The distribution of the total dividends for Markov-additive risk models under a barrier strategy

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Abstract

For a Markov-additive risk model under a barrier strategy of dividend payments, we derive the joint distribution of the total pay-out before ruin and the last time that dividends are paid. The method of analysis is based on two recent results in excursion theory for Markov-additive processes.

1 Introduction

Next to the time of ruin and the surplus immediately before, the question of dividend payments is one of the classical problems in the theory of insurance risk. Progress has been made bay analysing more general risk models and deriving higher moments or even the distribution of the total dividends paid before ruin, rather than mere expectations. The former allows for more realistic statistical model fitting, while the latter provides more detailed statements on the dividend payments before ruin (e.g. they will exceed a certain amount with a 90% probability). The recent crisis in the finance and insurance sector underlines the importance of realistic and mathematically sound models as a guideline for actuarial decision making.

Most models for dividend payment employ the so-called barrier strategy. As long as the risk reserve remains below a certain threshold, no dividends are paid out. Upon reaching this threshold, any additional premium income (that would lead to exceeding the threshold) is paid out as dividends immediately. This barrier strategy will be employed in the present paper, too. Since it implies that the admissible risk reserve has an upper bound, ruin will always be certain. The usual optimisation problem here is to set the threshold such that the dividend payments before ruin are maximised.

There has been a lot of recent work dedicated to the question of dividend payments. The barrier strategy has been shown to be optimal for a diffusion model in [7]. Newer results and further references on optimality can be found in [19]. For the compound Poisson model the expected discounted dividend payment before ruin have been derived in [13]. The moment generating function of the discounted dividends is studied in [14, 1, 18] for various generalisations of the compound Poisson model. The problem is approached in terms of Lévy processes by [17, 22]. Markov-modulation of the compound Poisson model with phase-type claim sizes yields stochastic fluid flow models, for which results are available in [8, 9]. The expected discounted dividend payments for Markov-additive risk models has been derived in [12] using vector-valued martingales. Dividend payments under strategies different from the barrier strategy have been analysed in [2, 3].

The present paper deals with Markov-additive rsik models. These combine the two generalising features of perturbation and Markov-modulation, with the single restriction that the claim sizes have phase-type distributions. A further advance of this paper may be seen in the result giving an explicit expression for the distribution of the total dividend payments before ruin, rather than expectations or higher moments only.

The paper is structured as follows. The following section defines the Markov-additive risk model, while section 3 contains preliminary results that will be needed in the sequel. The main result is presented in the last section.

2 The model

Consider a risk reserve process with initial capital $u \geq 0$ and claims occurring like a Markovian point process (MPP).[1] It is shown in [6] that the class of MPPs is dense within the class of marked point processes. Thus we incur no serious restriction in generality. Denote the claim arrival process by $(\mathcal{N}, \tilde{\mathcal{J}}) = ((N_t, \tilde{J}_t) : t \geq 0)$ and the phase space for $\tilde{\mathcal{J}}$ by $\tilde{E}$. Assume that the claim sizes have a phase–type distribution and denote the nth claim size by C_n, $n \in \mathbb{N}$. By [23] the class of phase–type distributions is dense within the class of all distributions on the positive real numbers.

We assume further that the premium income between claims can be modelled by a Brownian motion, where the parameters $\tilde{\mu}_i$ (drift) and $\tilde{\sigma}_i$ (variation) at time t may depend on the current phase $\tilde{J}_t = i$ of the claim arrival process. For insurance risk we typically

[1] see [21, 20]. This has traditionally been called Markovian arrival process and abbreviated as MAP. Since we use the shortcut MAP for the more general class of Markov–additive processes already, we prefer to use the term Markovian point process and the abbreviation MPP instead.

have $\tilde{\mu}_i > 0$ for all $i \in \tilde{E}$. We shall allow $\tilde{\sigma}_i = 0$ for some (or possibly all) phases, under which condition the Brownian motion becomes a linear drift. Then the process of premium income is a Markov–modulated Brownian motion which we denote by $(\mathcal{B}, \tilde{\mathcal{J}}) = ((B_t, \tilde{J}_t) : t \geq 0)$. We assume that $B_0 = 0$.

Note that $\tilde{\mathcal{J}}$ here is the same as for the claim arrival process $(\mathcal{N}, \tilde{\mathcal{J}})$. This is no restriction in modelling power as we can choose identical parameters $(\tilde{\mu}_i, \tilde{\sigma}_i) = (\tilde{\mu}_j, \tilde{\sigma}_j)$ for different phases $i \neq j \in \tilde{E}$ and map two different environments for the premium income and the claim arrivals by using Kronecker products. Rather on the contrary, a common phase space enables us to model correlations between claim arrivals, claim sizes, and the premium income.

