A generalised Gerber-Shiu function for Markov-additive risk processes

Lothar Breuer
University of Kent, UK

October 29, 2009

Abstract

We consider the following risk reserve process. Claims arrive according to a Markovian point process. Claim sizes are phase-type and the model allows for possible correlations among inter-claim times and claim sizes. The premium income between claim arrivals is modelled by Markov-modulated Brownian motion which may depend on the underlying phases of the claim arrival process. For this risk reserve model we derive the joint distribution of the time to ruin, the surplus immediately before ruin, the deficit at ruin, the minimal risk reserve before ruin, and the time until this minimum is attained. The analysis is based on first passage times for Markov-additive processes and their time reversals.

1 Introduction

In 1997, Gerber and Shiu published a seminal paper [17] in which they derived the joint distribution of the time to ruin, the surplus immediately before ruin, and the deficit at ruin. Their analysis covered the classical Poisson risk model

$$R_t = u + ct - \sum_{j=1}^{N_t} X_j$$

where $u \geq 0$ is the initial surplus, $c > 0$ is the rate of premium income, $(N_t : t \geq 0)$ is a Poisson process, and X_j, $j \in \mathbb{N}$, are iid positive random variables modelling the claim sizes. In this notation the time of ruin is given by

$$T = \inf\{t \geq 0 : R_t < 0\}$$

while the surplus immediately before ruin and the deficit at ruin are R_{T-} and $|R_T|$, respectively.

Given a discount rate $\delta \geq 0$ and a non-negative function $w(x,y)$ on $x,y \geq 0$, Gerber and Shiu [18] investigated the function

$$\phi(u) = \mathbb{E}(w(R_{T-}, |R_T|) \cdot e^{-\delta T} \cdot \mathbb{I}_{\{T<\infty\}}|R_0 = u)$$

where $\mathbb{I}_A$ denotes the indicator function of some set A. This function ϕ has found much atttention since then and was given the name Gerber-Shiu (GS) function or discounted penalty function. Many authors have contributed to its analysis, where the underlying risk reserve process has been generalised in several directions. The perturbed compound Poisson model has been considered in [16, 13, 24], while Markov-modulated (or regime switching) versions are analysed in [4, 27]. There are further related papers on the GS function for the Lévy risk process [15], the fluid flow model [10, 1], the Sparre Andersen model with Erlang inter-claim times [19] and its perturbed version [22]. One does not need to add more references to show that the GS function enjoys great popularity among the reserach community. The almost universal approach of analysis is the derivation of some (defective) renewal equations, coming from a set of integro-differential equations which are obtained via Itô's formula or the infinitesimal generator of the risk reserve process (see discussion to [23]).

The present paper deals with a generalisation of the GS function for Markov-additive risk processes. These combine the features of perturbation and Markov-modulation and render the aforementioned risk processes as special cases. The only restriction required for the analysis in this paper is that claim sizes have a phase-type distribution.

Rather than employing the mainstream approach of defective renewal equations, we shall generalise an idea presented in [14], where the joint distribution of the space-time positions of overshoots and undershoots has been derived for Lévy processes (theorem 3 therein, with example 8 dedicated to insurance risk). Their approach of analysis is more along the classical lines of fluctuation theory, using the ladder height process and time–reversal. The present paper aims to apply the classical approach of ladder heights and time–reversal to the class of Markov–additive risk processes with phase–type claims.

Of fundamental use in this paper will be the recent determination of the Laplace transform of first passage times for MAPs as given in [12]. A second pillar of the present work is theorem 2.5 in [4] (see also [2], theorem 3.1, for a queueing context), which yields a relation between an occupation measure and the ladder height via time–reversal. The original result was presented in the framework of the Markov–modulated compound Poisson model.

In the following section we shall present the model of Markov-additive risk processes and establish the relation to first passage times for Markov-additive processes (MAPs). In section 3 we collect all the necessary preliminary results for MAPs that we will need later on. In particular we simplify the results from [12] for the special kind of MAPs that we employ in this paper. Section 4 contains the main result with some corollaries. The final section compares the present result with existing ones in the literature.

2 The model

Consider a risk reserve process with initial capital $u \geq 0$ and claims occurring like a Markovian point process (MPP).[1] It is shown in [8] that the class of MPPs is dense within the class of marked point processes. Denote the claim arrival process by $(\mathcal{N}, \tilde{\mathcal{J}}) = ((N_t, \tilde{J}_t) : t \geq 0)$ and the phase space for $\tilde{\mathcal{J}}$ by $\tilde{E}$. Assume that the claim sizes have a phase–type distribution (with parameters that may depend on $\tilde{J}_t$ and $\tilde{J}_{t-}$ for a claim at time t). Denote the nth claim size by C_n, $n \in \mathbb{N}$. In [29] it is shown that the class of phase–type distributions is dense within the class of all distributions on the positive real numbers. We assume further that the premium income between claims can be modelled by a Brownian motion, where the parameters $\tilde{\mu}_i$ (drift) and $\tilde{\sigma}_i$ (variation) at time t may depend on the current phase $\tilde{J}_t = i$ of the claim arrival process. For insurance risk we typically have $\tilde{\mu}_i > 0$ for all $i \in \tilde{E}$. We shall allow $\tilde{\sigma}_i = 0$ for some (or possibly all) phases, under which condition the Brownian motion becomes a linear drift. However, the case $\tilde{\mu}_i = \tilde{\sigma}_i = 0$ of a constant (null) movement shall be excluded. Then the process of premium income is a Markov–modulated Brownian motion which we denote by $(\mathcal{B}, \tilde{\mathcal{J}}) = ((B_t, \tilde{J}_t) : t \geq 0)$. We assume that $B_0 = 0$.

Note that $\tilde{\mathcal{J}}$ here is the same as for the claim arrival process $(\mathcal{N}, \tilde{\mathcal{J}})$. This is no restriction in modelling power as we can choose identical parameters $(\tilde{\mu}_i, \tilde{\sigma}_i) = (\tilde{\mu}_j, \tilde{\sigma}_j)$ for different phases $i \neq j \in \tilde{E}$ and map two different environments for the premium income and the claim arrivals by using Kronecker products. Rather on the contrary, a common phase space enables us to model correlations between claim arrivals, claim sizes, and the premium income.

With the definitions above, the risk reserve process $\mathcal{R} = (R_t : t \geq 0)$ is given by

$$R_t = u + B_t - \sum_{n=1}^{N_t} C_n$$

for $t \geq 0$. Denote the net claim process by $\tilde{\mathcal{X}} = (\tilde{X}_t : t \geq 0)$ where $\tilde{X}_t := u - R_t$ for all $t \geq 0$. Then the process $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ is a MAP with phase–type jumps. Clearly, the time of

[1] see [26, 25]. This has traditionally been called Markovian arrival process and abbreviated as MAP. Since we use the shortcut MAP for the more general class of Markov–additive processes already, we prefer to use the term Markovian point process and the abbreviation MPP instead.

ruin is given by

$$T = \inf\{t \geq 0 : R_t < 0\} = \inf\{t \geq 0 : \tilde{X}_t > u\} =: \tilde{\tau}(u)$$

where $\tilde{\tau}(u)$ is the first passage time of the level u by the net claim process $\tilde{\mathcal{X}}$. The surplus immediately before ruin and the deficit at ruin are given by

$$R_{T-} = u - \tilde{X}_{\tilde{\tau}(u)-} \qquad \text{and} \qquad |R_T| = \tilde{X}_{\tilde{\tau}(u)} - u$$

respectively. Thus it suffices to look at undershoots and overshoots for MAPs in order to determine the GS function. In the present paper we shall go a step further and derive additionally the minimal surplus before ruin, i.e.

$$\min\{R_t : t < T\} = u - \max\{\tilde{X}_t : t < \tilde{\tau}(u)\} =: u - \tilde{M}_{\tilde{\tau}(u)}$$

and the time until this minimum is reached, i.e.

$$\tilde{G}_{\tilde{\tau}(u)} := \min\{t \geq 0 : \tilde{X}_t = \tilde{M}_{\tilde{\tau}(u)}\} = \max\{t \geq 0 : \tilde{X}_t = \tilde{M}_{\tilde{\tau}(u)}\} \tag{1}$$

The last equality holds because we have excluded constant (null) movements and allowed only phase-type claim sizes. The main result of this paper is an explicit formula for the function

$$\mathbb{E}\left(e^{-\gamma \tilde{G}_{\tilde{\tau}(u)} - \gamma^* \cdot (\tilde{\tau}(u) - \tilde{G}_{\tilde{\tau}(u)})}; \tilde{M}_T \in dz, R_{T-} \in dx, |R_T| \in dy\right)$$

where $\gamma, \gamma^* \geq 0$ are time discounting factors. This result provides all the information that is usually contained in the GS function. In addition, it yields the distributions of $\tilde{G}_{\tilde{\tau}(u)}$ and $\tilde{M}_{\tilde{\tau}(u)}$.

Looking at the problem from the angle described above, we first need to collect some necessary preliminary results for MAPs from existing literature. This shall be the purpose of the next section.

3 Preliminaries

3.1 Markov–additive processes with phase–type jumps

Let $\tilde{\mathcal{J}} = (\tilde{J}_t : t \geq 0)$ be an irreducible Markov (jump) process with finite state space $\tilde{E}$ and infinitesimal generator matrix $\tilde{Q} = (\tilde{q}_{ij})_{i,j \in \tilde{E}}$. We call $\tilde{J}_t$ the phase at time $t \geq 0$ (another common name is regime). Define the real–valued process $\tilde{\mathcal{X}} = (\tilde{X}_t : t \geq 0)$ as evolving like a Lévy process $\tilde{\mathcal{X}}^{(i)}$ with parameters $\tilde{\mu}_i$ (drift), $\tilde{\sigma}_i^2$ (variation), and $\tilde{\nu}_i$ (Lévy measure) during intervals when the phase equals $i \in \tilde{E}$. Whenever $\tilde{\mathcal{J}}$ jumps from a state

$i \in \tilde{E}$ to another state $j \in \tilde{E}$, $j \neq i$, this may be accompanied by a jump of $\tilde{\mathcal{X}}$ with some distribution function $\tilde{F}_{ij}$. Then the two–dimensional process $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ is called a Markov–additive process (or shortly MAP). A MAP can also be defined by the following property (see [5], section XI.2a): $(\mathcal{X}, \mathcal{J})$ is a Markov process such that

$$\mathbb{E}(f(X_{t+s} - X_t)g(J_{t+s})|\mathcal{F}_t, J_t = i) = \mathbb{E}(f(X_s)g(J_s)|X_0 = 0, J_0 = i)$$

holds for all $s, t > 0$ and $i \in E$, where f and g are measurable functions and $(\mathcal{F}_t : t \geq 0)$ denotes the canonical filtration of $(\mathcal{X}, \mathcal{J})$. For a textbook introduction to MAPs see [5], chapter XI.

Denote the indicator function of a set A by 1_A. We assume that the Lévy measures $\tilde{\nu}_i$ have the form

$$\tilde{\nu}_i(dx) = \lambda_i^+ 1_{x>0}\, \alpha^{(ii)+} \exp(T^{(ii)+}x)\eta^{(ii)+} dx \\ + \lambda_i^- 1_{x<0}\, \alpha^{(ii)-} \exp(-T^{(ii)-}x)\eta^{(ii)-} dx \quad (2)$$

for all $i \in \tilde{E}$, where $\lambda_i^\pm \geq 0$ and $(\alpha^{(ii)\pm}, T^{(ii)\pm})$ are representations of phase–type distributions without an atom at 0. The $\eta^{(ii)\pm} := -T^{(ii)\pm}\mathbf{1}$ are called the exit vectors, where $\mathbf{1}$ denotes a column vector of appropriate dimension with all entries being 1. This means that the jump process induced by the Lévy measure ν_i is compound Poisson with jump sizes of a doubly phase–type distribution. Denote the order of $PH(\alpha^{(ii)\pm}, T^{(ii)\pm})$ by $m_{ii}^\pm$. Further write $\lambda_i := \lambda_i^+ + \lambda_i^-$.

