# TWO EXAMPLES FOR COMPUTATIONALLY TRACTABLE PERIODIC QUEUES

L. BREUER

*Stochastic Modelling*
*Department of Computer Science*
*University of Trier*
*54286 Trier, Germany*
E-mail : breuer@info04.uni-trier.de

**Abstract:** In the present paper, the periodic $BMAP/M_t/c$ queue is analyzed. This queue has a periodic BMAP arrival process, which is defined in this paper, and exponential service time distributions with service rates varying periodically in time. The transient distribution and several formulae for its efficient computation are derived. Furthermore, a stability condition in terms of arrival and service rates is proven, and for the case of stability the asymptotic distribution is given explicitly. A special class of periodic $BMAP/M_t/c$ queues, which allows to reduce the complexity of computations to the level of homogeneous queues, is separated. The obtained results are applied to the spatial $SMAP/M_t/c/c$ variant of the queue, too. This variant is very useful as a model for mobile communication networks.

*Key Words:* Periodic Queues, BMAP, Markov Jump Processes, Spatial Queues, SMAP, Erlang Loss Formula, Matrix–Analytical Methods

## 1 INTRODUCTION

In queueing theory, most models are based on time-homogeneous arrival processes and service time distributions. However, a typical property of communication traffic is the dependence of its arrival rates on time. This aspect incites the use of time–inhomogeneous processes and queues for modelling communication networks. Typically, a periodic dependence of the arrival rates and/or the service time distribution can be assumed with period lengths of a day or a week. In order to reflect this property accurately, one needs to examine periodic rather than homogeneous queues.

While queues with periodic input naturally reflect the time-dependent amount of traffic that arrives in communication networks, the analysis of queues with inhomogeneous arrival rates is far less developed than the one for homogeneous queues. Some of the existing results in the literature are given in (Asmussen and Thorisson, 1987), (Bambos and Walrand, 1989), (Falin, 1989), (Harrison and Lemoine, 1977), (Hasofer, 1964), (Heyman and Whitt, 1984), (Lemoine, 1989; Lemoine, 1981), (Massey, 1981), (Rolski, 1987; Rolski, 1989), and (Willie, 1998). Although many types of stability conditions could be established, the first explicit formulae for asymptotic behaviour could be derived only recently and for Markovian queues, see (Breuer, 1999; Breuer, 2001; Breuer, 2002c).

The main objective of the present paper is to single out a class of periodic $BMAP/M_t/c$ queues which does not require more computational effort than homogeneous queues. This holds for the calculation of the transient as well as for the stationary distribution. Thus it is shown that Markovian queues can be modelled in a more realistic approach without any additional computational cost.

The paper is organized as follows. In the remainder of this section, the periodic $BMAP/M_t/c$ queue as well as the basic notations are defined. The next section contains the transient distributions at arbitrary times and their efficient computation in special cases. In section 3, a stability condition is given and the asymptotic distributions at any ”day time” of one period length are derived explicitly in the case of stability. For the special class to be examined, the asymptotic distributions will be constant over the period length and the queue will show the same asymptotic behaviour as a related homogeneous queue. In section 4 it is shown that the obtained results can be applied to the spatial $SMAP/M_t/c/c$ variant of the queue, too. Finally, conclusions and a view on future research are given.

One of the most important features to be exploited in the analysis of queueing systems is the Markov property which often appears after the construction of embedded

Markov chains. The search for Markovian but versatile arrival processes has led to the concept of batch Markovian arrival processes (BMAPs, see (Neuts, 1979) and (Lucantoni, 1991)) which allow for a phase process controlling the arrival rates. This arrival process is often used for modelling communication networks.

Like all Markov arrival processes in queueing theory, BMAPs are Markov jump processes. As main reference for the theory of Markov jump processes, the book by (Gikhman and Skorokhod, 1969) shall be referred to. Analogous to the definition of a BMAP (see (Lucantoni, 1991)), an inhomogeneous BMAP shall be defined by its time-dependent transition rate matrices $(D_n(t) : n \in \mathbb{N}_0, t \in \mathbb{R}_0^+)$, assuming that the continuity conditions for the transition rates of inhomogeneous Markov jump processes are satisfied (see (Gikhman and Skorokhod, 1969), p.362). Let the number of phases, which is the dimension of the square matrices $D_n(t)$, be denoted by $m \in \mathbb{N}$. Then the time–dependent generator of an inhomogeneous BMAP can be represented in a block matrix

$$\begin{pmatrix} D_0(t) & D_1(t) & D_2(t) & D_3(t) & \dots \\ & D_0(t) & D_1(t) & D_2(t) & \dots \\ & & D_0(t) & D_1(t) & \ddots \\ & & & D_0(t) & \ddots \\ & & & & \ddots \end{pmatrix}$$

with $m \times m$ matrices $D_n(t)$ depending on time $t$. Here, $D_n(t)(i,j)$ is the infinitesimal transition rate of observing $n$ arrivals at time $t$ while changing from phase $i$ to phase $j$. A periodic BMAP with period $T$ is an inhomogeneous BMAP with the property $D_n(s+T) = D_n(s)$ for all $n \in \mathbb{N}_0$ and $s \in [0, T[$. For further generalizations of the BMAP concept and more properties of inhomogeneous BMAPs see (Breuer, 2002b).