With the definitions above, the risk reserve process $\mathcal{R} = (R_t : t \geq 0)$ is given by

$$R_t = u + B_t - \sum_{n=1}^{N_t} C_n$$

for $t \geq 0$. The process $(\mathcal{R}, \tilde{\mathcal{J}})$ is a MAP with phase–type jumps.

Let $b \geq 0$ denote the barrier level beyond which the risk reserve is paid out as dividends immediately. We may assume $u \leq b$ without loss of generality since $b < u$ would entail an immediate pay-out of a risk reserve of $u - b$ in dividends and the risk process would continue with initial surplus b. Define the stopping times

$$\tau(0, b) := \inf\{t \geq 0 : R_t < 0 \quad \text{or} \quad R_t > b\} \tag{1}$$

and

$$\tau(0) := \inf\{t \geq 0 : R_t < 0\}$$

the latter being the time of ruin. Let D denote the total dividends paid until ruin, i.e. (with $\mathbb{I}_A$ denoting the indicator function of a set A)

$$D = \int_0^{\tau(0)} \mathbb{I}_{\{R_t = b\}} \, dt$$

is the local time of $\mathcal{R}$ at b before $\tau(0)$, cf. section 2.1 in [12]. Hence we can also write $D = L_{\tau(0)}(b)$, where $\mathcal{L}(b) = (L_t(b) : t \geq 0)$ is the local time process of $\mathcal{R}$ at the level b, i.e. $L_t(b) := \int_0^t \mathbb{I}_{\{R_t = b\}} \, dt$ for all $t \geq 0$. Denote the inverse local time by

$$L_x^{-1}(b) := \inf\{t \geq 0 : L_t(b) \geq x\}$$

for $x \geq 0$.

In the present paper we shall determine the joint distribution of D and $L_D^{-1}(b)$ in the form of an expression for

$$\bar{F}(x,\gamma) := \mathbb{E}\left(e^{-\gamma L_x^{-1}(b)}; D > x\right)$$

where $x \geq 0$ and $\gamma \geq 0$. Note that $L_D^{-1}(b)$ signifies the time of the last dividend payment and may be strictly smaller than $\tau(0)$, the time of ruin. Inequality implies the existence of phases with a diffusion component or a negative drift, i.e. $E_\sigma \cup E_n \neq \emptyset$.

Looking at the problem from the angle described above, we first need to collect some necessary preliminary results for MAPs from existing literature. This shall be the purpose of the next section.

3 Preliminaries

3.1 Markov–additive processes with phase–type jumps

Let $\tilde{\mathcal{J}} = (\tilde{J}_t : t \geq 0)$ be an irreducible Markov (jump) process with finite state space $\tilde{E}$ and infinitesimal generator matrix $\tilde{Q} = (\tilde{q}_{ij})_{i,j\in\tilde{E}}$. We call $\tilde{J}_t$ the phase at time $t \geq 0$ (another common name is regime). Define the real–valued process $\tilde{\mathcal{X}} = (\tilde{X}_t : t \geq 0)$ as evolving like a Lévy process $\tilde{\mathcal{X}}^{(i)}$ with parameters μ_i (drift), σ_i^2 (variation), and ν_i (Lévy measure) during intervals when the phase equals $i \in \tilde{E}$. Whenever $\tilde{\mathcal{J}}$ jumps from a state $i \in \tilde{E}$ to another state $j \in \tilde{E}$, this may be accompanied by a jump of $\tilde{\mathcal{X}}$ with some distribution function F_{ij}. Then the two–dimensional process $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ is called a Markov–additive process (or shortly MAP). A MAP can also be defined by the following property: $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$ is a Markov process such that

$$\mathbb{E}(f(\tilde{X}_{t+s} - \tilde{X}_t)g(\tilde{J}_{t+s})|\mathcal{F}_t, \tilde{J}_t = i) = \mathbb{E}(f(\tilde{X}_s)g(\tilde{J}_s)|\tilde{X}_0 = 0, \tilde{J}_0 = i)$$

holds for all $s, t > 0$ and $i \in E$, where f and g are measurable functions and $(\mathcal{F}_t : t \geq 0)$ is the canonical filtration of $(\tilde{\mathcal{X}}, \tilde{\mathcal{J}})$. For a textbook introduction to MAPs see [4], chapter XI.