Likewise, let p_{ij}^+ (resp. p_{ij}^-) denote the probability that a positive (resp. negative) jump is induced by a phase change from $i \in \tilde{E}$ to $j \in \tilde{E}$, and assume that these jumps have a $PH(\alpha^{(ij)\pm}, T^{(ij)\pm})$ distribution without an atom at 0. Note that $p_{ij}^+ + p_{ij}^- \leq 1$ for all $i, j \in \tilde{E}$. Let $m_{ij}^\pm$ denote the order of $PH(\alpha^{(ij)\pm}, T^{(ij)\pm})$ and define $\eta^{(ij)\pm} := -T^{(ij)\pm}\mathbf{1}$.

We shall exclude the case of $\tilde{\mu}_i = \tilde{\sigma}_i^2 = 0$ for any phase $i \in \tilde{E}$, which would govern the zero process or a pure Lévy measure. This avoids the awkward case of a non-unique time of attaining the minimal risk reserve $\tilde{G}_{\tilde{\tau}(u)}$ before ruin, see definition (1).

The class of Markov–additive processes with these assumptions of phase–type jumps is dense within the class of all MAPs, see [6], proposition 1. The main advantage of the restriction on the jump distributions is the possibility of transforming the jumps into a succession of linear pieces of exponential duration (each with slope 1 or -1) and retrieving the original process via a simple time change, see [7], section 8, or [9].

This is done in the following way. Without the jumps, the Lévy process $\tilde{\mathcal{X}}^{(i)}$ during a phase $i \in \tilde{E}$ is either a linear drift (of positive or negative slope $\tilde{\mu}_i \in \mathbb{R}$) or a Brownian motion (with parameters $\tilde{\sigma}_i > 0$ and $\tilde{\mu}_i \in \mathbb{R}$). Considering this MAP (without the jumps) we can partition its phase space $\tilde{E}$ into the subspaces E_p (for positive drifts), E_σ (for

Brownian motions), and E_n (for negative drifts). We thus define

$$E_p := \{i \in \tilde{E} : \tilde{\mu}_i > 0, \tilde{\sigma}_i = 0\}, E_n := \{i \in \tilde{E} : \tilde{\mu}_i < 0, \tilde{\sigma}_i = 0\}, E_\sigma := \{i \in \tilde{E} : \tilde{\sigma}_i > 0\} \tag{3}$$

Note that $\tilde{E} = E_p \cup E_\sigma \cup E_n$, since we have excluded the case of $\tilde{\mu}_i = \tilde{\sigma}_i^2 = 0$ for any phase $i \in \tilde{E}$. Then we introduce two new phase spaces

$$E_\pm := \{(i, j, k, \pm) : i, j \in E_p \cup E_\sigma \cup E_n, 1 \le k \le m_{ij}^\pm\} \tag{4}$$

to model the jumps. Define now the enlarged phase space

$$E = E_+ \cup \tilde{E} \cup E_- = E_+ \cup E_p \cup E_\sigma \cup E_n \cup E_-$$

We define the modified MAP $(\mathcal{X}, \mathcal{J})$ over the enlarged phase space E as follows. Set the parameters $(\mu_i, \sigma_i^2, \nu_i)$ for $i \in E$ as

$$(\mu_i, \sigma_i^2, \nu_i) := \begin{cases} (\pm 1, 0, \mathbf{0}), & i \in E_\pm \\ (\tilde{\mu}_i, \tilde{\sigma}_i, \mathbf{0}), & i \in \tilde{E} = E_p \cup E_\sigma \cup E_n \end{cases} \tag{5}$$

This leads to the cumulant functions

$$\psi_i(\alpha) = \begin{cases} \pm\alpha, & i \in E_\pm \\ \mu_i \alpha, & i \in E_p \cup E_n \\ \frac{1}{2}\sigma_i^2 \alpha^2 + \mu_i \alpha, & i \in E_\sigma \end{cases} \tag{6}$$

where the above exclusion of a phase $i \in \tilde{E}$ with $\tilde{\mu}_i = \tilde{\sigma}_i = 0$ yields $\mu_i > 0$ for $i \in E_p$ and $\mu_i < 0$ for $i \in E_n$. This excludes of course the possibility of a phase $i \in E$ with parameters $\mu_i = \sigma_i = 0$ which would govern the zero process. Since jumps are modelled by additional phases, this excludes further any phase $i \in \tilde{E}$ under which $\tilde{\mathcal{X}}^{(i)}$ would be a compound Poisson process.

We shall order the new phase space $E = E_+ \cup E_p \cup E_\sigma \cup E_n \cup E_-$ such that $i_+ < i_p < i_\sigma < i_n < i_-$ for phases $i_* \in E_*$. Let $E_c := E_p \cup E_\sigma \cup E_n = \tilde{E}$ denote the subspace of E that contains all phases under which the real time movements are continuous. The modified phase process $\mathcal{J}$ is determined by its generator matrix $Q = (q_{ij})_{i,j \in E}$. For this the construction above yields

$$q_{ih} = \begin{cases} \tilde{q}_{ii} - \lambda_i, & h = i \in E_c \\ \tilde{q}_{ih} \cdot (1 - p_{ih}^+ - p_{ih}^-), & h \in E_c, h \neq i \\ \lambda_i^\pm \alpha_k^{(ii)\pm}, & h = (i, i, k, \pm) \\ \tilde{q}_{ij} \cdot p_{ij}^\pm \cdot \alpha_k^{(ij)\pm}, & h = (i, j, k, \pm) \end{cases} \tag{7}$$

for $i \in E_c$ as well as

$$q_{(i,j,k,\pm),(i,j,l,\pm)} = T_{kl}^{(ij)\pm} \qquad \text{and} \qquad q_{(i,j,k,\pm),j} = \eta_k^{(ij)\pm} \tag{8}$$

for $i, j \in E_c$ and $1 \le k, l \le m_{ij}^{\pm}$. For later use we define $q_i := -q_{ii}$ for all $i \in E$.

We denote the MAP constructed in (5), (7), and (8) by $(\mathcal{X}, \mathcal{J})$. The original level process $\tilde{\mathcal{X}}$ is retrieved via the time change

$$c(t) := \int_0^t 1_{J_s \in E_c} \, ds \quad \text{and} \quad \tilde{X}_{c(t)} = X_t \tag{9}$$

for all $t \ge 0$. The inverses of the cumulant functions ψ_i can be given explicitly as

$$\phi_i(\beta) = \begin{cases} \pm\beta, & i \in E_\pm \\ \frac{\beta}{\mu_i}, & i \in E_p \cup E_n \\ \frac{1}{\sigma_i}\sqrt{2\beta + \frac{\mu_i^2}{\sigma_i^2}} - \frac{\mu_i}{\sigma_i^2}, & i \in E_\sigma \end{cases} \tag{10}$$

We shall need them only for the so–called ascending phases $i \in E_a := E_+ \cup E_p \cup E_\sigma$, in the same way as in [11], chapter VII.

Example 1 We consider the classical compound Poisson model. Inter–claim times and claim sizes are iid exponential with parameter $\lambda > 0$ and $\beta > 0$, respectively. The rate of premium income is $c > 0$. The net profit condition is then $\lambda/(c\beta) < 1$. This model has been examined in [18]. The net claim amount at time $t \ge 0$ is given by

$$\tilde{X}_t = \sum_{n=0}^{N_t} C_n - ct \tag{11}$$

where $(N_t : t \ge 0)$ is a Poisson process with intensity λ and the C_n, $n \in \mathbb{N}$, are iid random variables with exponential distribution of parameter β.

The net claim process $\tilde{\mathcal{X}} = (\tilde{X}_t : t \ge 0)$ can be analysed as a MAP with exponential (and hence phase–type) positive jumps with parameter β. For this, we would need only one phase, i.e. $|\tilde{E}| = 1$. This phase governs a Lévy process with parameters $\tilde{\sigma} = 0$, $\tilde{\mu} = -c$, and $\tilde{\nu}(dx) = \lambda e^{-\beta x}\beta dx$ for all $x > 0$.

We obtain the modified MAP $(\mathcal{X}, \mathcal{J})$ as follows. In equation (2) we have $\lambda_2^+ = \lambda$ and $\lambda_2^- = 0$. Further $m_{22}^+ = 1$ since the positive jumps have an exponential distribution. Hence the enlarged phase space is given by $E = \{1, 2\}$, where $E_+ = \{1\}$, $E_n = \{2\}$, and $E_\sigma = E_p = E_- = \emptyset$. Phase 1 was denoted as $(2, 2, 1, +)$ in definition (4). The parameters are given by $\sigma_1 = \sigma_2 = 0$, $\mu_1 = 1$, $\mu_2 = -c$, $\nu_1 = \nu_2 = \mathbf{0}$, according to (5). The generator matrix for the phase process $\mathcal{J}$ is given as

$$Q = \begin{pmatrix} -\beta & \beta \\ \lambda & -\lambda \end{pmatrix}$$

according to (7) and (8).

Example 2 The joint density function of undershoot and overshoot has been derived in [10] for the fluid flow case from an insurance perspective. The fluid queue $\{(L_t, J_t) : t \geq 0\}$ as defined on p. 434 therein is a MAP with phase space $S = S_1 \cup S_2$ and parameters

$$(\sigma_i, \mu_i, \nu_i) = \begin{cases} (0, 1, \mathbf{0}), & i \in S_1 \\ (0, -1, \mathbf{0}), & i \in S_2 \end{cases}$$

as well as generator matrix

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

Phases in S_1 are considered as premium income phases, while phases in S_2 pertain to claims. Not counting the time during claim phases (in S_2) and setting $L_0 := u$, which shall denote the initial risk reserve, $\{(L_t, J_t) : t \geq 0\}$ uniquely defines a risk reserve process $U_{c'(t)} := L_t$, $t \geq 0$, via the time change $c'(t) := \int_0^t 1_{J_s \in S_1} ds$. Thus the approach in [10] is very similar to the present paper, only the parameters are more restricted (they do not allow perturbations by diffusion).

It is therefore quite simple to compare results between [10] and the present paper. The net claim amount at time $t \geq 0$ is $\tilde{X}_t := u - U_t$. For the modified MAP $(\mathcal{X}, \mathcal{J})$, we obtain $E_p = E_\sigma = E_- = \emptyset$ and $E_+ = S_2$, $E_n = S_1$. Comparing the notations for the generator matrix of the phase process $\mathcal{J}$, we get the block partition

$$Q = \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix} = \begin{pmatrix} T_{22} & T_{21} \\ T_{12} & T_{11} \end{pmatrix}$$

The assumption $c = 1$ therein translates to $\mu_i = -1$ for all $i \in E_n$ in our notation.