Let $Q = (Q_t : t \in \mathbb{R}_0^+)$ denote a periodic $BMAP/M_t/c$ queue with a periodic BMAP arrival process, exponential service time distributions with periodic service rates, and $c$ servers. Arrival process and service rates shall have the same period $T$. Define the arrival process by its transition rate matrices $(D_n(t) : n \in \mathbb{N}_0, t \in \mathbb{R}_0^+)$ having dimension $m$ and period $T$. Every server is equal, and the service time distribution function $B_s$ for a user arriving at time $s \in \mathbb{R}_0^+$ is defined by

$$B_s(t) := 1 - e^{-\int_s^{s+t} \mu_u \, du}$$

for all $t \in \mathbb{R}_0^+$. This means that the service process without idle periods would be an inhomogeneous Poisson process with rates $(\mu_t : t \in \mathbb{R}_0^+)$. Periodicity of the service rates means $\mu_{s+T} = \mu_s$ for all $s \in [0, T[$. This queue has already been defined, but not analyzed, in (Breuer, 2001).

# 2 TRANSITION MATRICES

As a Markovian queue, the system can be analyzed like an (inhomogeneous) Markov jump process. The infinitesimal generator $G(t)$ of the queue process $Q$ at time $t \in \mathbb{R}_0^+$ can be written as an $\mathbb{N}_0 \times \mathbb{N}_0$ block matrix with entries being the $m \times m$ matrices

$$G_{kn}(t) = \begin{cases} 0 & \text{for} \quad k > n+1 \\ k\mu_t \cdot I & \text{for} \quad k = n+1 \le c \\ c\mu_t \cdot I & \text{for} \quad k = n+1 > c \\ D_0(t) - k\mu_t \cdot I & \text{for} \quad k = n \le c \\ D_0(t) - c\mu_t \cdot I & \text{for} \quad k = n > c \\ D_{n-k}(t) & \text{for} \quad k < n \end{cases}$$

for $k, n \in \mathbb{N}_0$, with 0 and $I$ denoting the zero and identity matrix, respectively. Denote the $(i,j)$th entry of the matrix $G_{kn}(t)$ by $G_{kn}(t)(i,j)$.

**Remark 1** The form of the generator shows that the model easily can be extended to the $BMAP/M_t/c$ queue with heterogeneous servers. Here, we simply would have different service rates $\mu_t^{(1)}, \dots, \mu_t^{(c)}$ for different servers. The state space would need to be extended such as to describe which of the servers are busy. Since this would complicate notations, we will pursue the analysis for the case of homogeneous servers only.

The multiplication of two $\mathbb{N}_0 \times \mathbb{N}_0$ block matrices $A$ and $B$ is defined by

$$(AB)_{kn}(i,j) := \sum_{l=0}^{\infty} \sum_{h=1}^{m} A_{kl}(i,h) B_{ln}(h,j)$$

for every $k, n \in \mathbb{N}_0$ and $i, j \in \{1, \dots, m\}$.
Define $P_{kn}(s,t)(i,j)$ as the probability of having $n \in \mathbb{N}_0$ users in the queue and being in phase $j$ at time $t > s$ under the condition of having $k \in \mathbb{N}_0$ users in the queue and being in phase $i$ at time $s$. Further define $P_{kn}(s,t)$ as the $m \times m$ matrix with entries $P_{kn}(s,t)(i,j)$ and $P(s,t) = (P_{kn}(s,t))_{k,n \in \mathbb{N}_0}$ as the $\mathbb{N}_0 \times \mathbb{N}_0$ block matrix with entries $P_{kn}(s,t)$.
Solving Kolmogorov's forward equations via the iteration method by Picard and Lindelöf (cf. (Gikhman and Skorokhod, 1969), p.317), it can be shown that the transition probability matrices of the queue can be written as

$$P(s,t) = \sum_{k=0}^{\infty} P^{(k)}(s,t)$$

with

$$P^{(k)}(s,t) = \underbrace{\int_s^t \int_s^{u_k} \dots \int_s^{u_2}}_{k \ integrals} G(u_1) \dots G(u_k) \, du_1 \dots du_k$$

for all $s < t \in \mathbb{R}_0^+$. For computing the terms $P(s,t)$, one can use the following iteration as given in (Bellman, 1997), p.168: Starting with $I_0(u) := Id$ for all $s \leq u \leq t$, the iteration

$$I_{n+1}(u) := \int_s^u I_n(v)G(v)dv + Id \qquad (1)$$

leads to the limit

$$P(s,t) = \lim_{n\to\infty} I_n(t)$$

for all $s < t \in \mathbb{R}_0^+$.
For homogeneous arrival and service rates, the generator $G(t) = G$ is constant in $t$ and the transition probability matrix reduces to the matrix exponential form $P(s,t) = e^{G\cdot(t-s)}$, which is well known for homogeneous Markov jump processes. If the equation

$$\frac{d}{dt}\left(\int_s^t G(u)du\right)^k = k\cdot\left(\int_s^t G(u)du\right)^{k-1} G(t) \quad (2)$$

holds for all $s < t \in \mathbb{R}_0^+$ and $k \in \mathbb{N}$, then the generator function $G(t)$ shall be called quasi–commutable. In this case, an effective reduction of the computational complexity is given in

**Theorem 1** *If the generator function $G(t)$ is quasi–commutable, then the transition probability matrix of the queue takes the form*

$$P(s,t) = \sum_{k=0}^{\infty} \frac{\left(\int_s^t G(u)du\right)^k}{k!} = e^{\int_s^t G(u)du}$$