Denote the indicator function of a set A by $\mathbb{I}_A$. We assume that the Lévy measures ν_i have the form

$$\nu_i(dx) = \lambda_i^+ \mathbb{I}_{\{x>0\}}\, \alpha^{(ii)+} \exp(T^{(ii)+}x)\eta^{(ii)+} + \lambda_i^- \mathbb{I}_{\{x<0\}}\, \alpha^{(ii)-} \exp(-T^{(ii)-}x)\eta^{(ii)-}$$

for all $i \in \tilde{E}$, where $\lambda_i^\pm \geq 0$ and $(\alpha^{(ii)\pm}, T^{(ii)\pm})$ are representations of phase–type distributions without an atom at 0. The $\eta^{(ii)\pm} := -T^{(ii)\pm}\mathbf{1}$ are called the exit vectors, where

$\mathbf{1}$ denotes a column vector of appropriate dimension with all entries being 1. This means that the jump process induced by the Lévy measure ν_i is compound Poisson with jump sizes of a doubly phase–type distribution. Denote the order of $PH(\alpha^{(ii)\pm}, T^{(ii)\pm})$ by $m_{ii}^{\pm}$. Further write $\lambda_i := \lambda_i^+ + \lambda_i^-$.

Likewise, let p_{ij}^+ (resp. p_{ij}^-) denote the probability that a positive (resp. negative) jump is induced by a phase change from $i \in \tilde{E}$ to $j \in \tilde{E}$, and assume that these jumps have a $PH(\alpha^{(ij)\pm}, T^{(ij)\pm})$ distribution without an atom at 0. Note that $p_{ij}^+ + p_{ij}^- \leq 1$ for all $i, j \in \tilde{E}$. Let $m_{ij}^{\pm}$ denote the order of $PH(\alpha^{(ij)\pm}, T^{(ij)\pm})$ and define $\eta^{(ij)\pm} := -T^{(ij)\pm}\mathbf{1}$.

The class of Markov–additive processes with these assumptions of phase–type jumps is dense within the class of all MAPs, see [5], proposition 1. The main advantage of the restriction on the jump distributions is the possibility of transforming the jumps into a succession of linear pieces of exponential duration (each with slope 1 or -1) and retrieving the original process via a simple time change.

This is done in the following way. Without the jumps, the Lévy process $\tilde{\mathcal{X}}^{(i)}$ during a phase $i \in \tilde{E}$ is either a linear drift (of positive or negative slope $\mu_i \in \mathbb{R}$) or a Brownian motion (with parameters $\sigma_i > 0$ and $\mu_i \in \mathbb{R}$). Considering this MAP (without the jumps) we can partition its phase space $\tilde{E}$ into the subspaces E_p (for positive drifts), E_σ (for Brownian motions), and E_n (for negative drifts). Then we introduce two new phase spaces

$$E_\pm := \{(i, j, k, \pm) : i, j \in E_p \cup E_\sigma \cup E_n, 1 \leq k \leq m_{ij}^{\pm}\}$$

to model the jumps. This leads to the cumulant functions

$$\psi_i(\alpha) = \begin{cases} \pm\alpha, & i \in E_\pm \\ \mu_i\alpha, & i \in E_p \cup E_n \\ \frac{1}{2}\sigma_i^2\alpha^2 + \mu_i\alpha, & i \in E_\sigma \end{cases} \tag{2}$$

where we assume that $\mu_i > 0$ for $i \in E_p$ and $\mu_i < 0$ for $i \in E_n$. Note that this assumption excludes the possibility of a phase i with parameters $\mu_i = \sigma_i = 0$ which would govern the zero process. Since jumps are modelled by additional phases, this excludes further any phase $i \in \tilde{E}$ under which $\tilde{\mathcal{X}}^{(i)}$ would be a compound Poisson process.

We shall order the new phase space $E = E_+ \cup E_p \cup E_\sigma \cup E_n \cup E_-$ such that $i_+ < i_p < i_\sigma < i_n < i_-$ for phases $i_* \in E_*$. Let $E_c := E_p \cup E_\sigma \cup E_n$ denote the subspace of E that contains all phases under which the real time movements are continuous. The modified phase process $\mathcal{J}$ is determined by its generator $Q = (q_{ij})_{i,j \in E}$. For this the construction above yields

$$q_{ih} = \begin{cases} \tilde{q}_{ii} - \lambda_i, & h = i \in E_c \\ \tilde{q}_{ij} \cdot (1 - p_{ij}^+ - p_{ij}^-), & h \in E_c, h \neq i \\ \lambda_i^{\pm} \alpha_k^{(ii)\pm}, & h = (i, i, k, \pm) \\ \tilde{q}_{ij} \cdot p_{ij}^{\pm} \cdot \alpha_k^{(ij)\pm}, & h = (i, j, k, \pm) \end{cases} \tag{3}$$