3.2 First passage times

Of central use in the present paper will be the recent derivation of the Laplace transforms for the first passage times of MAPs with phase–type jumps as given in [12]. We call the phases $i \in E_d := E_n \cup E_-$ descending. Define $\tilde{\tau}(x) := \inf\{t \geq 0 : \tilde{X}_t > x\}$ for all

$x \geq 0$ and assume that $X_0 = 0$. Note that this is the first passage time over the level x for the original MAP $\tilde{\mathcal{X}}$, meaning that we do not count the time spent in jump phases $i \in E_\pm$. This means that $\tilde{\tau}(x) = c(\tau(x)) = \int_0^{\tau(x)} 1_{J_s \in E_c} ds$, according to (9). In particular, we may compute expectations over $\tilde{\tau}(x)$ using the distribution of the modified MAP $(\mathcal{X}, \mathcal{J})$ only and without needing to recur to the original MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$. For $\gamma \geq 0$ denote

$$\mathbb{E}_{ij}(e^{-\gamma\tilde{\tau}(x)}) := \mathbb{E}(e^{-\gamma\tilde{\tau}(x)}; J_{\tau(x)} = j | J_0 = i, X_0 = 0)$$

for all $i, j \in E$. Note that the phases i, j may be taken from the enlarged phase space E, thus we include phases $i, j \in E_+ \cup E_-$ that model the jumps. Let $\mathbb{E}(e^{-\gamma\tilde{\tau}(x)})$ denote the matrix with these entries and write

$$\mathbb{E}(e^{-\gamma\tilde{\tau}(x)}) = \begin{pmatrix} \mathbb{E}_{(a,a)}(e^{-\gamma\tilde{\tau}(x)}) & \mathbb{E}_{(a,d)}(e^{-\gamma\tilde{\tau}(x)}) \\ \mathbb{E}_{(d,a)}(e^{-\gamma\tilde{\tau}(x)}) & \mathbb{E}_{(d,d)}(e^{-\gamma\tilde{\tau}(x)}) \end{pmatrix}$$

in obvious block notation with respect to the subspaces $E_a = E_+ \cup E_p \cup E_\sigma$ (ascending phases) and $E_d = E_n \cup E_-$ (descending phases).

Since a first passage to a level above cannot occur in a descending phase, we obtain first $\mathbb{P}(J_{\tau(x)} = j) = 0$ for all $j \in E_d$ and thus $\mathbb{E}_{(d,d)}(e^{-\gamma\tilde{\tau}(x)}) = \mathbb{E}_{(a,d)}(e^{-\gamma\tilde{\tau}(x)}) = \mathbf{0}$ where $\mathbf{0}$ denotes a zero matrix of suitable dimension. The exponential form

$$e^{-\gamma\tilde{\tau}(x)} = e^{-\gamma \int_0^{\tau(x)} 1_{J_s \in E_c} ds}$$

as well as path continuity of $\mathcal{X}$ and spatial homogeneity of $(\mathcal{X}, \mathcal{J})$ lead to the functional equation

$$\mathbb{E}_{ij}(x + y) = \sum_{k \in E_a} \mathbb{E}_{ik}(x)\mathbb{E}_{kj}(y)$$

for all $i \in E$ and $j \in E_a$. Hence we obtain

$$\mathbb{E}_{(d,a)}(e^{-\gamma\tilde{\tau}(x)}) = A(\gamma)e^{U(\gamma)x} \quad \text{and} \quad \mathbb{E}_{(a,a)}(e^{-\gamma\tilde{\tau}(x)}) = e^{U(\gamma)x} \tag{12}$$

for some sub–generator matrix $U(\gamma)$ of dimension $E_a \times E_a$ and a sub–transition matrix $A(\gamma)$ of dimension $E_d \times E_a$, cf. equation (6) in [12].

Remark 1 The matrix-exponential form in (12) holds for more general functionals, too. For instance, one might consider an E-dimensional vector $\mathbf{r} = (r_i : i \in E)$ with non-negative entries $r_i \geq 0$ for all $i \in E$. Define

$$\mathbb{E}_{ij}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right) := \mathbb{E}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}; J_{\tau(x)} = j | J_0 = i, X_0 = 0\right)$$

for $i,j \in E$ and $\mathbb{E}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right)$ as the $E \times E$-matrix with these entries. The same arguments as above would then yield $\mathbb{E}_{(d,d)}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right) = \mathbb{E}_{(a,d)}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right) = \mathbf{0}$ as well as

$$\mathbb{E}_{(d,a)}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right) = A(\mathbf{r})e^{U(\mathbf{r})x} \quad \text{and} \quad \mathbb{E}_{(a,a)}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right) = e^{U(\mathbf{r})x}$$

where the matrices $A(\mathbf{r})$ and $U(\mathbf{r})$ can be computed by similar formulas as in [12], theorem 3. In the present paper we are merely interested in the case

$$r_i = \begin{cases} \gamma, & i \in E_c \\ 0, & i \in E_+ \cup E_- \end{cases}$$

and therefore choose not to pursue this slightly more general point of view.

One additional remark, however, might be interesting. Let us add an absorbing phase, say Δ, to the phase space E to obtain $E' = E \cup \{\Delta\}$. Define a MAP $(\mathcal{X}', \mathcal{J}')$ on E' as follows. The generator matrix Q' of $\mathcal{J}'$ shall be given by

$$q'_{ij} := \begin{cases} q_{ij}, & i,j \in E, j \neq i \\ q_{ii} - r_i, & j = i \in E \\ r_i, & i \in E, j = \Delta \\ 0, & i = \Delta, j \in E' \end{cases}$$

Further let

$$(\mu'_i, \sigma'_i, \nu'_i) := \begin{cases} (\mu_i, \sigma_i, \nu_i), & i \in E \\ (0, 0, \mathbf{0}), & i = \Delta \end{cases}$$

which means that the phase Δ governs the zero process. Let $\tau_\Delta := \min\{t \geq 0 : J'_t = \Delta\}$ denote the time until absorption in Δ and $\tau'(x) := \inf\{t \geq 0 : X'_t > x\}$ the first passage time of $\mathcal{X}'$ over the level $x \geq 0$. Then

$$\mathbb{E}_{ij}\left(e^{-\int_0^{\tau(x)} r_{J_s} ds}\right) = \mathbb{P}\left(\tau'(x) < \tau_\Delta, J'_{\tau'(x)} = j | J'_0 = i, X'_0 = 0\right)$$

for $i,j \in E' \setminus \{\Delta\}$, i.e. the generalised Laplace transforms of the first passage times $\tau(x)$ can be seen as transition probabilities among the transient phases $i,j \in E' \setminus \{\Delta\}$ for the phase process $\mathcal{J}'$ which is killed by a constant rate r_i during $\{t \geq 0 : J'_t = i\}$. This is the point of view taken in [3], albeit that $\mathbf{r} = \mathbf{0}$ is trivial therein.

Remark 2 The remainder of this subsection concerns the computation of the matrices $U(\gamma)$ and $A(\gamma)$ appearing in (12). We shall briefly specify an iteration, previously derived in [12], to the particular case that is relevant for the present paper. It should be noted, however, that in special cases the Laplace transforms in (12) can be expressed in terms of linear combinations of exponential functions $e^{\gamma_k x}$, where the (complex) numbers γ_k are obtained as solutions of a Cramér-Lundberg equation. This is described in [21, 20].

Write $\Delta_q := diag(q_i)_{i\in E}$ where $q_i := -q_{ii}$ for all $i \in E$ and let $P = \Delta_q^{-1}Q + I$ denote the transition matrix of phase changes. Note that $p_{ii} = 0$ for all $i \in E$. Further let e_i' denote the ith canonical row base vector, according to context either on E, on E_a, or on E_d. According to theorem 3 in [12], $A(\gamma)$ and $U(\gamma)$ satisfy the following equations:

$$e_h' U(\gamma) = \sum_{l=1}^{m_{ij}^+} T_{kl}^{(ij)+} e_{(i,j,l,+)}' + \eta_k^{(ij)+} e_j' \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} \qquad \text{for } h = (i,j,k,+) \in E_+,$$

$$e_i' U(\gamma) = -\phi_i(q_i+\gamma) e_i' + \phi_i(q_i) \sum_{j\in E} p_{ij}\, e_j' \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} L_i(-U(\gamma)) \quad \text{for } i \in E_p \cup E_\sigma,$$

$$e_i' A(\gamma) = \sum_{j\in E, j\neq i} q_{ij} e_j' \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} \left((q_i+\gamma)I - \psi_i(-U(\gamma))\right)^{-1} \qquad \text{for } i \in E_n,$$

$$e_i' A(\gamma) = \sum_{j\in E, j\neq i} q_{ij} e_j' \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} \left(q_i I - \psi_i(-U(\gamma))\right)^{-1} \qquad \text{for } i \in E_-.$$

For the MAP $(\mathcal{X}, \mathcal{J})$ with continuous level process, the matrix function

$$L_i(-U(\gamma)) = \frac{q_i}{\phi_i(q_i)} \cdot \left(\phi_i(q_i+\gamma)I + U(\gamma)\right) \cdot \left((q_i+\gamma)I - \psi_i(-U(\gamma))\right)^{-1}$$

can be simplified considerably. For $i \in E_\sigma$, the same arguments as in [12], example 2, lead to

$$L_i(-U(\gamma)) = \phi_i^*(q_i) \cdot \left(\phi_i^*(q_i+\gamma)I - U(\gamma)\right)^{-1} \tag{13}$$

with

$$\phi_i^*(\beta) = \frac{1}{\sigma_i}\sqrt{2\beta + \frac{\mu_i^2}{\sigma_i^2}} + \frac{\mu_i}{\sigma_i^2} \tag{14}$$

Furthermore, $L_i(-U(\gamma)) = I$ for $i \in E_p$ (see example 3 in [12]), while according to (6) $\psi_i(-U(\gamma)) = -\mu_i U(\gamma)$ for $i \in E_n$, and $\psi_i(-U(\gamma)) = U(\gamma)$ for $i \in E_-$. Hence the equations above involve rather simple expressions only.

Considering (10), the matrices $A(\gamma)$ and $U(\gamma)$ can be determined by successive approximation as the limit of the sequence $((A_n, U_n) : n \geq 0)$ with initial values $A_0 := \mathbf{0}$, $U_0 := -diag(\phi_i(q_i+\gamma)1_{i\in E_\sigma \cup E_p} + \phi_i(q_i)1_{i\in E_+})_{i\in E_a}$ and the following iteration:

$$e_h' U_{n+1} = \sum_{l=1}^{m_{ij}^+} T_{kl}^{(ij)+} e_{(i,j,l,+)}' + \eta_k^{(ij)+} e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} \qquad \text{for } h = (i,j,k,+) \in E_+,$$

$$e_i' U_{n+1} = -\frac{q_i+\gamma}{\mu_i} e_i' + \frac{1}{\mu_i} \sum_{j\in E, j\neq i} q_{ij}\ e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} \qquad \text{for } i \in E_p,$$

$$e_i' A_{n+1} = \sum_{j\in E, j\neq i} q_{ij} e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} \left((q_i+\gamma)I + \mu_i U_n\right)^{-1} \qquad \text{for } i \in E_n,$$

$$e_i' A_{n+1} = \sum_{j\in E, j\neq i} q_{ij} e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} (q_i I - U_n)^{-1} \qquad \text{for } i \in E_-, \text{ and}$$

$$e_i' U_{n+1} = -\phi_i(q_i+\gamma) e_i' + \frac{2}{\sigma_i^2} \sum_{j\in E, j\neq i} q_{ij}\ e_j' \begin{pmatrix} I_a \\ A_n \end{pmatrix} (\phi_i^*(q_i+\gamma)I - U_n)^{-1} \tag{15}$$

for $i \in E_\sigma$. For the last equality the relation $\phi_i(q_i)\phi_i^*(q_i) = 2q_i/\sigma_i^2$ has been used. Note that the only difference between the iterations for E_n and E_- is the missing γ in the last factor for E_-, reflecting that we do not discount the time for phases $i \in E_-$ as they are jump phases in real time.