*for all $s < t \in \mathbb{R}_0^+$.*

**Proof:** For every $k \in \mathbb{N}$ it is shown that

$$P^{(k)}(s,t) = \frac{\left(\int_s^t G(u)du\right)^k}{k!}$$

is implied by equation (2). For $k = 1$ this is obvious. The induction step for $k+1$ is seen by first applying the induction hypothesis and then equation (2) in

$$\begin{aligned}
&\int_s^t \int_s^{u_{k+1}} \dots \int_s^{u_2} G(u_1)\dots G(u_{k+1})du_1\dots du_{k+1} \\
&= \int_s^t \frac{\left(\int_s^{u_{k+1}} G(u)du\right)^k}{k!} G(u_{k+1})du_{k+1} \\
&= \frac{1}{k!}\int_s^t \frac{1}{k+1}\frac{d}{du_{k+1}}\left(\int_s^{u_{k+1}} G(u)du\right)^{k+1} du_{k+1} \\
&= \frac{1}{(k+1)!}\left(\int_s^t G(u)du\right)^{k+1}
\end{aligned}$$

which completes the induction step.
☺

Sufficient conditions for equation (2) are given if the phase space $\Lambda$ has only one element or if for every $t \in \mathbb{R}_0^+$ the kernel matrices $G(t)$ and $\int_s^t G(u)du$ are commutable. The latter holds in particular if there are some constant parameter matrices $(D_n : n \in \mathbb{N}_0)$ for a BMAP and a service rate $\mu \in \mathbb{R}^+$, along with a bounded function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, such that

$$D_n(t) = f(t)\cdot D_n \qquad \text{and} \qquad \mu_t = f(t)\cdot\mu \qquad (3)$$

for all $n \in \mathbb{N}_0$, $t \in \mathbb{R}_0^+$. Then we also can write

$$G(t) = f(t)\cdot G$$

for the generator function of the queue, as can be readily verified from the definition of $G(t)$ at the beginning of the present section. This case yields especially simple solutions.
The following theorem provides an expression which allows a computation of the transient distribution at any time $t \in \mathbb{R}^+$ without needing to integrate over ranges larger than the period $T$.

**Theorem 2** *Define*

$$\lfloor t/T \rfloor := \max\{n \in \mathbb{N}_0 : nT \leq t\}$$

*as the number of period lengths that have passed until time $t \in \mathbb{R}^+$. Then the transition probability matrices of $Q$ can be represented as*

$$P(0,t) = (P(0,T))^{\lfloor t/T \rfloor} P(0, t - \lfloor t/T \rfloor \cdot T)$$

*for all $t \in \mathbb{R}^+$.*

**Proof:** The periodicity of the generator function implies

$$\begin{aligned}
P(0,nT) &= P(0,(n-1)T)P((n-1)T,nT) \\
&= P(0,(n-1)T)P(0,T) = P(0,T)^n
\end{aligned}$$

for all $n \in \mathbb{N}$. It further yields

$$P(\lfloor t/T \rfloor \cdot T, t) = P(0, t - \lfloor t/T \rfloor \cdot T)$$

for all $t \in \mathbb{R}^+$.
☺

Combining periodicity and condition (3) leads to the following theorem which defines a class of periodic queues for which the computation of the transient distribution does not require any more effort than for homogeneous queues.

**Theorem 3** *If the above condition (3) holds with a $T$–periodic function $f$, then the transition probability matrices of $Q$ are determined by*

$$P(0,t) = e^{G\cdot\int_0^t f(u)du} = \left(e^{G\cdot F(T)}\right)^{\lfloor t/T \rfloor} e^{G\cdot\int_0^r f(u)du}$$

*with $F(T) := \int_0^T f(u)du$ and $r := t - \lfloor t/T \rfloor \cdot T$.*

The proof follows obviously from the relations collected above.

**Example 1** Assume that $f(t) := \left(1 + \sin\left(\frac{2\pi}{T}t\right)\right)$ for all $t \in I\!R_0^+$. Then the terms $P(0,r) = e^{G \cdot \int_0^r f(u)du}$ can be expressed by

$$\begin{aligned} P(0,r) &= \exp\left(\left(r + \int_0^r \sin\left(\frac{2\pi}{T}u\right) du\right) \cdot G\right) \\ &= \exp\left(\left(r + \frac{T}{2\pi}\left(1 - \cos\left(\frac{2\pi}{T}r\right)\right)\right) \cdot G\right) \end{aligned}$$

for all $r \leq T$. Furthermore, we obtain

$$P(0,T) = e^{G \cdot F(T)} = e^{G \cdot T}$$

which illustrates that computations in the above theorem are not more complex than in the homogeneous case.

# 3 STABILITY AND ASYMPTOTIC DISTRIBUTIONS

The asymptotic analysis of the $BMAP/M_t/c$ queue first requires the derivation of an ergodicity condition, before an asymptotic distribution can be given. After that, the concept of an asymptotic distribution (which is valid for homogeneous queues) is adapted to periodic queues by the definition of a periodic family of asymptotic distributions. This is given explicitly for the case of ergodicity at the end of this section.

Define $Y = (Y_n : n \in I\!N_0)$ as the homogeneous Markov chain with transition probability matrix $P(0,T)$ and let $Y^\mu = (Y_n^\mu : n \in I\!N)$ denote the version of $Y$ with initial distribution $\mu$. Furthermore, denote the phase process of $Q$, which coincides with that of the BMAP, by $J$. Assume in the following that the homogeneous Markov chain $J^d := (J_{nT} : n \in I\!N)$ is irreducible. Since the phase space is finite, the chain $J^d$ has a stationary distribution, which shall be denoted by $\pi = (\pi_1, \ldots, \pi_m)$. Finally, let $1_m$ denote the $m$–dimensional column vector with all entries being 1 and denote the $i$th canonical row base vector of $I\!R^m$ by $e_i$.

**Theorem 4** *The Markov chain $Y$ is ergodic if and only if the stability condition*

$$\int_0^T \pi \sum_{n=1}^\infty n D_n(t) 1_m \, dt < \int_0^T c \cdot \mu_t \, dt \qquad (4)$$

*holds.*

The proof is completely analogous to the proof of theorem 2 in (Breuer, 2001).

