Chapter 3: Homogeneous Markov Processes on Discrete State Spaces

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Further definitions

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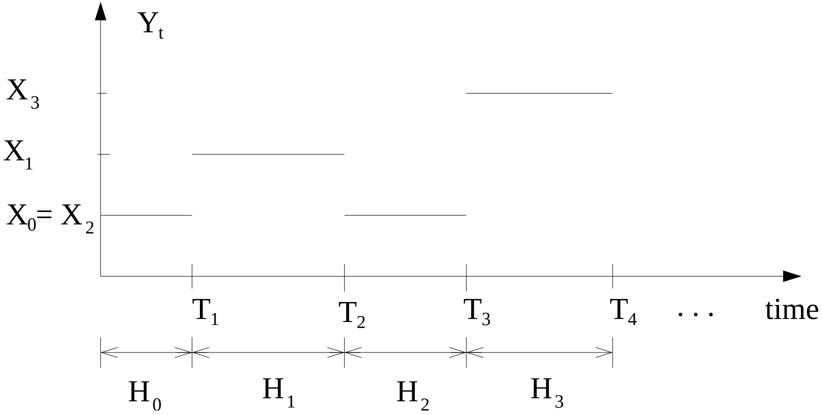
Typical path



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This is due to the memoryless property of the holding times and the Markov property of the embedded chain $\mathcal{X}$.

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for all $i, j \in E$ and $s < t$, which we call **homogeneity** of $\mathcal{Y}$. For every $t \geq 0$, define the **transition probability matrix** $P(t)$ in time t by its entries

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holds for all $i \in E$. Unless stated otherwise, we shall always assume $p_{ii} = 0$ for all $i \in E$.

Transition graph

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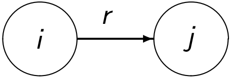

for $g_{ij} = r > 0$ and $j \neq i$.

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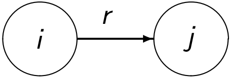
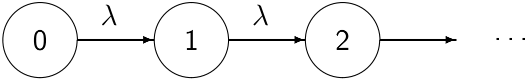


for $g_{ij} = r > 0$ and $j \neq i$. The value $r = g_{ij}$ is called the **infinitesimal transition rate** from state i to state j.

Example 3.5: Poisson process

The transition graph of the Poisson process with intensity λ is given by



Theorem 3.6: Kolmogorov differential equations

The transition probabilities $P_{ij}(t)$ of a Markov process satisfy the systems

$$\frac{dP_{ij}(t)}{dt} = \sum_{k \in E} P_{ik}(t) g_{kj} = \sum_{k \in E} g_{ik} P_{kj}(t)$$

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$$P_{ij}(t) = e^{-\lambda_i \cdot t} \cdot \delta_{ij} + \int_0^t e^{-\lambda_i \cdot s} \lambda_i \sum_{k \in E} p_{ik} P_{kj}(t-s)\, ds$$

Proof of theorem 3.6 (contd.)

We obtain further

$$P_{ij}(t) = e^{-\lambda_i \cdot t} \cdot \left(\delta_{ij} + \int_0^t e^{+\lambda_i \cdot u} \lambda_i \sum_{k \in E} p_{ik} P_{kj}(u)\, du \right)$$

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by substituting $u = t - s$ in the integral. We can differentiate $P(t)$ as

$$\frac{dP_{ij}(t)}{dt} = -\lambda_i e^{-\lambda_i \cdot t} \cdot \left(\delta_{ij} + \int_0^t f(u) \, du \right) + e^{-\lambda_i \cdot t} \cdot f(t)$$

with f denoting the integrand function.

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which proves the backward equation. For the forward equations, one only needs to use the Chapman–Kolmogorov equations and apply the backward equations in

Proof of theorem 3.6 (contd.)

Thus

$$\frac{dP_{ij}(t)}{dt} = -\lambda_i P_{ij}(t) + \lambda_i \sum_{k \in E} p_{ik} P_{kj}(t)$$

$$= -\lambda_i (1 - p_{ii}) \cdot P_{ii}(t) + \sum_{k \neq i} g_{ik} P_{kj}(t)$$

which proves the backward equation. For the forward equations, one only needs to use the Chapman–Kolmogorov equations and apply the backward equations in

$$\frac{dP_{ij}(t)}{dt} = \lim_{h \to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lim_{h \to 0} \sum_{k \in E} P_{ik}(t) \frac{P_{kj}(h) - \delta_{kj}}{h}$$

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$$= \sum_{k\in E} P_{ik}(t) \lim_{h\to 0} \frac{P_{kj}(h) - P_{kj}(0)}{h} = \sum_{k\in E} P_{ik}(t) g_{kj}$$

Theorem 3.7

The transition probability matrices can be expressed in terms of the generator by

$$P(t) = e^{G \cdot t} := \sum_{n=0}^{\infty} \frac{t^n}{n!} G^n$$

for all $t \geq 0$, with G^n denoting the nth power of the matrix G.

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$$\frac{d}{dt} e^{G \cdot t} = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n}{n!} G^n = \sum_{n=1}^{\infty} G^n \frac{d}{dt} \frac{t^n}{n!} = \sum_{n=1}^{\infty} G^n \frac{t^{n-1}}{(n-1)!} = G e^{G \cdot t}$$

Proof of theorem 3.7 (contd.)

Furthermore, it is obvious that

$$Ge^{G\cdot t} = G\sum_{n=0}^{\infty}\frac{t^n}{n!}G^n = \left(\sum_{n=0}^{\infty}\frac{t^n}{n!}G^n\right)G = e^{G\cdot t}G$$

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and thus $P(t) = e^{G\cdot t}$ is a solution of Kolmogorov's forward and backward equations. Uniqueness of the solution follows from the initial condition

$$P(0) = I$$

Remarks

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Hence the generator of a Markov process uniquely determines all its transition matrices. This can also be seen from the definition, if we agree (without loss of generality) upon the convention $p_{ii} = 0$ for all $i \in E$. Then the parameters for the definition of the Markov process can be recovered by

$$\lambda_i = -g_{ii} \qquad \text{and} \qquad p_{ij} = \frac{g_{ij}}{-g_{ii}}$$

for all $i \neq j \in E$.

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As in the discrete time case of Markov chains, Markov processes are not completely determined by their transition probability matrices only.

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$$\pi_i = \mathbb{P}(Y_0 = X_0 = i)$$

for all $i \in E$.

Theorem 3.8: Finite–dimensional marginal distributions

For a Markov process $\mathcal{Y}$ with initial distribution π and time instances $0 < t_1 < \ldots < t_n$, $n \in \mathbb{N}$,

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$$\mathbb{P}(Y_{t_1} = j_1, \ldots, Y_{t_n} = j_n)$$

$$= \sum_{i \in E} \pi_i P_{i,j_1}(t_1) P_{j_1,j_2}(t_2 - t_1) \ldots P_{j_{n-1},j_n}(t_n - t_{n-1})$$

holds for all $j_1, \ldots, j_n \in E$.

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Proof: as an exercise (induction on n, use the Markov property)