Example 3 Continuing example 1, first note that phase 1 represents the upwards jumps and we will not discount the time during sojourns in it. As shown in [12], example 5, the Laplace transform of the first passage time $\tilde{\tau}(x) := \inf\{t \geq 0 : \tilde{X}_t > x\}$ to a level $x > 0$ is given by

$$\mathbb{E}(e^{-\gamma\tilde{\tau}(x)}) = A(\gamma)e^{U(\gamma)x} \quad \text{where} \quad A(\gamma) = \frac{\beta - R}{\beta}, \quad U(\gamma) = -R$$

and

$$-R = \frac{1}{2c}\left(\lambda + \gamma - c\beta - \sqrt{(c\beta - \gamma - \lambda)^2 + 4c\beta\gamma}\right)$$

which coincides with equation (4.24) in [18], noting that γ is denoted as δ there.

3.3 Time–reversed MAPs

Denote the number of phases in E by $m := |E|$. Let $\pi = (\pi_1, \ldots, \pi_m)$ denote the stationary phase distribution, which can be computed by $\pi Q = \mathbf{0}$ and $\pi\mathbf{1} = \sum_{i=1}^m \pi_i = 1$, where **0** denotes the zero row vector and **1** the column vector with all entries being one. Define the matrix $Q^* = (q^*_{ij})_{i,j\in E}$ by $q^*_{ij} := \pi_j q_{ji}/\pi_i$ for all $i, j \in E$ or in shorter notation $Q^* := \Delta_\pi^{-1} Q' \Delta_\pi$, where $\Delta_\pi = diag(\pi_1, \ldots, \pi_m)$ is the diagonal matrix with entry π_i in its ith row and the superscript $'$ denotes transposition of a matrix. Then the Markov process with state space E and generator matrix Q^* is a time–reversed version of the original phase process $\mathcal{J}$. We denote it by $\mathcal{J}^* = (J^*_t : t \geq 0)$.

Based on $\mathcal{J}^*$ we define a time-reversal $(\mathcal{X}^*, \mathcal{J}^*)$ of the original MAP $(\mathcal{X}, \mathcal{J})$ by the rule that $\mathcal{X}^*$ evolves like a Lévy process with parameters $-\mu_i$ (drift) and σ_i^2 (variation) during intervals when the time–reversed phase J^*_t equals $i \in E$. Note that the sign change of the μ_i leads to $E^*_\pm = E_\mp$, $E^*_p = E_n$, $E^*_n = E_p$, and $E^*_\sigma = E_\sigma$. We denote the first passage times for $(\mathcal{X}^*, \mathcal{J}^*)$ by $\tau^*(x) := \inf\{t \geq 0 : X^*_t > x\}$ for any level $x \geq 0$.

The same arguments as for equation (3.3) in [2] yield the following relation between the occupation measure (before $\tau(0)$) for the MAP $(\mathcal{X}, \mathcal{J})$ and the first passage time for its time–reversal $(\mathcal{X}^*, \mathcal{J}^*)$:

$$\begin{aligned}\pi_j \mathbb{P}(X^*_t \in dx, X^*_t > X^*_u \; \forall u < t, J^*_t = i | X^*_0 = 0, J^*_0 = j) \\ = \pi_i \mathbb{P}(X_t \in -dx, \tau(0) > t, J_t = j | X_0 = 0, J_0 = i)\end{aligned}$$

where $dx = [x, x + \varepsilon)$ for a small $\varepsilon > 0$. Multiplying by $e^{-\gamma t}$ and integrating over t yields in the limit $\varepsilon \to 0$

$$\begin{aligned}&\pi_i \mathbb{E}\left(\int_0^{\tau(0)} 1_{X_t=-x, J_t=j} \; e^{-\gamma t} \; dt | X_0 = 0, J_0 = i \right) \\ &\quad = \pi_j \mathbb{E}\left(\int_0^{\infty} 1_{X^*_t=x, X^*_t>X^*_u \; \forall u<t, J^*_t=i} \; e^{-\gamma t} \; dt | X^*_0 = 0, J^*_0 = j \right) \\ &\quad = \pi_j \mathbb{E}\left(e^{-\gamma \tau^*(x)}; J^*_{\tau^*(x)} = i | X^*_0 = 0, J^*_0 = j \right) \qquad (16)\end{aligned}$$

A well-known result that we shall use in the next section is the following lemma which is theorem VI.5(i) and theorem VII.4(i) of [11] applied to Brownian motion and its time–reversal. This lemma also yields an alternative explanation for (13).

Lemma 1 Let $\mathcal{B} = (B_t : t \geq 0)$ denote a Brownian motion with drift $\mu \in \mathbb{R}$ and variation $\sigma^2 > 0$. Assume that $B_0 = 0$. Further let $\mathcal{E}(q)$ denote a random variable which is independent of $\mathcal{B}$ and has an exponential distribution with parameter $q > 0$. Write

$$\bar{B}_{\mathcal{E}(q)} := \max_{0 \leq t \leq \mathcal{E}(q)} B_t \qquad \text{and} \qquad \bar{B}^*_{\mathcal{E}(q)} := \bar{B}_{\mathcal{E}(q)} - B_{\mathcal{E}(q)}$$

as well as

$$G_{\mathcal{E}(q)} := \sup\{t < \mathcal{E}(q) : B_t = \bar{B}_{\mathcal{E}(q)}\} \qquad \text{and} \qquad G^*_{\mathcal{E}(q)} := \mathcal{E}(q) - G_{\mathcal{E}(q)}$$

Then the pairs $(G_{\mathcal{E}(q)}, \bar{B}_{\mathcal{E}(q)})$ and $(G^*_{\mathcal{E}(q)}, \bar{B}^*_{\mathcal{E}(q)})$ are independent with respective measures

$$\mathbb{E}\left(e^{-\gamma G_{\mathcal{E}(q)}}; \bar{B}_{\mathcal{E}(q)} \in dx\right) = \phi(q)e^{-\phi(q+\gamma)x}dx$$

and

$$\mathbb{E}\left(e^{-\gamma G^*_{\mathcal{E}(q)}}; \bar{B}^*_{\mathcal{E}(q)} \in dy\right) = \phi^*(q)e^{-\phi^*(q+\gamma)y}dy$$

for $\gamma \geq 0$, where $\phi(\beta)$ and $\phi^*(\beta)$ for $\beta > 0$ are given in the last line of (10) and in (14), respectively.

4 Main result

Let $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}}) = ((\tilde{X}_t, \tilde{J}_t) : t \geq 0)$ denote a MAP with phase–type jumps and assume that $\tilde{X}_0 = 0$. Denote the phase space of $\tilde{\mathcal{J}}$ by $\tilde{E}$ and its generator matrix by $\tilde{Q}$. Let

$$\tilde{\tau}(u) := \inf\{t > 0 : \tilde{X}_t > u\}$$

denote the first passage time over some level $u \geq 0$. Write

$$\tilde{M}_{\tilde{\tau}(u)} := \sup\{\tilde{X}_t : t < \tilde{\tau}(u)\}$$

for the maximum of $\tilde{\mathcal{X}}$ before the first passage over u. We necessarily have $0 \leq \tilde{M}_{\tilde{\tau}(u)} \leq u$, where $\tilde{M}_{\tilde{\tau}(u)} = 0$ means that $\tilde{\mathcal{X}}$ does not exceed its initial value 0 before it jumps (from a negative value) over the threshold $u \geq 0$. The case $\tilde{M}_{\tilde{\tau}(u)} = u$ means that passage occurs by creeping, i.e. the threshold u is not passed by a jump but continuously. We are further interested in

$$\tilde{G}_{\tilde{\tau}(u)} := \sup\{t < \tilde{\tau}(u) : \tilde{X}_t = \tilde{M}_{\tilde{\tau}(u)}\}$$

which is the time of attaining the maximum before passage over u (cf. lemma 2 regarding its uniqueness). Finally, we wish to determine the density function of the undershoot and the overshoot, defined as

$$R_{T-} := u - \tilde{X}_{\tilde{\tau}(u)-} \qquad \text{and} \qquad |R_T| := \tilde{X}_{\tilde{\tau}(u)} - u$$

respectively. Our aim is to derive a computable expression for the joint law of these five variables in terms of the measure

$$\mathbb{E}\left(e^{-\gamma \tilde{G}_{\tilde{\tau}(u)} - \gamma^*(\tilde{\tau}(u) - \tilde{G}_{\tilde{\tau}(u)})}; \tilde{M}_{\tilde{\tau}(u)} \in dz, R_{T-} \in dx, |R_T| \in dy\right)$$

where $\gamma, \gamma^* \geq 0$ are arguments for the double Laplace transform, $x, y \geq 0$, and $0 \leq z \leq u$. Note that necessarily $x \geq u - z$.

The approach of analysis in this paper is the same as in [14] for Lévy processes. We divide the sample paths into three parts: the path until the supremum $\tilde{M}_{\tilde{\tau}(u)} < u$ is attained, the path from $\tilde{M}_{\tilde{\tau}(u)}$ to $\tilde{X}_{\tilde{\tau}(u)-}$, and the final jump which leads to an overshoot of the level u. The second part can be determined by (4) via the time–reversed process. Between the three parts we need to take possible (and necessary) phase changes into account. This reasoning will be similar to [10]. A difference to [14] is that we measure the times $\tilde{G}_{\tilde{\tau}(u)}$ and $\tilde{\tau}(u) - \tilde{G}_{\tilde{\tau}(u)}$ in terms of their Laplace transforms. This enables us to provide an explicit formula with expressions that can be readily computed.

One preliminary lemma that we need for the main result concerns the time of attaining the maximum $\tilde{M}_{\tilde{\tau}(u)}$. See [28], lemma 2, for the equivalent statement regarding Lévy processes.

Lemma 2 Define $\tilde{G}'_{\tilde{\tau}(u)} := \inf\{t < \tilde{\tau}(u) : \tilde{X}_t = \tilde{M}_{\tilde{\tau}(u)}\}$. Then $\tilde{G}_{\tilde{\tau}(u)} = \tilde{G}'_{\tilde{\tau}(u)}$ almost surely.

Proof: Since all possible jumps are phase–type and the zero process as well as compound Poisson processes are excluded under any regime, the transition probabilities between levels of local maxima of $\tilde{\mathcal{X}}$ are absolutely continuous. Thus the probability of attaining the same local maximum level twice is 0.
□

Given the MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$, we construct the modified MAP $(\mathcal{X}, \mathcal{J})$ from it as described in section 3.1. Recall that $P = \Delta_q^{-1} Q + I$, where Q denotes the generator matrix of $\mathcal{J}$, see (7) and (8), and Δ_q is the diagonal matrix with entries $q_i = -q_{ii}$ for all $i \in E$. Define $p_{ij}^{(+,-)} := \delta_{ij}$ for $i \in E_\sigma$ and $p_{ij}^{(+,-)} := p_{ij}$ for $i \in E_+ \cup E_p$, $j \in E_n \cup E_-$. Further define

$$P^{(+,-)} := \left(p_{ij}^{(+,-)}\right)_{i \in E_a, j \in E_\sigma \cup E_d} \qquad \text{and} \qquad P^{(c,+)} := (p_{ij} 1_{i \in E_c})_{i \in E, j \in E_+} \tag{17}$$

The matrices $P^{(+,-)}$ and $P^{(c,+)}$ subsume the transition probabilities from ascending to descending phases and from continuous to positive jump phases, respectively. Write

$$\Delta_\phi := diag(\phi_i(q_i))_{i \in E_a} \quad \text{and} \quad \Delta_{\phi^*} := diag(\phi_i(q_i) 1_{i \in E_p} + \phi^*(q_i) 1_{i \in E_\sigma \cup E_n})_{i \in E} \tag{18}$$

and define the block diagonal matrix $T = diag\left(T^{(ij)}\right)_{(i,j) \in E_c \times E_c}$ and the block column vector $\eta = \left(\eta^{(ij)}\right)_{(i,j) \in E_c \times E_c}$. Here $E_c \times E_c$ must be ordered in some way, say lexicographically. Note that this order must be inherited from the order on E_+. Finally, define the diagonal matrices

$$\Pi_a^* = diag(1/\pi_i)_{i \in E_\sigma \cup E_d} \qquad \text{and} \qquad \Pi_c = diag(\pi_j 1_{j \in E_c})_{j \in E} \tag{19}$$

Now we can state the main result. In theorem 1 we consider the perhaps typical case where the first passage over u is incurred by a jump of $\tilde{\mathcal{X}}$, i.e. R_{T-} and $|R_T|$ are positive, and the maximum before $\tilde{\tau}(u)$ is positive, i.e. $\tilde{M}_{\tilde{\tau}(u)} > 0$. Then an example shall demonstrate how to use theorem 1. After that, some corollaries shall describe special events. Corollary 1 deals with the case that $\tilde{M}_{\tilde{\tau}(u)} = 0$, i.e. the level never exceeds the initial value 0 before u is passed by jumping. Corollary 2 treats the case of ruin by diffusion, i.e. $R_{T-} = |R_T| = 0$, also called the creeping case. Finally, corollary 3 considers the case that passage over u happens by a jump from a current maximum, i.e. $R_{T-} = u - \tilde{M}_{\tilde{\tau}(u)}$.