**Remark 2** For the case of heterogeneous servers as described in remark 1, the stability condition would assume the form

$$\int_0^T \pi \sum_{n=1}^\infty n D_n(t) 1_m \, dt < \sum_{i=1}^c \int_0^T \mu_t^{(i)} \, dt$$

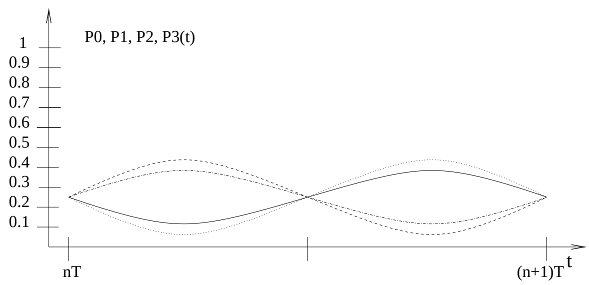
with $\mu_t^{(i)}$ denoting the rate of the $i$the server at time $t$.

Now the concept of an asymptotic distribution shall be adapted to periodic queues. After that, the main theorem of this paper gives an explicit formula for the asymptotic distribution of a stable periodic $BMAP/M_t/c$ queue.

A family $(q_s : s \in [0,T[)$ of probability distributions shall be called a periodic family of asymptotic distributions for $Q$ if

$$q_s = \lim_{n \to \infty} Q_{nT+s}$$

does exist for every $s \in [0,T[$. Here, the limit shall be defined in terms of weak convergence and the distribution of the random variable $Q_{nT+s}$ is denoted by $Q_{nT+s}$ itself. It is trivial to verify that in case a periodic family of asymptotic distributions exists, it will be unique.




Example for a periodic family $p(t) = (p_0(t), p_1(t), p_2(t), p_3(t))$ of asymptotic distributions (for sufficiently large $n$)

**Theorem 5** *Let $Q$ denote a periodic $BMAP/M_t/c$ queue with period $T$. $Q$ has a periodic family of asymptotic distributions if and only if the stability condition (4) holds. In this case, the periodic family of asymptotic distributions is uniquely determined by*

$$q_s = \int dq \; P(0,s)$$

*for all $s \in [0,T[$, with $q$ being the stationary distribution of the homogeneous Markov chain $Y = (Y_n : n \in I\!N_0)$ with transition probability matrix $P(0,T)$.*

**Proof:** Let $\pi$ denote the initial distribution of $Q$ at time $t_0 := 0$ and assume that $(q_s : s \in [0,T[)$ is a periodic family of asymptotic distributions. A necessary property of $q_0$ is

$$\begin{aligned} q_0 &= \lim_{n\to\infty} \int d\pi P(0, nT) \\ &= \left( \lim_{n\to\infty} \int d\pi P(0, (n-1)T) \right) P(0, T) \\ &= \int dq_0 P(0, T) \end{aligned}$$

which means that $q = q_0$ is a stationary distribution of $Y$. Now let $q$ be the asymptotic distribution of $Y$. Then

$$\begin{aligned} q_0 &= \lim_{n\to\infty} \int d\pi P(0, nT) \\ &= \lim_{n\to\infty} \int d\pi P(0, T)^n \\ &= \lim_{n\to\infty} Y_n^\pi = q \end{aligned}$$

does exist.
Using this distribution $q$, the periodic family of asymptotic distributions $(q_s : s \in [0, T])$ is given by

$$\begin{aligned} q_s &= \lim_{n\to\infty} \int d\pi P(0, nT) P(nT, nT + s) \\ &= \left( \lim_{n\to\infty} \int d\pi P(0, nT) \right) P(0, s) \\ &= \int dq P(0, s) \end{aligned}$$

for all $s \in [0, T]$.
☺

As intuitively plausible, this theorem shows that periodic Markovian arrival rates and service capacity yield a periodic asymptotic behaviour of the queue. The stability condition coincides with intuition, too, as it compares the accrued workload and service capacity over one period length. For the special case of $m = 1$ and constant service capacity, the results yield an analysis of the periodic $M_t/M/c$ queue. This extends the results of (Heyman and Whitt, 1984).
In the special case that condition (3) holds, we obtain a simpler stability condition and a particularly simple stationary distribution:

**Theorem 6** *Let $Q$ denote a periodic $BMAP/M_t/c$ queue with parameters satisfying condition (3). Then the stability condition (4) reduces to*

$$\pi \sum_{n=1}^{\infty} n D_n 1_m < c \cdot \mu$$

*In case of stability, the periodic family $(q_s : 0 \le s < T)$ of stationary distributions is constant and equals the stationary distribution $q$ of the homogeneous queue with generator matrix $G$.*

**Proof:** The transformation of the stability condition is obvious. Because of condition (4), there is a stationary distribution $q$ for the imbedded chain $Y$ with transition probability matrix $P(0, T) = e^{G \cdot F(T)}$. This is unique and hence determined by $qG = 0$. Now theorem 3 shows that $q$ is stationary for the process $Q$.
☺

**Example 2** Resuming example 1 and further assuming that $G$ is recurrent and there is a measure $q$ with $qG = 0$ (which implies $qe^{T \cdot G} = q$), we get a periodic family of asymptotic distributions $(q_s : s \in [0, T])$ given by

$$\begin{aligned} q_s &= q \, \exp\left( \left( s + \frac{T}{2\pi} \left( 1 - \cos\left( \frac{2\pi}{T} s \right) \right) \right) \cdot G \right) \\ &= q \end{aligned}$$

for all $s \in [0, T[$.

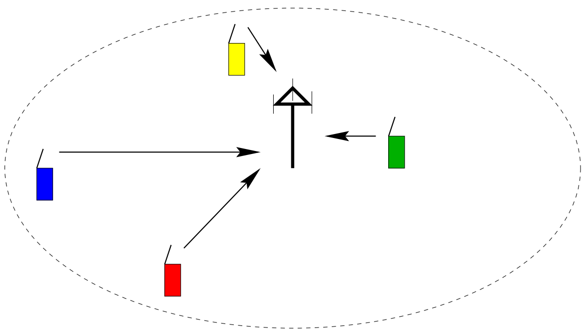
Thus in case of condition (3), the asymptotic behaviour of the periodic queue is the same as for the respective homogeneous queue. As in the statements on the computation of the transient distribution, this shows that the class of periodic queues satisfying condition (3) does not involve higher computational complexity than its homogeneous counterpart.