for $i \in E_c$ as well as

$$q_{(i,j,k,\pm),(i,j,l,\pm)} = T_{kl}^{(ij)\pm} \qquad \text{and} \qquad q_{(i,j,k,\pm),j} = \eta_k^{(ij)\pm} \tag{4}$$

for $i, j \in E_c$ and $1 \leq k, l \leq m_{ij}^{\pm}$. For later use we define $q_i := -q_{ii}$ for all $i \in E$.

Denote the MAP constructed in such a way by $(\mathcal{X}, \mathcal{J})$. The original level process $\tilde{\mathcal{X}}$ is retrieved via the time change

$$c(t) := \int_0^t 1_{J_s \in E_c} \, ds \quad \text{and} \quad \tilde{X}_{c(t)} = X_t \tag{5}$$

for all $t \geq 0$ Denote the generalised inverse of the function c by c^{-1}.

The inverses of the cumulant functions ψ_i can be given explicitly as

$$\phi_i(\beta) = \begin{cases} \pm\beta, & i \in E_{\pm} \\ \frac{\beta}{\mu_i}, & i \in E_p \cup E_n \\ \frac{1}{\sigma_i}\sqrt{2\beta + \frac{\mu_i^2}{\sigma_i^2}} - \frac{\mu_i}{\sigma_i^2}, & i \in E_\sigma \end{cases} \tag{6}$$

We shall, however, use them only for the so–called ascending phases $i \in E_a := E_+ \cup E_p \cup E_\sigma$, cf. [10], chapter VII.

Example 1 We consider the classical compound Poisson model. Inter–claim times and claim sizes are iid exponential with parameter $\lambda > 0$ and $\beta > 0$, respectively. The rate of premium income is $c > 0$. The net profit condition is then $\lambda/(c\beta) < 1$. Denote the initial risk reserve by $u \geq 0$. This model has been examined in [13]. The risk reserve at time $t \geq 0$ is given by

$$\tilde{X}_t = u + ct - \sum_{n=0}^{N_t} C_n \tag{7}$$

where $(N_t : t \geq 0)$ is a Poisson process with intensity λ and the C_n, $n \in \mathbb{N}$, are iid random variables with exponential distribution of parameter β.

The process of accumulated claims can be analysed as a MAP with exponential (and hence phase–type) positive jumps with parameter β. We further obtain the MAP $(\mathcal{X}, \mathcal{J})$

as follows. Let the phase space be given by $E_p = \{1\}$, $E_- = \{2\}$, and $E_\sigma = \emptyset$. The parameters are given by $\sigma_1 = \sigma_2 = 0$, $\mu_1 = c$, $\mu_2 = -1$, $\nu_1 = \nu_2 = \mathbf{0}$, and

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \beta & -\beta \end{pmatrix}$$

The initial state is $(X_0, J_0) = (u, 1)$.

3.2 First passage times

Of central use in the present paper will be the recent derivation of the Laplace transforms for the first passage times of MAPs with phase–type jumps as given in [11]. We call the phases $i \in E_d := E_n \cup E_-$ descending. Define $\tilde{\tau}(x) := \inf\{t \geq 0 : \tilde{X}_t > x\}$ for all $x \geq 0$ and assume that $X_0 = 0$. Note that this is the first passage time over the level x for the original MAP $\tilde{\mathcal{X}}$, meaning that we do not count the time spent in jump phases $i \in E_\pm$. We write $\tau(x) = c^{-1}(\tilde{\tau}(x))$ for the corresponding first passage time of the modified level process $\mathcal{X}$. For $\gamma \geq 0$ denote

$$\mathbb{E}_{ij}(e^{-\gamma\tilde{\tau}(x)}) := \mathbb{E}(e^{-\gamma\tilde{\tau}(x)}; J_{\tau(x)} = j | J_0 = i, X_0 = 0)$$

for all $i, j \in E$. Let $\mathbb{E}(e^{-\gamma\tilde{\tau}(x)})$ denote the matrix with these entries and write

$$\mathbb{E}(e^{-\gamma\tilde{\tau}(x)}) = \begin{pmatrix} \mathbb{E}_{(a,a)}(e^{-\gamma\tilde{\tau}(x)}) & \mathbb{E}_{(a,d)}(e^{-\gamma\tilde{\tau}(x)}) \\ \mathbb{E}_{(d,a)}(e^{-\gamma\tilde{\tau}(x)}) & \mathbb{E}_{(d,d)}(e^{-\gamma\tilde{\tau}(x)}) \end{pmatrix}$$

in obvious block notation with respect to the subspaces $E_a = E_+ \cup E_p \cup E_\sigma$ (ascending phases) and $E_d = E_n \cup E_-$ (descending phases).