Theorem 1 *Let $\tilde{\alpha}$ denote the initial phase distribution of a MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ with phase–type jumps. Define the row vector $\alpha = (\alpha_i : i \in E)$ on the phase space E of the enlarged MAP $(\mathcal{X}, \mathcal{J})$ by $\alpha_i := \tilde{\alpha}_i$ for all $i \in E_c = \tilde{E}$ and $\alpha_i := 0$ for $i \in E_+ \cup E_-$. Then*

$$\begin{aligned} &\mathbb{E}\left(e^{-\gamma \tilde{G}_{\tilde{\tau}(u)} - \gamma^*(\tilde{\tau}(u) - \tilde{G}_{\tilde{\tau}(u)})}; \tilde{M}_{\tilde{\tau}(u)} \in dz, R_{T-} \in dx, |R_T| \in dy\right) \\ &\qquad = \alpha \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} e^{U(\gamma)z} \Delta_\phi P^{(+,-)} \\ &\qquad\qquad \left(\Pi_c \begin{pmatrix} A^*(\gamma^*) \\ I_{E_\sigma \cup E_d} \end{pmatrix} e^{U^*(\gamma^*)\cdot(z-(u-x))} \Pi_a^*\right)' \Delta_{\phi^*} P^{(c,+)} e^{T\cdot(x+y)} \eta \;\; dx\, dy\, dz \end{aligned}$$

for all $\gamma, \gamma^* \geq 0$, $0 < z < u$, $x > u - z$, and $y > 0$.

Proof: We consider all possible paths leading to the event

$$\{\tilde{M}_{\tilde{\tau}(u)} \in dz, R_{T-} \in dx, |R_T| \in dy\}$$

Since the paths of $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ can be retrieved from those of $(\mathcal{X}, \mathcal{J})$, we may restrict our attention to $(\mathcal{X}, \mathcal{J})$. We shall, however, speak of jumps whenever we refer to linear movements governed by phases in $E_- \cup E_+$. Recall the first passage time

$$\tau(u) := \inf\{t \geq 0 : X_t > u\} = \min\{t \geq 0 : X_t = u\}$$

where the last equality holds because of path continuity of $\mathcal{X}$. Define the times

$$\sigma(u) := \sup\{t < \tau(u) : J_t \notin E_+\} \quad \text{and} \quad \sigma'(u) := \inf\{t > \tau(u) : J_t \notin E_+\}$$

The assumption $z < u$ implies $x \geq u - z > 0$. Further, $y > 0$ almost surely since positive jumps are phase-type and hence absolutely continuous. The time $\sigma(u)$ denotes the instant when the final positive jump, which leads $\mathcal{X}$ over the threshold u, begins. The time

$\sigma'(u)$ indicates the instant when this final jump ends and we can measure the overshoot $|R_T| = \tilde{X}_{\tau(\tilde{u})} - u = X_{\sigma'(u)} - u$. We further obtain

$$\tilde{M}_{\tilde{\tau}(u)} = M_{\sigma(u)} := \sup\{X_t : t \leq \sigma(u)\}$$

such that we can write

$$\{\tilde{M}_{\tilde{\tau}(u)} \in dz, R_{T-} \in dx, |R_T| \in dy\} = \{M_{\sigma(u)} \in dz, X_{\sigma(u)} \in u - dx, X_{\sigma'(u)} \in u + dy\}$$

We shall employ the following path decomposition. First we consider the path up to its maximum (which is attained at a unique time, due to lemma 2). The second part to consider is the path strictly between the time of maximum and $\sigma(u)$. The last part is the jump.

The initial phase distribution is denoted by the row vector $\alpha = (\alpha_i : i \in E)$ where $\alpha_i = \mathbb{P}(J_0 = i)$ for all $i \in E$. In order to attain $M_{\sigma(u)} \in dz$, a path must first pass the level z. This happens at the time $\tau(z) := \inf\{t \geq 0 : X_t > z\}$. The Laplace transform of $\tilde{\tau}(z) := \int_0^{\tau(z)} 1_{J_s \in E_c} ds$, restricted to $\{J_{\tau(z)} = i\}$ and with argument $\gamma \geq 0$, is given by

$$\mathbb{E}\left(e^{-\gamma\tilde{\tau}(z)}; J_{\tau(z)} = i | X_0 = 0\right) = \alpha \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} e^{U(\gamma)z} e_i$$

where e_i denotes the ith canonical base column vector of dimension E_a.

In order to satisfy $M_{\sigma(u)} \in dz$, the upward movement of $\mathcal{X}$ needs to stop in dz. This happens with infinitesimal rate $\phi_i(q_i)dz$ on $\{J_{\tau(z)} = i\}$, see lemma 1. Then the path continues differently, according to $i \in E_+ \cup E_p$ or $i \in E_\sigma$.

Case 1: $i = J_{\tau(z)} \in E_+ \cup E_p$

This means that $\mathcal{X}$ has attained the level z in a linear upward movement which has stopped in dz. In order to satisfy $M_{\sigma(u)} \in dz$ and $x > u - z$, i.e. $X_{\sigma(u)} < z$, the phase must now change from i to some phase $j \in E_d = E_n \cup E_-$. This happens with probability $p_{ij} = p_{ij}^{(+,-)}$. Up to this point, the Laplace transform of $\tilde{\tau}(z)$ on $\{J_{\tau(z)+} = j\}$, $j \in E_d$, is subsumed under

$$\mathbb{E}\left(e^{-\gamma\tilde{\tau}(z)}; J_{\tau(z)+} = j | X_0 = 0\right) = \alpha \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} e^{U(\gamma)z} \Delta_\phi P^{(+,-)} e_j dz$$

for $j \in E_d$, where e_j denotes the jth canonical base column vector of dimension $E_\sigma \cup E_d$.

The second part of the path consists of a movement from the level z at time $\tau(z)+$ to the level $u - x$ at time $\sigma(u)-$ without crossing the level z again. Consider the event

$$\{X_s < z \; \forall s \in]\tau(z), \rho], X_\rho = u - x\}$$

for some random time $\rho > \tau(z)$. Denote the phase at time ρ by $k := J_\rho$. If we want that $\rho = \sigma(u)$, then $k \in E_c$ is a necessity. The Laplace transform of $\int_{\tau(z)}^{\rho} 1_{J_s \in E_c} ds$ with argument γ^* is, according to (16),

$$\mathbb{E}\left(\int_0^{\tau(z)} 1_{X_t = u-x, J_t = k}\, e^{-\gamma^* \int_0^t 1_{J_s \in E_c} ds} dt | X_0 = z, J_0 = j \right)$$
$$= \frac{\pi_k}{\pi_j} \mathbb{E}\left(e^{-\gamma^* \tilde{\tau}^*(z-(u-x))}; J_{\tau^*(z-(u-x))} = j | X_0^* = 0, J^* = k \right) \quad (20)$$

where the following notation is used. The MAP $(\mathcal{X}^*, \mathcal{J}^*)$ is the time-reversal of $(\mathcal{X}, \mathcal{J})$, constructed as in section 3.3. The row vector $\pi = (\pi_i : i \in E)$ is the stationary distribution of $\mathcal{J}$. Further, $\tau^*(x) := \inf\{t \geq 0 : X_t^* > x\}$ and $\tilde{\tau}^*(x) := \int_0^{\tau^*(x)} 1_{J_s^* \in E_c} ds$.

We can compute the expectation on the right-hand side of (20) by determining the matrices $A^*(\gamma^*)$ and $U^*(\gamma*)$ for $(\mathcal{X}^*, \mathcal{J}^*)$ just as $A(\gamma^*)$ and $U(\gamma^*)$ are determined for $(\mathcal{X}, \mathcal{J})$, see section 3.2. Then

$$\mathbb{E}\left(e^{-\gamma^* \tilde{\tau}^*(z-(u-x))}; J_{\tau^*(z-(u-x))} = j | X_0^* = 0, J^* = k \right)$$
$$= e_k' \begin{pmatrix} A^*(\gamma^*) \\ I_{E_\sigma \cup E_d} \end{pmatrix} e^{U^*(\gamma^*) \cdot (z-(u-x))} e_j \quad (21)$$

where e_k' denotes the kth canonical base row vector on E and e_j is the jth canonical base column vector of dimension $E_\sigma \cup E_d$. Note that the ascending phases for the time-reversed MAP $(\mathcal{X}^*, \mathcal{J}^*)$ are the ones in $E_a^* := E_\sigma \cup E_d$. Thus the matrix $A^*(\gamma^*)$ has dimension $(E_+ \cup E_p) \times E_a^*$ and $U^*(\gamma*)$ has dimension $E_a^* \times E_a^*$.

Altogether we obtain for the double Laplace transform of $\left(\tilde{\tau}(z), \int_{\tau(z)}^{\rho} 1_{J_s \in E_c} ds\right)$ in case 1, i.e. for $J_{\tau(z)} \in E_+ \cup E_p$,

$$\mathbb{E}\left(e^{-\gamma \tilde{\tau}(z) - \gamma^* \int_{\tau(z)}^{\rho} 1_{J_s \in E_c} ds}; J_{\tau(z)} \in E_+ \cup E_p, X_s < z \; \forall s \in]\tau(z), \rho], X_\rho = u - x, J_\rho = k \right)$$
$$= \alpha \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} e^{U(\gamma) z} \Delta_\phi P^{(+,-)} I_1 \left(\Pi_c \begin{pmatrix} A^*(\gamma^*) \\ I_{E_\sigma \cup E_d} \end{pmatrix} e^{U^*(\gamma^*) \cdot (z-(u-x))} \Pi_a^* \right)' e_k \, dz \quad (22)$$

where I_1 is a matrix of dimension $(E_\sigma \cup E_d)^2$ with (i, j)th entry $\delta_{ij} 1_{i \in E_d}$ and the superscript $(\ldots)'$ indicates transposition of the matrix within the brackets. Further, Π_a^* is a diagonal matrix of dimension $E_a^* = E_\sigma \cup E_d$ with ith entry $1/\pi_i$ and Π_c is the E-dimensional diagonal matrix with ith entry $\pi_i 1_{i \in E_c}$. Finally, e_k denotes the kth canonical base column vector of dimension E.

In order that $\rho = \sigma(u)$, the phase process $\mathcal{J}$ must change from $k \in E_c$ to some phase $l \in E_+$ while the level process $\mathcal{X}$ is still in the vicinity of $u - x$. This happens with probability

$$e_k' \Delta_{\phi^*} P^{(c,+)} e_l \, dx = \begin{cases} \phi_k(q_k) dx \cdot p_{kl}, & k \in E_p \\ \phi_k^*(q_k) dx \cdot p_{kl}, & k \in E_\sigma \cup E_n \end{cases}$$

where e_k' has dimension E and e_l has dimension E_+.