# 4 THE SPATIAL VARIANT

The technical availability of wireless data transmission opened up the way for mobile communication systems, which today are used by increasingly many subscribers. At the same time, this development exhibited the need for queueing models that can reflect the spatial distribution of the users in the system. These are indispensable in order to model phenomena like spatially dependent service time distributions, user movements or the spatial distribution of users in the system. The analysis of spatial queues yields such essential values as the necessary cell capacities needed to guarantee a low outage probability or the outage probability of an existing network. Such knowledge greatly improves the competence of planning and operating mobile communication networks.
In (Breuer, 2000), a groundwork has been lain for designing queueing models for the application field of mobile communication networks. Its main pillar consists of a spatial generalization of the BMAP concept, called Spatial Markovian Arrival Process (SMAP, see (Breuer, 2000), chapter 3, or (Breuer, 2003), chapter 7). Although this class of arrival processes is much more general than the class of BMAPs, it retains the same crucial properties that make their analysis tractable (see (Breuer, 2000), chapters 4 and 5).
In this section it shall be shown how to apply the simplifications obtained in sections 2 and 3 to a spatial queue of

high utility for the modeling of communication networks. Furthermore, as a new result for this queue, an expression for the loss probability of incoming users will be derived. For a thorough understanding of this section, it might be unavoidable to read the presentation of SMAPs in (Breuer, 2000) or (Breuer, 2003).

The $SMAP/M_t/c/c$ queue is a natural model for a cell of an FDMA or a TDMA mobile communication network. For such a cell, which remains of constant size over time (unlike a CDMA cell), the arrival space $R$ of the queue would be the landscape covered by the cell. The number $c$ of servers would be the number of available channels, i.e. the ratio of the total band width to a single frequency lot assigned to every user (in the FDMA case). The assumption that no user is waiting appears natural, as in mobile communication such a user would prefer to dial into another network rather than wait. Furthermore, usually the arrival process will be periodic with the period length being a day or a week.



Model for a cell of a wireless communication network (with four users and base station in the center)

The $SMAP/M_t/c/c$ queue is a loss system. This means that arriving users who find all servers busy are rejected and since they do not retry to enter the queue, they can be regarded as lost. One of the most important performance measures of loss systems is the loss probability, which is defined as the long time fraction of lost, i.e. rejected users to all users having arrived. For the application field of mobile communcation networks, the meaning of this probability as a criterion of quality is immediate.

In order to derive the simplest possible form of a loss formula (see theorem 9 below), the rejection policy of the communication network to be modelled shall be defined as follows. A rejection shall be conceived to occur only in the event that the system capacity is totally filled, i.e. that there are $c$ users in the system. Thus, if there are $k < c$ users in the system and a batch arrival of size $h$ with $k + h > c$ occurs, then this will not be rejected although it cannot be totally served by the system. In this case, the queue will be filled to its full capacity $c$. This can be interpreted as reflecting the interest of the network provider to partially serve an incoming user instead of rejecting him, even if the total amount of service that the user requires would exceed the system capacity. It is possible to derive transient and asymptotic distributions as well as a loss formula for any other rejection policy using the same methods, but this would result in more complicated formulae.

Let $(N, J)$ denote the SMAP of the queue with arrival space $(R, \mathcal{R})$ and

$$(\Delta(t; S_1, \ldots, S_n) : t \in I\!R_0^+, n \in I\!N, S_1, \ldots, S_n \in \mathcal{R})$$

denoting the generating sequences of $(N, J)$ (see definition 3.4 in (Breuer, 2000)). Further, let $(\mu_t : t \in I\!R_0^+)$ denote the time-dependent service rate, which shall be equal for all servers.

For ease of notation, the queue distribution shall be derived only for one-dimensional marginal processes, i.e. only one subset $S \in \mathcal{R}$ will be described. For higher-dimensional marginal distributions on $(S_1, \ldots, S_n)$ with $n > 1$, the method of analysis is completely analogous.

Denote the joint queueing process in the subsets $R$ and $S$ by $Q = (Q_t : t \in I\!R_0^+)$. For this the generating sequence at time $t$ is given by

$$\Delta(t; R, S) = (D_{k,l}(t; R, S) : k, l \in I\!N_0)$$

with $D_{k,l}(t; R, S)$ signifying the rate matrix that in $]t, t+dt]$ there will be a batch arrival of $k$ users in the whole arrival space $R$, leaving $l$ of these in the subset $S$. Of course, we have $D_{k,l}(t; R, S) = 0$ for $k < l$. Likewise is the meaning of the sequence $\Delta(t; R) = (D_n(t; R) : n \in I\!N_0)$.

The infinitesimal generator $G(t)$ of $Q$ can be written in a block matrix with entries $G_{(h,k),(l,n)}(t) = k\mu_t \cdot I$ for $h = l + 1 \leq c, k = n + 1 \leq l + 1$, $G_{(h,k),(l,n)}(t) = (h - k)\mu_t \cdot I$ for $h = l + 1 \leq c, k = n \leq l$, $G_{(h,k),(l,n)}(t) = D_{0,0}(t; R, S) - h\mu_t \cdot I$ for $h = l < c, k = n \leq l$, $G_{(h,k),(l,n)}(t) = D_{0,0}(t; R, S) - h\mu_t \cdot I + \sum_{i=1}^{\infty} D_i(t; R)$ for $h = l = c, k = n \leq l$, $G_{(h,k),(l,n)}(t) = D_{l-h,n-k}(t; R, S)$ for $h < l < c, k \leq n \leq l$, $G_{(h,k),(l,n)}(t) = \sum_{i=c-h}^{\infty} D_{i,n-k}(t; R, S)$ for $h < l = c, k \leq n \leq l$, and $G_{(h,k),(l,n)}(t) = 0$ otherwise, with 0 and $I$ denoting the zero and identity kernel, respectively.