Since a first passage to a level above cannot occur in a descending phase, we obtain first $\mathbb{P}(J_{\tau(x)} = j) = 0$ for all $j \in E_d$ and thus $\mathbb{E}_{(d,d)}(e^{-\gamma\tilde{\tau}(x)}) = \mathbb{E}_{(a,d)}(e^{-\gamma\tilde{\tau}(x)}) = \mathbf{0}$ where $\mathbf{0}$ denotes a zero matrix of suitable dimension. Equation (6) in [11] states that

$$\mathbb{E}_{(d,a)}(e^{-\gamma\tilde{\tau}(x)}) = A(\gamma)e^{U(\gamma)x} \quad \text{and} \quad \mathbb{E}_{(a,a)}(e^{-\gamma\tilde{\tau}(x)}) = e^{U(\gamma)x} \tag{8}$$

for some sub–generator matrix $U(\gamma)$ of dimension $E_a \times E_a$ and a sub–transition matrix $A(\gamma)$ of dimension $E_d \times E_a$. Altogether we can write

$$\mathbb{E}(e^{-\gamma\tilde{\tau}(x)}) = \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} \begin{pmatrix} e^{U(\gamma)x} & \mathbf{0} \end{pmatrix} \tag{9}$$

where I_a denotes the identity matrix of dimension $E_a \times E_a$.

Write $\Delta_q := diag(q_i)_{i\in E}$ and let $P = \Delta_q^{-1}Q + I$ denote the transition matrix of phase changes. Note that $p_{ii} = 0$ for all $i \in E$. According to theorem 3 in [11], $A(\gamma)$ and $U(\gamma)$ satisfy the following equations:

$$e_h'U(\gamma) = \sum_{l=1}^{m_{ij}^+} T_{kl}^{(ij)+} e_{(i,j,l,+)}' + \eta_k^{(ij)+} e_j' \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} \qquad \text{for } h = (i,j,k,+) \in E_+,$$

$$e_i'U(\gamma) = -\phi_i(q_i+\gamma)e_i' + \phi_i(q_i) \sum_{j\in E} p_{ij}\; e_j' \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} L_i(-U(\gamma)) \quad \text{for } i \in E_p \cup E_\sigma,$$

$$e_i'A(\gamma) = \sum_{j\in E, j\neq i} q_{ij} e_j' \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} \left((q_i+\gamma)I - \psi_i(-U(\gamma))\right)^{-1} \qquad \text{for } i \in E_n,$$

$$e_i'A(\gamma) = \sum_{j\in E, j\neq i} q_{ij} e_j' \begin{pmatrix} I_a \\ A(\gamma) \end{pmatrix} \left(q_i I - \psi_i(-U(\gamma))\right)^{-1} \qquad \text{for } i \in E_-.$$

For the MAP $(\mathcal{X}, \mathcal{J})$ with continuous level process, the matrix function

$$L_i(-U(\gamma)) = \frac{q_i}{\phi_i(q_i)} \cdot (\phi_i(q_i+\gamma)I + U(\gamma)) \cdot ((q_i+\gamma)I - \psi_i(-U(\gamma)))^{-1}$$

can be simplified considerably. For $i \in E_\sigma$, the same arguments as in [11], example 2, lead to

$$L_i(-U(\gamma)) = \phi_i^*(q_i) \cdot (\phi_i^*(q_i+\gamma)I - U(\gamma))^{-1} \tag{10}$$

with

$$\phi_i^*(\beta) = \frac{1}{\sigma_i}\sqrt{2\beta + \frac{\mu_i^2}{\sigma_i^2}} + \frac{\mu_i}{\sigma_i^2} \tag{11}$$

Furthermore, $L_i(-U(\gamma)) = I$ for $i \in E_p$ (see example 3 in [11]), while according to (2) $\psi_i(-U(\gamma)) = -\mu_i U(\gamma)$ for $i \in E_n$, and $\psi_i(-U(\gamma)) = U(\gamma)$ for $i \in E_-$. Hence the equations above involve rather simple expressions only.