After this, the phase process needs to stay in E_+ for at least x time units. This happens with probability $e_l' e^{Tx} e_h$, where e_h has dimension E_+. At this point the time $\tau(u)$ is reached, with $J_{\tau(u)} = h \in E_+$, which completes all conditions for $\rho = \sigma(u)$. Furthermore, for any path considered above, the equations

$$\tilde{\tau}(z) = \tilde{G}_{\tilde{\tau}(u)} \quad \text{and} \quad \tilde{\tau}(u) - \tilde{G}_{\tilde{\tau}(u)} = \int_{\tau(z)}^{\rho} 1_{J_s \in E_c} ds$$

hold. Thus we obtain for case 1, i.e. $J_{\tau(z)} \in E_+ \cup E_p$,

$$\begin{aligned} &\mathbb{E}\left(e^{-\gamma \tilde{G}_{\tilde{\tau}(u)} - \gamma^*(\tilde{\tau}(u) - \tilde{G}_{\tilde{\tau}(u)})}; J_{\tau(z)} \in E_+ \cup E_p, \tilde{M}_{\tilde{\tau}(u)} \in dz, R_{T-} \in dx\right) \\ &= \mathbb{E}\Big(e^{-\gamma \tilde{\tau}(z) - \gamma^* \int_{\tau(z)}^{\rho} 1_{J_s \in E_c} ds}; J_{\tau(z)} \in E_+ \cup E_p, X_s < z \ \forall s \in]\tau(z), \rho], X_\rho = u - x, \\ &\qquad J_t \in E_+ \ \forall t \in]\rho, \tau(u)], J_{\tau(u)} = h\Big) \\ &= \alpha \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} e^{U(\gamma) z} \Delta_\phi P^{(+,-)} I_1 \left(\Pi_c \begin{pmatrix} A^*(\gamma^*) \\ I_{E_\sigma \cup E_d} \end{pmatrix} e^{U^*(\gamma^*) \cdot (z - (u - x))} \Pi_a^* \right)' \\ &\qquad \Delta_{\phi^*} P^{(c,+)} e^{Tx} e_h \, dx dz \end{aligned}$$

where e_h denotes the hth canonical base column vector of dimension E_+.

Considering that the remaining part of the positive jumps, starting from the phase $J_{\tau(u)} = h$, constitutes the severity of ruin $|R_T|$, we obtain

$$\mathbb{P}\left(|R_T| \in dy | J_{\tau(u)} = h\right) = e_h' e^{Ty} \eta \, dy$$

which together with the previous formula yields the statement for case 1.

Case 2: $i = J_{\tau(z)} \in E_\sigma$

In this case there will be no phase change immediately after $\tau(z)$. The remaining path during phase i will stay below the level z and is independent of the path until $\tau(z)$, see lemma 1. There are two possibilities for $\mathcal{X}$ to reach the level $u - x$.

The first one is that this happens without a phase change, i.e. the event

$$\{X_s < z, J_s = i \ \forall s \in]\tau(z), \rho], X_\rho = u - x\}$$

occurs for some random time $\rho > \tau(z)$. This path corresponds to a path of the time-reversed MAP $(\mathcal{X}^*, \mathcal{J}^*)$ starting with $X_0^* = 0$, $J_0^* = i$, and first hitting the level $z-(u-x)$ at a time ρ, without changing its phase in between.

The second possibility of reaching level $u - x$ is that $\mathcal{X}$ moves to some level w while $\mathcal{J}$ remains in i, then a phase change occurs, and $\mathcal{X}$ reaches $u - x$ after that. This is the event

$$\{X_s < z \; \forall s \in]\tau(z), \rho], X_\rho = u - x, J_s = i \; \forall s \in]\tau(z), \rho'[, J_{\rho'} \neq i, J_\rho = k\}$$

where $\rho' < \rho$ is the time of the first phase change. This corresponds to a path of the time-reversed MAP $(\mathcal{X}^*, \mathcal{J}^*)$ starting with $X_0^* = 0$, $J_0^* = k$, and first hitting the level $z - (u - x)$ in phase i at a time ρ, with at least one phase change in between.

Together, the Laplace transform of $\int_{\tau(z)}^{\rho} 1_{J_s \in E_c} ds$ with argument $\gamma^* \geq 0$ is the same as in (20), namely

$$\frac{\pi_k}{\pi_i} \mathbb{E}\left(e^{-\gamma^* \tilde{\tau}^*(z-(u-x))}; J_{\tau^*(z-(u-x))} = i | X_0^* = 0, J_0^* = k\right)$$
$$= e_i' \left(\Pi_c \begin{pmatrix} A^*(\gamma^*) \\ I_{E_\sigma \cup E_d} \end{pmatrix} e^{U^*(\gamma^*) \cdot (z-(u-x))} \Pi_a^* \right)' e_k \quad (23)$$

with the same notations as in (20) and (21-22). After that the arguments are exactly the same as for case 1.
□

Example 4 The following example shall serve as a manual to theorem 1. It explains where to find the relevant formulas needed to compute the ingredients for it and the corollaries below.

Consider a positive linear drift with slope $c > 0$ which is superposed by two compound Poisson processes. One of them has positive jumps of exponential size with parameter $\beta^+ > 0$ and jump intensity $\lambda^+ > 0$. The other one has negative jumps of exponential size with parameter $\beta^- > 0$ and jump intensity $\lambda^- > 0$. Together this forms a Lévy process $\tilde{\mathcal{X}}$ with parameters $\mu = c$ for the drift, $\sigma^2 = 0$ (i.e. there is no diffusion part), and Lévy measure

$$\nu(dx) = \lambda^+ \cdot 1_{x>0} \cdot e^{-\beta^+ x} \beta^+ \, dx + \lambda^- \cdot 1_{x<0} \cdot e^{-\beta^- x} \beta^- \, dx$$

The process $\tilde{\mathcal{X}}$ can of course be represented as a MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ with trivial phase space $E = 1$ and the trivial generator matrix $\tilde{Q} = 0$ for $\mathcal{J}$. According to section 3.1 we construct the MAP $(\mathcal{X}, \mathcal{J})$ as follows. The enlarged phase space E consists of the subsets $E_p = \{1\}$, $E_+ = \{(1, 1, 1, +)\}$, and $E_- = \{(1, 1, 1, -)\}$, according to (3) and (4). The subsets E_σ

and E_n are empty. For ease of notation, denote the jump phases by $1+ = (1,1,1,+)$ and $1- = (1,1,1,-)$. The parameters for each phase are set according to (5) as

$$(\mu_i, \sigma_i, \nu_i) = \begin{cases} (1, 0, \mathbf{0}), & i = 1+ \\ (c, 0, \mathbf{0}), & i = 1 \\ (-1, 0, \mathbf{0}), & i = 1- \end{cases}$$

The generator matrix Q for $\mathcal{J}$ is given by (7 - 8) as

$$Q = \begin{pmatrix} -\beta^+ & \beta^+ & 0 \\ \lambda^+ & -\lambda & \lambda^- \\ 0 & \beta^- & -\beta^- \end{pmatrix}$$

where $\lambda = \lambda^+ + \lambda^-$. This completely defines the MAP $(\mathcal{X}, \mathcal{J})$. The matrices $A(\gamma)$ and $U(\gamma)$ have dimension 1×2 and 2×2, respectively. According to (15), they are determined as the limit $(U(\gamma), A(\gamma)) = \lim_{n\to\infty}(U_n, A_n)$, with initial values $A_0 = \mathbf{0}$ and $U_0 = diag(-\beta^+, -(\lambda+\gamma)/c)$ and iteration

$$\begin{aligned} e'_{1+}U_{n+1} &= -\beta^+ e'_{1+} + \beta^+ e'_1 = (-\beta^+, \beta^+) \\ e'_1 U_{n+1} &= -\frac{\lambda+\gamma}{c} e'_1 + \frac{1}{c}\left(\lambda^+ e'_{1+} + \lambda^- A_n\right) \\ A_{n+1} &= \beta^- e'_1 \left(\beta^- I - U_n\right)^{-1} \end{aligned} \tag{24}$$

Here e'_{1+} and e'_1 are the canonical row base vectors on the space $E_a = \{1+, 1\}$. Note that the first equation is constant in n, i.e. the first row of $U(\gamma)$, pertaining to phase $1+$, is given as $(-\beta^+, \beta^+)$. Alternatively to this iteration, one could solve the corresponding fixed point equation. Write $A(\gamma) = (a_1, a_2)$. Then

$$U(\gamma) = \begin{pmatrix} -\beta^+ & \beta^+ \\ \frac{\lambda^+}{c} + \frac{\lambda^-}{c}a_1 & \frac{\lambda^-}{c}a_2 - \frac{\lambda+\gamma}{c} \end{pmatrix}$$

and due to the third equation (24) we have to solve

$$(a_1, a_2)\begin{pmatrix} \beta^- + \beta^+ & -\beta^+ \\ -\frac{\lambda^+}{c} - \frac{\lambda^-}{c}a_1 & \beta^- + \frac{\lambda+\gamma}{c} - \frac{\lambda^-}{c}a_2 \end{pmatrix} = (0, \beta^-)$$

The entries of the diagonal matrix Δ_ϕ are given in (10) and yield $\Delta_\phi = diag(\beta^+, \lambda/c)$ according to (18). The transition matrix of phase changes is given by

$$P = \Delta_q^{-1} Q + I = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\lambda^+}{\lambda} & 0 & \frac{\lambda^-}{\lambda} \\ 0 & 1 & 0 \end{pmatrix}$$

According to (17) we obtain the column vector $P^{(+,-)} = (0, \lambda^-/\lambda)'$. The matrices Π_c and Π_a^* are determined by the stationary row vector π which satisfies $\pi Q = \mathbf{0}$ and $\pi \mathbf{1} = 1$. We thus obtain

$$\pi_{1+} = \left(1 + \frac{\beta^+}{\lambda^+} + \frac{\beta^+}{\lambda^+}\frac{\lambda^-}{\beta^-}\right)^{-1}, \qquad \pi_1 = \pi_{1+}\frac{\beta^+}{\lambda^+}, \qquad \pi_{1-} = \pi_{1+}\frac{\beta^+}{\lambda^+}\frac{\lambda^-}{\beta^-}$$

which determines $\Pi_c = diag(0, \pi_1, 0)$ and $\Pi_a^* = 1/\pi_{1-}$ according to (19). In order to compute the matrices $A^*(\gamma^*)$ and $U^*(\gamma^*)$, we first need to determine the time-reversion $(\mathcal{X}^*, \mathcal{J}^*)$ of $(\mathcal{X}, \mathcal{J})$. This is described in section 3.3. The generator matrix Q^* of $\mathcal{J}^*$ is given by $Q^* = \Delta_\pi^{-1} Q' \Delta_\pi$. It turns out that $Q^* = Q$, i.e. $\mathcal{J}$ is reversible. The other parameters of $(\mathcal{X}^*, \mathcal{J}^*)$ are

$$(\mu_i^*, \sigma_i^*, \nu_i^*) = \begin{cases} (-1, 0, \mathbf{0}), & i = 1+ \\ (-c, 0, \mathbf{0}), & i = 1 \\ (1, 0, \mathbf{0}), & i = 1- \end{cases}$$