Here, the first two cases represent service completions within $S$ or outside of $S$, respectively. The next two cases contain the rates for arrivals into the queue, in case 3 leaving some servers idle and in case 4 filling all system capacity. Finally, cases 5 and 6 represent the diagonal elements of the generator, representing phase changes without arrivals and service completions.

Exactly as in the non–spatial case, the solution of Kolmogorov's forward equations via the iteration method leads to

**Theorem 7** *The transition probability matrix $P_{st}^{R,S}$ of the queue process $Q^{R,S}$ from time $s$ to time $t$ can be writ-*

*ten as*

$$P_{st}^{R,S} = \sum_{k=0}^{\infty} \underbrace{\int_s^t \int_s^{u_k} \dots \int_s^{u_2}}_{k\ integrals} G_{u_1}^{R,S} \dots G_{u_k}^{R,S} \, du_1 \dots du_k$$

*defining the summand for $k = 0$ as the identity matrix on the state space $\mathbb{N}_0 \times \mathbb{N}_0 \times \Phi$ of $Q^{R,S}$.*

Furthermore, in order to compute the transition probability matrices the same kind of iteration as given in formula (1) for the non–spatial case can be used here, too.
Knowing the generator $G$, we can verify that it is quasi–commutable if

$$\Delta(t) = f(t) \cdot \Delta \quad \text{and} \quad \mu_t = f(t) \cdot \mu$$

which is completely analogous to condition (3) for the non–spatial queue. If furthermore the function $f$ is periodic with some period length $T > 0$, then all simplifications from sections 2 and 3 hold for the spatial queue, too.

**Example 3** Assume that the parameters are given as follows: Let $\Delta$ denote the characteristic sequence of a homogeneous SMAP and choose service rates $\mu_1, \mu_2 \in \mathbb{R}^+$. Further, let $T > 0$ denote a period length and set

$$\Delta(t) = \left(1 + \sin\left(\frac{2\pi}{T}t\right)\right) \cdot \Delta$$

as well as

$$\mu_t = \begin{cases} \mu_1 & \text{for} \quad t < T/2 \\ \mu_2 & \text{for} \quad t \geq T/2 \end{cases}$$

for all $t \in [0, T[$. Then the generators are periodic with period $T$ and piecewise quasi–commutable (on $[0, T/2[$ and on $[T/2, T[$). Denote $G_1 := G_0^{R,S}$ as well as $G_2 := G_{T/2}^{R,S}$. Acccording to example 1, we then obtain

$$P_{0s} = \exp\left(\left(s + \frac{T}{2\pi}\left(1 - \cos\left(\frac{2\pi}{T}s\right)\right)\right) \cdot G_1\right)$$

for $s < T/2$ and

$$P_{0s} = \exp\left(\left(\frac{T}{2} + \frac{T}{\pi}\right) G_1\right) \times \exp\left(\left(s - \frac{T}{2} - \frac{T}{2\pi}\left(1 + \cos\left(\frac{2\pi}{T}s\right)\right)\right) \cdot G_2\right)$$

for $s \geq T/2$.
This example is typical for the structure of parameters in real communication systems: The rates of the input stream of arriving system requirements usually change periodically in a continuous way. On the other hand, technical adaptations of the system to the changes in input intensity are abrupt (i.e. purely discontinuous) because they usually involve switching on (resp. off) additional service capacity. This reserve capacity is used for the system in times of high traffic, but can be used otherwise in times of lower traffic intensity.

Since the $SMAP/M_t/c/c$ queue as described above has only finitely many states, it will always find an equilibrium, and we do not even need a stability condition. Let $Y^{R,S} = (Y_n^{R,S} : n \in \mathbb{N}_0)$ be the homogeneous discrete time Markov chain with transition kernel $P_{0T}^{R,S}$. Then the complete analogue of the proof for the non–spatial case leads to

**Theorem 8** *The chain $Y^{R,S}$ has an asymptotic distribution $Q^{R,S}$, and the marginal queue process $Q^{R,S}$ has a periodic family $(q_s^{R,S} : s \in [0, T[)$ of asymptotic distributions assuming the form*

$$q_s^{R,S} = q^{R,S} P_{0s}^{R,S}$$

*for all $s \in [0, T[$. The distribution $q^{R,S}$ is determined by the equilibrium equation*

$$q^{R,S} = q^{R,S} P_{0T}^{R,S}$$

The remainder of the section thus will be dedicated to a derivation of the loss probability of $Q$. Denote the number of users that have arrived at $Q$ in subset $S$ until time $t \in \mathbb{R}_0^+$ by $N_t$ and the number of rejected users in subset $S$ until time $t \in \mathbb{R}_0^+$ by $L_t$. Assume that $Q$ has an asymptotic distribution. Then the loss probability $p_l$ of $Q$ is defined as the fraction

$$p_l := \lim_{t \to \infty} \frac{L_t}{N_t}$$

of the number of rejected users over the number of all users having arrived in the long run.
Now let $Q$ denote the periodic $SMAP/M_t/c/c$ queue with arrival space $(R, \mathcal{R})$, finite phase space $\Phi = \{1, \dots, m\}$ and period length $T > 0$. Denote the SMAP arrival process on the subset $S$ by $(N(S), J)$ and define it by its generating sequence $(D_n(S) : l, n \in \mathbb{N}_0)$. Then $D := \sum_{n=0}^{\infty} D_n(S)$ is the generator of the phase process $J$. This has a periodic family $(\pi_s : s \in [0, T[)$ of asymptotic distributions, since the phase space $\Phi$ is finite. Define the column vector $1_m \in \mathbb{R}^m$ as the one with all entries being 1. Finally, assume that the asymptotic mean arrival rate during one period length