Considering (6), the matrices $A(\gamma)$ and $U(\gamma)$ can be determined by successive approximation as the limit of the sequence $((A_n, U_n) : n \geq 0)$ with initial values $A_0 := \mathbf{0}$,

$U_0 := -diag(\phi_i(q_i + \gamma))_{i\in E}$ and the following iteration:

$$e'_h U_{n+1} = \sum_{l=1}^{m_{ij}^+} T_{kl}^{(ij)+} e'_{(i,j,l,+)} + \eta_k^{(ij)+} e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} \qquad \text{for } h = (i,j,k,+) \in E_+,$$

$$e'_i U_{n+1} = -\frac{q_i + \gamma}{\mu_i} e'_i + \frac{1}{\mu_i} \sum_{j\in E, j\neq i} q_{ij}\; e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} \qquad \text{for } i \in E_p,$$

$$e'_i A_{n+1} = \sum_{j\in E, j\neq i} q_{ij} e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} ((q_i + \gamma)I + \mu_i U_n)^{-1} \qquad \text{for } i \in E_n,$$

$$e'_i A_{n+1} = \sum_{j\in E, j\neq i} q_{ij} e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} (q_i I - U_n)^{-1} \qquad \text{for } i \in E_-, \text{ and}$$

$$e'_i U_{n+1} = -\phi_i(q_i + \gamma) e'_i + \frac{2}{\sigma_i^2} \sum_{j\in E, j\neq i} q_{ij}\; e'_j \begin{pmatrix} I_a \\ A_n \end{pmatrix} (\phi_i^*(q_i + \gamma)I - U_n)^{-1}$$

for $i \in E_\sigma$. For the last equality the relation $\phi_i(q_i)\phi_i^*(q_i) = 2q_i/\sigma_i^2$ has been used. Note that the only difference between the iterations for E_n and E_- is the missing γ in the last factor for E_-, reflecting that we do not discount the time for phases $i \in E_-$ as they are jump phases in real time.

Example 2 Continuing example 1, first note that phase 2 represents the downward jumps and we will not discount the time during sojourns in it. According to the formulas above, the Laplace transform of the first passage time $\tilde{\tau}(x) := \inf\{t \geq 0 : \tilde{X}_t > x\}$ to a level $x > u$ is given by

$$\mathbb{E}(e^{-\gamma\tilde{\tau}(x)}) = e^{U(\gamma)\cdot(x-u)} \quad \text{where} \quad U(\gamma) = -\frac{\lambda + \gamma}{c} + \frac{\lambda}{c} A(\gamma), \quad A(\gamma) = \frac{\beta}{\beta - U(\gamma)}$$

Noting that $U(\gamma)$ must be negative, this resolves as

$$U(\gamma) = \frac{1}{2c}\left(c\beta - \gamma - \lambda - \sqrt{(c\beta - \gamma - \lambda)^2 + 4c\beta\gamma}\right)$$

cf. equation (3.12) in [13], noting that γ is denoted as δ there.

3.3 The two-sided exit problem

Now we consider the exit time $\tau(0,b)$ from the interval $[0,b]$ as defined in (1). We aim to find an expression for

$$\Psi_{ij}^+(b|x) := \mathbb{E}\left(e^{-\gamma\tau(0,b)}; X_{\tau(0,b)} = b, J_{\tau(0,b)} = j | J_0 = i, X_0 = x\right)$$

for $x \in [0,b]$ and $i,j \in E$. Clearly $\Psi^+_{ij}(b|x) = 0$ for $j \in E_d$ since an exit over the upper boundary can occur only in an ascending phase. Define the matrix $\Psi^+(b|x) := (\Psi^+_{ij}(b|x))_{i\in E, j\in E_a}$. A formula for $\Psi^+(b|x)$ has been derived in [15]. In order to state it we need some additional notation.

Let $(\mathcal{X}^+, \mathcal{J})$ denote the MAP as constructed in section 3.1 and define the process $\mathcal{X}^- = (X^-_t : t \geq 0)$ by $X^-_t := -X^+_t$ for all $t > 0$ given that $X^+_0 = X^-_0 = 0$. Thus $(\mathcal{X}^-, \mathcal{J})$ is the negative of $(\mathcal{X}^+, \mathcal{J})$. The two processes have the same generator matrix Q for $\mathcal{J}$, but the cumulant functions of the Lévy process governed by phase $i \in E$ are different and relate as $\psi^-_i(\alpha) = \psi^+_i(-\alpha)$. Denoting variation and drift parameters for $\mathcal{X}^\pm$ by $\sigma^\pm_i$ and $\mu^\pm_i$, respectively, this means $\sigma^+_i = \sigma^-_i$ and $\mu^-_i = -\mu^+_i$ for all $i \in E$. This of course implies that phases $i \in E_+ \cup E_p$ (resp. $i \in E_- \cup E_n$) are descending (resp. ascending) phases for $\mathcal{X}^-$.