Thus $E_a^* = E_+^* = \{1-\}$, which means that $U^*(\gamma^*)$ is a number and $A^*(\gamma^*)$ is a column vector of dimension 2. According to (15), the pair $(U^*(\gamma^*), A^*(\gamma^*))$ is determined as the limit $\lim_{n\to\infty}(U_n^*, A_n^*)$ with initial values $A_0^* = \mathbf{0}$ and $U_0^* = -\beta^-$ and iteration

$$\begin{aligned} U_{n+1}^* &= -\beta^- + \beta^- e_1' \begin{pmatrix} A_n^* \\ 1 \end{pmatrix} \\ e_1' A_{n+1}^* &= \lambda^+ e_{1+}' \begin{pmatrix} A_n^* \\ 1 \end{pmatrix} (\lambda + \gamma^* - cU_n^*)^{-1} + \lambda^- e_{1-}' \begin{pmatrix} A_n^* \\ 1 \end{pmatrix} (\lambda + \gamma^* - cU_n^*)^{-1} \\ e_{1+}' A_{n+1}^* &= \beta^+ e_1' \begin{pmatrix} A_n^* \\ 1 \end{pmatrix} \left(\beta^+ - U_n^*\right)^{-1} \end{aligned}$$

where e_i' denotes the ith canonical row vector on the phase space $E = \{1+, 1, 1-\}$ for the right-hand sides of all equations, while on the left-hand sides e_i' denotes the ith canonical row vector on the space $\{1+, 1\}$. Writing $A^*(\gamma^*) = (a_{1+}^*, a_1^*)'$ and $U^* = U^*(\gamma^*)$, we can alternatively solve the fixed point equation

$$U^* = \beta^- a_1^* - \beta^-, \quad a_1^* = \frac{\lambda^+ a_1^* + \lambda^-}{\lambda + \gamma^* - cU^*}, \quad a_{1+}^* = \frac{\beta^+ a_1^*}{\beta^+ - U^*}$$

This determines all ingredients we need from the time-reversed MAP $(\mathcal{X}^*, \mathcal{J}^*)$. According to (18), $\Delta_{\phi^*} = diag(0, \lambda/c, 0)$. The matrix $P^{(c,+)}$ is a column vector of dimension 3. We obtain $P^{(c,+)} = (0, \lambda^+/\lambda, 0)'$ according to (17). Regarding the positive jump part, we obtain finally the parameters $T = -\beta^+$ and $\eta = \beta^+$. This completes the derivations that we need for theorem 1.

If the process starts with a negative drift, then the singular case $\tilde{M}_{\tilde{\tau}(u)} = 0$ is possible. This implies $\tilde{G}_{\tilde{\tau}(u)} = 0$ and $x > u$. The remaining quadruple law is given in the following corollary. For $\gamma^* = 0$ and $E_\sigma = \emptyset$ it yields equation (3.6) in [2] and theorem 1 in [27].

Corollary 1 Let α be an initial phase distribution with support on E_n. Then

$$\mathbb{E}\left(e^{-\gamma^*\tilde{\tau}(u)}; \tilde{M}_{\tilde{\tau}(u)} = 0, R_{T-} \in dx, |R_T| \in dy\right) \\ = \alpha\left(\Pi_c \begin{pmatrix} A^*(\gamma^*) \\ I_{E_\sigma \cup E_d} \end{pmatrix} e^{U^*(\gamma^*)\cdot(x-u)}\Pi_a^*\right)' \Delta_{\phi^*} P^{(c,+)} e^{T\cdot(x+y)}\eta \, dxdy$$

Two other singular cases that may arise are given in the following corollaries. The reasoning for them is the same as for theorem 1.

Corollary 2 Ruin by diffusion
Let α denote the initial phase distribution of a MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ with phase–type jumps. Then

$$\mathbb{E}\left(e^{-\gamma\tilde{\tau}(u)}; R_{T-} = 0, |R_T| = 0\right) = \alpha\begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} e^{U(\gamma)u}\mathbf{1}_{E_p \cup E_\sigma}$$

where $\mathbf{1}_{E_p \cup E_\sigma}$ is a column vector of dimension E_a with ith entry being 0 for $i \in E_+$ and 1 for $i \in E_p \cup E_\sigma$.

Corollary 3 Ruin by jump from a maximal net claim
Let α denote the initial phase distribution of a MAP $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ with phase–type jumps. Then

$$\mathbb{E}\left(e^{-\gamma\tilde{\tau}(u)}; \tilde{M}_{\tilde{\tau}(u)} \in dz, R_{T-} \in u - dz, |R_T| \in dy\right) \\ = \alpha\begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} e^{U(\gamma)z}\Delta_\phi P^{(p,+)} e^{T\cdot(u-z+y)}\eta \, dydz$$

where $P^{(p,+)} = \left(p_{ij}^{(p,+)}\right)_{i \in E_a, j \in E_+}$ with $p_{ij}^{(p,+)} = 0$ for $i \in E_+ \cup E_\sigma$ and $p_{ij}^{(p,+)} = p_{ij}$ for $i \in E_p$ and $j \in E_+$.

5 Comparison to existing results

Example 5 We continue example 1 and 3. The time-reversed process has a positive drift ($E_p^* = E_n = \{2\}$) and negative jumps ($E_-^* = E_+ = \{1\}$). Instead of evaluating the rather complicated expression (23), we can treat it as a spectrally negative MAP directly. Its cumulant function is given by

$$\psi^*(x) = cx - \lambda + \lambda \frac{\beta}{\beta + x}$$

Hence we find for the inverse of ψ^*

$$\phi^*(\gamma) = \frac{1}{2c}\left(\lambda + \gamma - c\beta + \sqrt{(c\beta - \lambda - \gamma)^2 + 4c\beta\gamma}\right)$$

which is denoted by ρ in [18], equation (3.12). We thus obtain

$$\left(\Pi_c \begin{pmatrix} A^*(\gamma) \\ I_{E_n} \end{pmatrix} e^{U^*(\gamma)\cdot(z-(u-x))}\Pi_a^*\right)' = e^{-\rho\cdot(z-(u-x))} \tag{25}$$

Further we shall need $\phi_1(q_1) = \beta$ and $\phi_2^*(q_2) = \lambda/c$.

In order to find the function $f(x|u) := \mathbb{E}\left(e^{-\gamma\tilde{\tau}(u)}; R_{T-} \in dx\right)$, we consider the cases $x \le u$ and $x > u$ separately. We always have $M_{\tilde{\tau}(u)} \ge u - x$. From theorem 1 and the above equation (25) we compute the marginal density function

$$\begin{aligned}
\mathbb{E}(e^{-\gamma\tilde{\tau}(u)}; R_{T-} \in dx, x \le u) &= \int_{u-x}^{u} A(\gamma)e^{U(\gamma)z}\phi_1(q_1)e^{-\rho\cdot(z-u+x)}\phi_2^*(q_2)\, dz\; e^{Tx} \\
&= \int_{u-x}^{u} \frac{\beta - R}{\beta} e^{-Rz}\beta e^{-\rho\cdot(z-u+x)}\frac{\lambda}{c}\, dz\; e^{-\beta x} \\
&= \frac{\lambda}{c}(\beta - R)e^{-\rho\cdot(x-u)}e^{-\beta x}\int_{u-x}^{u} e^{-(R+\rho)z}\, dz \\
&= \frac{\lambda}{c}\frac{\beta - R}{R+\rho}e^{-(\rho+\beta)x}e^{\rho u}\left(e^{-(R+\rho)\cdot(u-x)} - e^{-(R+\rho)\cdot u}\right) \\
&= \frac{\lambda}{c}\frac{\beta - R}{R+\rho}e^{-(\rho+\beta)x}\left(e^{(R+\rho)\cdot x} - 1\right)e^{-Ru}
\end{aligned}$$

which coincides with equation (6.40) in [18]. Now for the case $x > u$. This means that the event $M_{\tilde{\tau}(u)} = 0$ may attain a positive probability. Corollary 1 and equation (25) yield for this case

$$\begin{aligned}
\mathbb{E}(e^{-\gamma\tilde{\tau}(u)}, R_{T-} \in dx, M_{\tilde{\tau}(u)} = 0) &= e^{-\rho\cdot(x-u)}\phi_2^*(q_2)e^{Tx} = e^{-\rho\cdot(x-u)}\frac{\lambda}{c}e^{-\beta x} \\
&= \frac{\lambda}{c}e^{-(\rho+\beta)x}e^{\rho u}
\end{aligned}$$

For the case of $x > u$ and $M_{\tilde{\tau}(u)} > 0$, theorem 1 yields

$$\begin{aligned}\mathbb{E}(e^{-\gamma\tilde{\tau}(u)}, R_{T-} \in dx, M_{\tilde{\tau}(u)} > 0, x > u) &= \int_0^u \frac{\beta - R}{\beta} e^{-Rz} \beta e^{-\rho\cdot(z-u+x)} \frac{\lambda}{c}\, dz\, e^{-\beta x} \\ &= \frac{\lambda}{c} \frac{\beta - R}{R+\rho} e^{-(\rho+\beta)x} e^{\rho u} \left(1 - e^{-(R+\rho)\cdot u}\right) \\ &= \frac{\lambda}{c} \frac{\beta - R}{R+\rho} e^{-(\rho+\beta)x} \left(e^{\rho u} - e^{-Ru}\right)\end{aligned}$$

Adding these results we obtain finally

$$\begin{aligned}\mathbb{E}(e^{-\gamma\tilde{\tau}(u)}, R_{T-} \in dx, x > u) &= \frac{\lambda}{c} e^{-(\rho+\beta)x} \left(e^{\rho u} + \frac{\beta - R}{R+\rho}(e^{\rho u} - e^{-Ru})\right) \\ &= \frac{\lambda}{c\cdot(R+\rho)} e^{-(\rho+\beta)x} \left((\beta+\rho)e^{\rho u} - (\beta - R)e^{-Ru}\right)\end{aligned}$$

which coincides with (6.39) in [18].

Example 6 The same net claim process as in (11) can be analysed by the approach in [14], example 8. There the time aspect is neglected and thus we set $\gamma = \gamma^* = 0$. Then we obtain $\rho = 0$ and $-R = (\lambda - c\beta)/c = \lambda/c - \beta$ by (3.12) and (4.24) in [18]. This implies $\beta - R = \lambda/c$, and theorem 1 now yields for $0 < z < u$

$$\begin{aligned}\mathbb{P}(M_{\tilde{\tau}(u)} \in dz, R_{T-} \in dx, |R_T| \in dy) &= \frac{\beta - R}{\beta} e^{-Rz} \beta\, e^{-\rho\cdot(z-(u-x))} \frac{\lambda}{c} e^{-\beta(x+y)} \beta \\ &= \frac{\lambda}{c} e^{-(\beta-\lambda/c)z} \frac{\lambda}{c} e^{-\beta\cdot(x+y)} \beta\end{aligned}$$

Corollary 1 yields further for $x > u$

$$\mathbb{P}(M_{\tilde{\tau}(u)} = 0, R_{T-} \in dx, |R_T| \in dy) = e^{-\rho\cdot(x-u)} \frac{\lambda}{c} e^{-\beta\cdot(x+y)} \beta = \frac{\lambda}{c} e^{-\beta\cdot(x+y)} \beta$$

The form obtained in [14], p.101, is

$$\mathbb{P}(M_{\tilde{\tau}(u)} \in dz, R_{T-} \in dx, |R_T| \in dy) = \frac{1}{c} \sum_{n\geq 0} \nu^{*n}(dz) \Pi_X(x+dy)dx$$

for $0 \leq z \leq u$, $x \geq u - z$ and $y > 0$, where $\Pi_X(dx) = \lambda e^{-\beta x}\beta\, dx$ is the Lévy measure, $\nu^{*0} = \delta_0$ (the Dirac measure on 0), and ν^{*n} denotes the n–fold convolution of the measure

$$\nu(dx) = \frac{1}{c}\Pi_X(x,\infty)dx = \frac{\lambda}{c} e^{-\beta x} dx$$

It is immediate that the results coincide for the singular case $M_{\tilde{\tau}(u)} = 0$. In order to show agreement for the case $M_{\tilde{\tau}(u)} > 0$, it suffices to show that

$$\sum_{n\geq 1} \nu^{*n}(dz) = \frac{\lambda}{c} e^{-(\beta-\lambda/c)z} dz \tag{26}$$

holds. Taking Laplace transforms on both sides, we obtain on the left–hand side

$$\int e^{-\alpha z} \sum_{n\geq 1} \nu^{*n}(dz) = \sum_{n\geq 1} \int e^{-\alpha z} \nu^{*n}(dz) = \sum_{n\geq 1} (L_\nu(\alpha))^n = \frac{L_\nu(\alpha)}{1 - L_\nu(\alpha)}$$

where $L_\nu(\alpha) = \lambda/c\ (\alpha+\beta)^{-1}$, $\alpha > 0$, is the Laplace transform of ν. On the right–hand side we obtain

$$\frac{\lambda}{c} \int e^{-\alpha z} e^{-(\beta-\lambda/c)z} dz = \frac{\lambda}{c} \cdot \frac{1}{\alpha+\beta-\frac{\lambda}{c}} = \frac{\frac{\lambda}{c} \cdot \frac{1}{\alpha+\beta}}{1 - \frac{\lambda}{c} \cdot \frac{1}{\alpha+\beta}}$$

which yields (26).