$$\lambda(S) := \frac{1}{T} \int_0^T \pi_t \sum_{n=1}^{\infty} n D_n(t; S) 1_m dt < \infty$$

is finite. Then the main result is

**Theorem 9** *Let $Q$ denote a periodic $SMAP/M_t/c/c$ loss system with finite phase space $\Phi = \{1, \dots, m\}$, arrival space $(R, \mathcal{R})$ and period length $T > 0$. Let the SMAP of $Q$ have a generating sequence $(D_n(S) : n \in \mathbb{N}_0)$ on the subset $S \in \mathcal{R}$. Assume that $Q(R)$ has a periodic family $(q_s : s \in [0, T[)$ of asymptotic distributions and define $q_t(c) \in \mathbb{R}^m$ as the row vector with $i$th entry $q_t(c, i)$. Then the loss probability of $Q$ on the subset $S$ is*

$$p_l = \frac{\int_0^T q_t(c) \sum_{n=1}^\infty nD_n(t; S) 1_m dt}{\int_0^T \pi_t \sum_{n=1}^\infty nD_n(t; S) 1_m dt}$$

*for all $S \in \mathcal{R}$.*

**Proof:** For ease of notation, we shall write $N_t := N_t(S)$, $L_t := L_t(S)$ and $D_n(t) := D_n(t; S)$ in this proof. By definition, the arrival process $N(S)$ in $S$ is a one-dimensional Markovian arrival process. Thus, the strong law of large numbers (theorem 3.13 in (Breuer, 2000)) applies and yields

$$\lim_{t \to \infty} \frac{N_t}{t} = \frac{1}{T} \int_0^T \pi_t \sum_{n=1}^\infty nD_n(t) 1_m dt$$

almost surely.
The process $L(S) = (L_t : t \in \mathbb{R}_0^+)$ which counts the number of rejected users can be described by a thinned Markovian arrival process. The infinitesimal transition rates of $L(S)$ are given by

$$R_n(t; i, j) = D_n(t; i, j) \cdot P(Q_t(R) = (c, i) | J_t = i)$$

for every $n \in \mathbb{N}$, since the arrival process $(N(S), J)$ only depends on the phase but not on the number of users in $R$. Furthermore, we have

$$\begin{aligned} R_0(t; i, j) &= D_0(t; i, j) + \\ &+ \sum_{n=1}^\infty D_n(t; i, j) \cdot (1 - P(Q_t(R) = (c, i) | J_t = i)) \end{aligned}$$

Writing $t = kT + s$ and letting $k \to \infty$, the asymptotic rates are

$$\lim_{k \to \infty} R_n(kT + s; i, j) = D_n(s; i, j) \cdot \frac{q_s(c, i)}{\pi_s(i)}$$

for all $s \in [0, T[$, $n \in \mathbb{N}$ and $i, j \in \Phi$ and

$$\begin{aligned} \lim_{k \to \infty} R_0(kT + s; i, j) &= D_0(s; i, j) + \\ &+ \sum_{n=1}^\infty D_n(s; i, j) \cdot \left(1 - \frac{q_s(c, i)}{\pi_s(i)}\right) \end{aligned}$$

for all $s \in [0, T[$ and $i, j \in \Phi$. Obviously, the generator of the phase process of $R(S)$ equals

$$\sum_{n=0}^\infty R_n(t; i, j) = D(t; i, j)$$

which is the generator of $J$. Hence, by theorem 3.13 in (Breuer, 2000), the strong law of large numbers assures the convergence

$$\begin{aligned} \lim_{t \to \infty} \frac{L_t}{t} &= \frac{1}{T} \int_0^T \pi_t \sum_{n=1}^\infty nR_n(t) 1_m dt \\ &= \frac{1}{T} \int_0^T q_t(c) \sum_{n=1}^\infty nD_n(t) 1_m dt \end{aligned}$$

almost surely.
Combining the above results, the loss probability is determined by

$$p_l = \lim_{t \to \infty} \frac{L_t}{t} \cdot \frac{t}{N_t} = \frac{\int_0^T q_t(c) \sum_{n=1}^\infty nD_n(t) 1_m dt}{\int_0^T \pi_t \sum_{n=1}^\infty nD_n(t) 1_m dt}$$

which proves the statement.
☺

As usual, the case of a homogeneous queue can be regarded as the special periodic queue with arbitrary period length $T > 0$ and a generator which is constant in time. Thus, the result for homogeneous queues follows immediately from the above theorem:

**Theorem 10** *Let $Q$ denote a homogeneous SMAP/M/c/c loss system with arrival space $(R, \mathcal{R})$. Let the SMAP of $Q$ have a characteristic sequence $(D_n^S : n \in \mathbb{N}_0)$ on the subset $S \in \mathcal{R}$. Assume that $Q^R$ has an asymptotic distribution $q$ and define $q(c) \in \mathbb{R}^m$ as the row vector with $i$th entry $q(c, i)$. Then the loss probability of $Q$ on the subset $S$ is*

$$p_l = \frac{q(c) \sum_{n=1}^\infty nD_n^S 1_m}{\pi \sum_{n=1}^\infty nD_n^S 1_m}$$

*for all $S \in \mathcal{R}$.*

**Remark 3** The classical Erlang loss formula can be derived from the above formula via the following specifications: Since Erlang's queueing models were homogeneous and non–spatial, we obtain the formula

$$p_l = \frac{q(c) \sum_{n=1}^\infty nD_n 1_m}{\pi \sum_{n=1}^\infty nD_n 1_m}$$

which is an Erlang loss formula for homogeneous queues with BMAP input. Reducing the number of phases to one (i.e. $m := 1$), we obtain $\pi = 1$, and the expressions $\sum_{n=1}^\infty nD_n$ and $q(c)$ are positive numbers. Thus we obtain

$$p_l = \frac{q(c) \sum_{n=1}^\infty nD_n}{\sum_{n=1}^\infty nD_n} = q(c)$$

which is the classical Erlang loss formula, valid also for batch arrivals.