Let $A^\pm(\gamma)$ and $U^\pm(\gamma)$ denote the matrices that determine the first passage times in (9). In order to simplify notation, we shall from now on write $A^\pm = A^\pm(\gamma)$ and $U^\pm = U^\pm(\gamma)$ except in cases when we wish to underline the dependence on γ.

Example 3 If $(\mathcal{X}^+, \mathcal{J})$ is the MAP as constructed in example 1, then $(\mathcal{X}^-, \mathcal{J})$ would be the net claim process for the compound Poisson model. As shown in [11], example 5, the Laplace transform of the first passage time $\tilde{\tau}^-(x) := \inf\{t \geq 0 : \tilde{X}^-_t > x\}$ to a level $x > 0$ is given by

$$\mathbb{E}(e^{-\gamma\tilde{\tau}^-(x)}) = A^- e^{U^- x} \quad \text{where} \quad A^- = \frac{\beta - R}{\beta}, \quad U^- = -R$$

and

$$-R = \frac{1}{2c}\left(\lambda + \gamma - c\beta - \sqrt{(c\beta - \gamma - \lambda)^2 + 4c\beta\gamma}\right)$$

which coincides with equation (4.24) in [13], noting that γ is denoted as δ there.

Define the matrices

$$C^+ := \begin{pmatrix} \mathbf{0} & I_{E_\sigma} \\ A^+ & \end{pmatrix} \qquad \text{and} \qquad C^- := \begin{pmatrix} A^- & \\ I_{E_\sigma} & \mathbf{0} \end{pmatrix}$$

of dimensions $(E_\sigma \cup E_d) \times E_a$ and $E_a \times (E_\sigma \cup E_d)$, respectively. Further define

$$W^+ := \begin{pmatrix} I_{E_a} \\ A^+ \end{pmatrix} \qquad \text{and} \qquad W^- := \begin{pmatrix} A^- \\ I_{E_\sigma \cup E_d} \end{pmatrix}$$

which are matrices of dimensions $E \times E_a$ and $E \times (E_\sigma \cup E_d)$. Finally, let $Z^\pm := C^\pm e^{U^\pm \cdot b}$. Then equation (23) in [15] states that

$$\Psi^+(b|x) = \left(W^+ e^{U^+\cdot(b-x)} - W^- e^{U^-\cdot x} Z^+\right) \cdot \left(I - Z^- Z^+\right)^{-1} \tag{12}$$

for $0 \le x \le b$.

Remark 1 Noting that $\left(I - Z^- Z^+\right)^{-1} = \sum_{n=0}^{\infty} \left(Z^- Z^+\right)^n$ and $Z^- Z^+$ represents a crossing of the interval $[0, b]$ from b to 0 and back, this formula has a clear probabilistic interpretation. The term $W^+ e^{U^+ \cdot (b-x)}$ simply yields the event that $\mathcal{X}$ exits from b. The correction term $W^- e^{U^- \cdot x} Z^+$ refers to the event that $\mathcal{X}$ descends below 0 before exiting from b. Multiplication by $\left(I - Z^- Z^+\right)^{-1}$ yields all possible combinations with any number of subsequent (down and up) crossings over the complete interval $[0, b]$.

Remark 2 Since $Z^+ = C^+ e^{U^+ \cdot b}$ we can write $\Psi^+(b|x)$ in the form

$$\Psi^+(b|x) = \left(W^+ e^{-U^+ \cdot x} - W^- e^{U^- \cdot x} C^+\right) \cdot \left(e^{-U^+ \cdot b} - C^- e^{U^- \cdot b} C^+\right)^{-1}$$

This comes closer to the usual expression of the exit time distribution in terms of scale functions. For instance, if $\mathcal{X}$ is a standard Wiener process, specified by $|E| = 1$, $\sigma = 1$, $\mu = 0$, then $U^+ = U^- = -\sqrt{2\gamma}$ and

$$\Psi^+(b|x) = \frac{e^{\sqrt{2\gamma} x} - e^{-\sqrt{2\gamma} x}}{e^{\sqrt{2\gamma} b} - e^{-\sqrt{2\gamma} b}} = \frac{\sinh\left(\sqrt{2\gamma} x\right)}{\sinh\left(\sqrt{2\gamma} b\right)}$$

cf. [16], exercise 8.2(iv), which states that the γ-scale function for the standard Wiener process is $W(x) = \sqrt{2/\gamma} \cdot \sinh\left(\sqrt{2\gamma} x\right)$.