Example 7 We continue example 2. Since we do not need to determine the time variables, we can set $\gamma = \gamma^* = 0$. First of all we observe that Γ in [10], p.436, equals $A(0)$ by definition. Then equation (16) in [12] yields

$$U(0) = Q_{++} + Q_{+-}A(0) = T_{22} + T_{21}\Gamma = H$$

which is the notation in [10], p.437. Furthermore, $\phi(q_i) = q_i$ for all $i \in E_+$ and $P^{(+,-)} = P_{+-}$, which yields $\Delta_\phi P^{(+,-)} = Q_{+-} = T_{21}$. Regarding the time reversal $(\mathcal{X}^*, \mathcal{J}^*)$ of $(\mathcal{X}, \mathcal{J})$, the phase subspaces translate as $E_p^* = E_n$ and $E_-^* = E_+$. Write

$$Q^* = \Delta_\pi^{-1} Q' \Delta_\pi = \begin{pmatrix} Q_{--}^* & Q_{-+}^* \\ Q_{+-}^* & Q_{++}^* \end{pmatrix}$$

such that $Q_{++}^* = \Delta_{\pi_-}^{-1} Q'_{--} \Delta_{\pi_-}$ and $Q_{+-}^* = \Delta_{\pi_-}^{-1} Q'_{-+} \Delta_{\pi_+}$, denoting $\Delta_{\pi_-} = diag(\pi_i)_{i\in E_n}$ and $\Delta_{\pi_+} = diag(\pi_i)_{i\in E_+}$. Then $U^*(0) = Q_{++}^* + Q_{+-}^* A^*(0)$, where $A^*(0)$ has dimension $E_+ \times E_n$. Since $E_\sigma = E_\sigma^* = \emptyset$, we obtain $A^*(0) = \Delta_{\pi_+}^{-1} A(0)' \Delta_{\pi_-}$. Thus

$$U^*(0) = \Delta_{\pi_-}^{-1} \left(Q'_{--} + Q'_{-+} A(0)' \right) \Delta_{\pi_-}$$

and

$$\begin{aligned} \left(\Delta_{\pi_-} e^{U^*(0)\cdot(z-(u-x))} \Delta_{\pi_-}^{-1}\right)' &= \left(e^{(Q'_{--}+Q'_{-+}A(0)')\cdot(z-(u-x))}\right)' \\ &= e^{(Q_{--}+A(0)Q_{+-})\cdot(z-(u-x))} \\ &= e^{(T_{11}+\Gamma T_{21})\cdot(z-(u-x))} = e^{K\cdot(z-(u-x))} \end{aligned} \tag{27}$$

in the notation of [10], p.436. Finally, $\phi^*(q_i) = q_i$ for all $E_p^* = E_n$ and $P^{(c,+)} = P_{-+}$ such that $\Delta_{\phi^*} P^{(c,+)} = Q_{-+} = T_{12}$. For the jump the notations translate as $T = Q_{++} = T_{22}$ and $\eta = t_2$.

Hence the density function $h(u, x, y)$ of the surplus prior to ruin (the undershoot x) and the deficit at ruin (the overshoot y) is given by

$$h(u, x, y) = \int_{z=u-x}^{u} \mathbb{P}(M_{\tilde{\tau}(u)} \in dz, R_{T-} \in dx, |R_T| \in dy)$$

for $x < u$. Theorem 1 yields

$$\begin{aligned} h(u, x, y) &= \alpha A(0) \int_{z=u-x}^{u} e^{U(0)z} \Delta_\phi P^{(+,-)} \left(\Delta_{\pi_-} e^{U^*(\gamma^*)\cdot(z-(u-x))} \Delta_{\pi_-}^{-1}\right)' \, dz \\ &\quad \Delta_{\phi^*} P^{(c,+)} e^{T\cdot(x+y)} \eta \\ &= \alpha \Gamma e^{H\cdot(u-x)} \int_{w=0}^{x} e^{Hw} T_{21} e^{Kw} dw \; T_{12} e^{T_{22}\cdot(x+y)} t_2 \end{aligned}$$

after a substituion $w = z - (u - x)$. This coincides with equation (17) in [10], noting the definition (19) therein. The case $x > u$ includes the singular event $\{M_{\tilde{\tau}(u)} = 0\}$. For this, corollary 1 and (27) yield

$$\begin{aligned} \mathbb{P}(M_{\tilde{\tau}(u)} = 0, R_{T-} \in dx, |R_T| \in dy) &= \alpha \left(\Delta_{\pi_-} e^{U^*(\gamma^*)\cdot(z-(u-x))} \Delta_{\pi_-}^{-1}\right)' \\ &\quad \Delta_{\phi^*} P^{(c,+)} e^{T\cdot(x+y)} \eta \\ &= \alpha e^{K\cdot(x-u)} T_{12} e^{T_{22}\cdot(x+y)} t_2 \end{aligned} \tag{28}$$

The other part can be determined via theorem 1 as

$$\begin{aligned} \mathbb{P}(M_{\tilde{\tau}(u)} = 0, R_{T-} \in dx, |R_T| \in dy) &= \alpha \Gamma \int_{z=0}^{u} e^{Hz} T_{21} e^{K\cdot(z-(u-x))} dz T_{12} e^{T_{22}\cdot(x+y)} t_2 \\ &= \alpha \Gamma R(u) e^{K\cdot(x-u)} T_{12} e^{T_{22}\cdot(x+y)} t_2 \end{aligned} \tag{29}$$

where $R(u) = \int_{w=0}^{u} e^{Hw} T_{21} e^{Kw} dw$ is the notation in [10], equation (19). Adding the two results (28) and (29) yields the same expression as (18) in [10].

References

[1] S. Ahn and A. L. Badescu. On the analysis of the Gerber-Shiu discounted penalty function for risk processes with Markovian arrivals. *Insurance: Mathematics and Economics*, 41(2):234–249, 2007.

[2] S. Asmussen. Ladder heights and the Markov-modulated M/G/1 queue. *Stoch. Proc. Appl.* , 37:313–326, 1991.

[3] S. Asmussen. Stationary distributions for fluid flow models with or without Brownian motion. *Stochastic Models*, 11:1–20, 1995.

[4] S. Asmussen. *Ruin probabilities*. Singapur: World Scientific, 2000.

[5] S. Asmussen. *Applied Probability and Queues*. New York etc.: Springer, 2003.

[6] S. Asmussen, F. Avram, and M. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and their Applications*, 109:79–111, 2004.

[7] S. Asmussen and O. Kella. A multi-dimensional martingale for Markov additive processes and its applications. *Adv. Appl. Prob.*, 32:376–393, 2000.

[8] S. Asmussen and G. Koole. Marked point processes as limits of Markovian arrival streams. *J. Appl. Probab.*, 30(2):365–372, 1993.

[9] A. Badescu, L. Breuer, A. da Silva Soares, G. Latouche, M.-A. Remiche, and D. Stanford. Risk processes analyzed as fluid queues. *Scandinavian Actuarial Journal*, pages 127–141, 2005.

[10] A. Badescu, L. Breuer, S. Drekic, G. Latouche, and D. Stanford. The surplus prior to ruin and the deficit at ruin for a correlated risk process. *Scandinavian Actuarial Journal*, pages 433–445, 2005.

[11] J. Bertoin. *Lévy Processes*. Cambridge University Press, Cambridge etc., 1996.

[12] L. Breuer. First passage times for Markov–additive processes with positive jumps of phase–type. *J. Appl. Prob.*, 45(3):779–799, 2008.

[13] S. Chiu and C. Yin. The time of ruin, the surplus prior to ruin and the deficit at ruin for the classical risk process perturbed by diffusion. *Insurance: Mathematics and Economics*, 33:59–66, 2003.

[14] R. Doney and A. Kyprianou. Overshoots and undershoots of Lévy processes. *Annals of Applied Probability*, 16(1):91–106, 2006.

[15] J. Garrido and M. Morales. On the expected discounted penalty function for Lévy risk processes. *North American Actuarial Journal*, 10(4):196–216, 2006.

[16] H. Gerber and B. Landry. On the discounted penalty at ruin in a jump-diffusion and the perpetual put option. *Insurance: Mathematics and Economics*, 22:263–276, 1998.

[17] H. Gerber and E. Shiu. The joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. *Insurance: Mathematics and Economics*, 21:129–137, 1997.

[18] H. Gerber and E. Shiu. On the time value of ruin. *North American Actuarial Journal*, 2(1):48–78, 1998.

[19] H. Gerber and E. Shiu. The time value of ruin in a Sparre Andersen model. *North American Actuarial Journal*, 9(2):49–84, 2005.

[20] M. Jacobsen. Martingales and the distribution of the time to ruin. *Stoch. Process. Appl.*, 107:29–51, 2003.

[21] M. Jacobsen. The time to ruin for a class of Markov additive risk processes with two-sided jumps. *Adv. Appl. Prob.*, 37:963–992, 2005.

[22] S. Li and J. Garrido. The Gerber-Shiu function in a Sparre Andersen risk process perturbed by diffusion. *Scand. Act. J.*, pages 161–186, 2005.

[23] Y. Lu and C. C.-L. Tsai. The expected discounted penalty at ruin for a Markov-modulated risk process perturbed by diffusion. *North American Actuarial Journal*, 11(2):136–152, 2007.

[24] Y. Lu, R. Wu, and R. Xu. The joint distributions of some extrema for the classical risk process perturbed by diffusion. *Chinese Journal of Engineering Mathematics*, 23(2):355–360, 2006.

[25] D. M. Lucantoni. New results on the single server queue with a batch Markovian arrival process. *Commun. Stat., Stochastic Models*, 7(1):1–46, 1991.

[26] M. F. Neuts. A versatile Markovian point process. *J. Appl. Probab.*, 16:764–774, 1979.

[27] A. Ng and H. Yang. On the joint distribution of surplus before and after ruin under a Markovian regime switching model. *Stoch. Proc. Appl.*, 116:244–266, 2006.

[28] E. Pecherskii and B. Rogozin. On joint distributions of random variables associated with fluctuations of a process with independent increments. *Theory of Probability and its Applications*, 14(3):410–423, 1969.

[29] R. Schassberger. *Warteschlangen.* Wien-New York: Springer-Verlag, 1973.