid, exponentially distributed with rate $\mu > 0$

1 server

The arrival process and the service times are independent.

Example: M/M/1 queue

Arrivals: Poisson process with rate $\lambda > 0$

Service times: iid, exponentially distributed with rate $\mu > 0$

1 server

The arrival process and the service times are independent.

Let Y_t denote the number of users in the system at time t.

Example: M/M/1 queue

Arrivals: Poisson process with rate $\lambda > 0$

Service times: iid, exponentially distributed with rate $\mu > 0$

1 server

The arrival process and the service times are independent.

Let Y_t denote the number of users in the system at time t.

State space $E = \mathbb{N}_0$

The M/M/1 queue as a Markov process

Holding time in state 0:

The M/M/1 queue as a Markov process

Holding time in state 0: $H_0 \sim Exp(\lambda)$

The M/M/1 queue as a Markov process

Holding time in state 0: $H_0 \sim Exp(\lambda)$

$p_{01} = 1$

The M/M/1 queue as a Markov process

Holding time in state 0: $H_0 \sim Exp(\lambda)$

$p_{01} = 1$

Holding time in state $i \geq 1$:

The M/M/1 queue as a Markov process

Holding time in state 0: $H_0 \sim Exp(\lambda)$

$p_{01} = 1$

Holding time in state $i \geq 1$:
$H_i = \min(A, S)$, where $A \sim Exp(\lambda)$ and $S \sim Exp(\mu)$

The M/M/1 queue as a Markov process

Holding time in state 0: $H_0 \sim Exp(\lambda)$

$p_{01} = 1$

Holding time in state $i \geq 1$:
$H_i = \min(A, S)$, where $A \sim Exp(\lambda)$ and $S \sim Exp(\mu)$

Hence, $H_i \sim Exp(\lambda + \mu)$ for $i \geq 1$.

The M/M/1 queue as a Markov process

Holding time in state 0: $H_0 \sim Exp(\lambda)$

$p_{01} = 1$

Holding time in state $i \geq 1$:
$H_i = \min(A, S)$, where $A \sim Exp(\lambda)$ and $S \sim Exp(\mu)$

Hence, $H_i \sim Exp(\lambda + \mu)$ for $i \geq 1$.

Further

$$p_{ij} = \begin{cases} \mathbb{P}(A < S) = \frac{\lambda}{\lambda+\mu}, & j = i+1 \\ \mathbb{P}(S < A) = \frac{\mu}{\lambda+\mu}, & j = i-1 \end{cases}$$

for $i \geq 1$.

The generator matrix

$$g_{01} = \lambda \cdot p_{01} = \lambda$$

The generator matrix

$g_{01} = \lambda \cdot p_{01} = \lambda$ and for $i \geq 1$

The generator matrix

$g_{01} = \lambda \cdot p_{01} = \lambda$ and for $i \geq 1$

$$g_{ij} = \begin{cases} (\lambda + \mu) \cdot \frac{\lambda}{\lambda+\mu} = \lambda, & j = i + 1 \\ (\lambda + \mu) \cdot \frac{\mu}{\lambda+\mu} = \mu, & j = i - 1 \end{cases}$$

The generator matrix

$g_{01} = \lambda \cdot p_{01} = \lambda$ and for $i \geq 1$

$$g_{ij} = \begin{cases} (\lambda + \mu) \cdot \frac{\lambda}{\lambda+\mu} = \lambda, & j = i+1 \\ (\lambda + \mu) \cdot \frac{\mu}{\lambda+\mu} = \mu, & j = i-1 \end{cases}$$

Hence,

$$G = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ \mu & -\lambda-\mu & \lambda & 0 & \ddots \\ 0 & \mu & -\lambda-\mu & \lambda & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Stationary distribution

Stationary distribution

$\mathcal{Y} = (Y_t : t \geq 0)$ is a skip-free Markov process on $E = \mathbb{N}_0$

Stationary distribution

$\mathcal{Y} = (Y_t : t \geq 0)$ is a skip-free Markov process on $E = \mathbb{N}_0$ with arrival rates $\lambda_i = \lambda$

Stationary distribution

$\mathcal{Y} = (Y_t : t \geq 0)$ is a skip-free Markov process on $E = \mathbb{N}_0$ with arrival rates $\lambda_i = \lambda$ and departure rates $\mu_i = \mu$.

Stationary distribution

$\mathcal{Y} = (Y_t : t \geq 0)$ is a skip-free Markov process on $E = \mathbb{N}_0$ with arrival rates $\lambda_i = \lambda$ and departure rates $\mu_i = \mu$.

Thus the stationary distribution π is given by

$$\pi_0 = \left(\sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}} \right)^{-1} = \left(\sum_{n=0}^{\infty} \rho^n \right)^{-1} = (1 - \rho)$$

if $\rho := \lambda / \mu < 1$

Stationary distribution

$\mathcal{Y} = (Y_t : t \geq 0)$ is a skip-free Markov process on $E = \mathbb{N}_0$ with arrival rates $\lambda_i = \lambda$ and departure rates $\mu_i = \mu$.

Thus the stationary distribution π is given by

$$\pi_0 = \left(\sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}} \right)^{-1} = \left(\sum_{n=0}^{\infty} \rho^n \right)^{-1} = (1 - \rho)$$

if $\rho := \lambda / \mu < 1$ and

$$\pi_i = \pi_0 \rho^i$$

for $i \geq 1$.

Stationary distribution

$\mathcal{Y} = (Y_t : t \geq 0)$ is a skip-free Markov process on $E = \mathbb{N}_0$ with arrival rates $\lambda_i = \lambda$ and departure rates $\mu_i = \mu$.

Thus the stationary distribution π is given by

$$\pi_0 = \left(\sum_{n \in E} \prod_{j=0}^{n-1} \frac{\lambda_j}{\mu_{j+1}} \right)^{-1} = \left(\sum_{n=0}^{\infty} \rho^n \right)^{-1} = (1 - \rho)$$

if $\rho := \lambda / \mu < 1$ and

$$\pi_i = \pi_0 \rho^i$$

for $i \geq 1$. For $\rho \geq 1$ there is no stationary distribution.