# 5 CONCLUSIONS AND FUTURE RESEARCH

In this paper, formulae for the transient as well as for asymptotic distributions for the periodic $BMAP/M_t/c$ queue with a common period length $T$ for arrival and service rates are given, extending the results in (Heyman and Whitt, 1984) which were obtained for the periodic $M_t/M/c$ queue. The stability condition that has been derived is easy to check, since it is written in terms of arrival and service rates and asymptotic distributions of the phase process. A structural result is that the family of asymptotic distributions of the examined periodic queue is again periodic.

For an important subclass of these queues, the expressions for the distributions can be simplified considerably. These simplifications lead to a complexity of the numerical evaluation which is at the same level as computations of the distributions of homogeneous queues. Thus the paper demonstrates that for a large class of queueing models, the additional modeling power which is due to periodicity of the arrival and service rates does not need to be paid off by higher computation costs.

The method applied in this paper seems to work only for Markovian queues. In order to model phase–type service time distributions, one can apply the same method to the periodic $BMAP/PH/c$ queue, which has been described in (Breuer, 2001). Furthermore, the same method has been applied towards the spatial Markovian $SMAP/M_t/c/c$ queue with a periodic spatial Markovian arrival process (SMAP) and a limited number of users in the queue. Spatial queues are used for modelling mobile communication networks and take the spatial location of their users in account.

Further research and applications to real communication networks need to start from statistical estimates of the parameters for periodic BMAP streams. A first step in this direction has been taken in (Breuer, 2002a) with an estimation procedure for homogeneous BMAPs. However, an extension of this to the class of BMAPs used in the present paper still needs to be developed.

# References

Asmussen, S. and Thorisson, H. (1987). A Markov chain approach to periodic queues. *J. Appl. Probab.*, 24:215–225.

Bambos, N. and Walrand, J. (1989). On queues with periodic inputs. *J. Appl. Probab.*, 26(2):381–389.

Bellman, R. (1997). *Introduction to matrix analysis*. Philadelphia, PA: SIAM.

Breuer, L. (1999). Markovian Spatial Queues with Periodic Arrival and Service Rates. In Baum et al., editor, *Messung, Modellierung und Bewertung von Rechen- und Kommunikationssystemen. Kurzvorträge der 10.GI/ITG - Fachtagung MMB '99*, pages 97–102. Research Report No.99-17, Dept. of Mathematics and Computer Science, University of Trier, Germany.

Breuer, L. (2000). *Spatial Queues*. PhD thesis, University of Trier, Germany.

Breuer, L. (2001). The Periodic BMAP/PH/c Queue. *Queueing Systems*, 38(1):67–76.

Breuer, L. (2002a). An EM Algorithm for Batch Markovian Arrival Processes and its Comparison to a Simpler Estimation Procedure. *Annals of Operations Research*, 112:123–138.

Breuer, L. (2002b). On Markov–Additive Jump Processes. *Queueing Systems*, 40(1):75–91.

Breuer, L. (2002c). Simple Solutions for a Class of Periodic Queues. In Amborski, K. and Meuth, H., editors, *Modelling and Simulation 2002*, pages 505–512, Ghent, Belgium. SCS Europe.

Breuer, L. (2003). *From Markov Jump Processes to Spatial Queues*. Kluwer, Dordrecht (Netherlands).

Falin, G. (1989). Periodic queues in heavy traffic. *Adv. Appl. Probab.*, 21(2):485–487.

Gikhman, I. and Skorokhod, A. (1969). *Introduction to the theory of random processes*. Saunders.

Harrison, J. and Lemoine, A. J. (1977). Limit theorems for periodic queues. *J. Appl. Probab.*, 14:566–576.

Hasofer, A. (1964). On the single-server queue with non-homogeneous Poisson input and general service time. *J. Appl. Probab.*, 1:369–384.

Heyman, D. P. and Whitt, W. (1984). The asymptotic behavior of queues with time-varying arrival rates. *J. Appl. Probab.*, 21:143–156.

Lemoine, A. J. (1981). On queues with periodic Poisson input. *J. Appl. Probab.*, 18:889–900.

Lemoine, A. J. (1989). Waiting time and workload in queues with periodic Poisson input. *J. Appl. Probab.*, 26(2):390–397.

Lucantoni, D. M. (1991). New results on the single server queue with a batch Markovian arrival process. *Commun. Stat., Stochastic Models*, 7(1):1–46.

Massey, W. (1981). *Non–Stationary Queues*. PhD thesis, Stanford University.

Neuts, M. F. (1979). A versatile Markovian point process. *J. Appl. Probab.*, 16:764–774.

Rolski, T. (1987). Approximation of periodic queues. *Adv. Appl. Probab.*, 19:691–707.

Rolski, T. (1989). Queues with nonstationary inputs. *Queueing Syst.*, 5(1-3):113–129.

Willie, H. (1998). Periodic steady state of loss systems with periodic inputs. *Adv. Appl. Probab.*, 30(1):152–166.

# AUTHOR BIOGRAPHY

**LOTHAR BREUER** received his Ph.D. at the University of Trier, Germany, in 2000. He works in the fields of queueing theory and stochastic processes. He is employed as an assistant professor at the University of Trier. Currently, he is further affiliated with the Université Libre de Bruxelles, Belgium, via the "Prix Ouverture Internationale".