Example 4 Another example is Brownian motion with variation $\sigma > 0$ and drift μ. We then obtain

$$U^+ = \frac{\mu - \sqrt{\mu^2 + 2\gamma\sigma^2}}{\sigma^2} =: -r \qquad \text{and} \qquad U^- = \frac{-\mu - \sqrt{\mu^2 + 2\gamma\sigma^2}}{\sigma^2} =: s$$

and

$$\Psi^+(b|x) = \frac{e^{rx} - e^{sx}}{e^{rb} - e^{sb}}$$

cf. [14], (2.12 - 2.15), where a proportional equivalent of the γ-scale function is given as $g(x) = e^{rx} - e^{sx}$.

Example 5 Yet another example is the classical compound Poisson model with exponential jumps. Denote the intensity for claim arrivals by $\lambda > 0$ and the parameter for claim sizes by $\beta > 0$. Let $c > 0$ denote the rate of premium income. This continues examples 1, 2, and 3. We then obtain

$$U^+ = \frac{1}{2c}\left(c\beta - \gamma - \lambda - \sqrt{(c\beta - \gamma - \lambda)^2 + 4c\beta\gamma}\right) =: -\rho$$
$$U^- = \frac{1}{2c}\left(\lambda + \gamma - c\beta - \sqrt{(c\beta - \gamma - \lambda)^2 + 4c\beta\gamma}\right) =: -R$$

cf. [13], equations (3.12) and (4.24) with $\delta = \gamma$. Section 3.2 further yields $A^+ = \beta/(\beta+\rho)$ and $A^- = (\beta - R)/\beta$. Thus

$$\begin{aligned}\Psi^+(b|x) &= \left(e^{-U^+\cdot x} - A^- e^{U^-\cdot x} A^+\right) \cdot \left(e^{-U^+\cdot b} - A^- e^{U^-\cdot b} A^+\right)^{-1} \\ &= \frac{e^{\rho x} - \frac{\beta-R}{\beta} e^{-Rx} \frac{\beta}{\beta+\rho}}{e^{\rho a} - \frac{\beta-R}{\beta} e^{-Ra} \frac{\beta}{\beta+\rho}} = \frac{e^{\rho x} - \psi(x)}{e^{\rho b} - \psi(b)}\end{aligned}$$

if we write $\psi(x) := e^{-Rx} \cdot (\beta - R)/(\beta + \rho)$, cf. equation (6.37) in [13]. This coincides with formula (6.25) in [13], where $\Psi^+(b|x)$ is denoted by $B(0, b|x)$.

4 Main result

Starting with an initial reserve $u < b$ or with $u = b$ but in a descending phase, there is a positive probability of obtaining no dividends at all before ruin. Let α denote the initial phase distribution of $(\mathcal{X}, \mathcal{J})$, i.e. $\alpha_i = \mathbb{P}(J_0 = i)$ for all $i \in E$. Then equation (12) yields

$$\mathbb{P}(D = 0) = \begin{cases} 1 - \alpha\Psi^+(b|u)\mathbf{1}, & u < b \\ 1 - \alpha \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{E_d} \end{pmatrix} \Psi^+(b|b)\mathbf{1}, & u = b \end{cases}$$

where I_{E_d} denotes the identity matrix of dimension $|E_d|$. Clearly the event $D = 0$ means that $\mathcal{X}$ exits the interval $[0, b]$ at the lower boundary first. We further observe that dividends can be collected only in ascending phases, i.e. on the time set $\{t \geq 0 : J_t \in E_a\}$. We wish to derive an expression for the function

$$\bar{F}(x, \gamma) := \mathbb{E}\left(e^{-\gamma L_x^{-1}(b)}; D > x\right)$$

where $x, \gamma \geq 0$. The strong Markov property and the fact that an exit from $[0, b]$ at the upper boundary can occur only in an ascending phase yield together

$$\bar{F}(x, \gamma) = \Psi^+(b|u)\, \mathbb{E}\left(e^{-\gamma L_x^{-1}(b)}; D > x|X_0 = b\right)$$

where the last factor (written as an expectation) is an $E_a \times E_a$ matrix with entries

$$\mathbb{E}_{ij}\left(e^{-\gamma L_x^{-1}(b)}; D > x|X_0 = b\right) = \mathbb{E}\left(e^{-\gamma L_x^{-1}(b)}; D > x, J_{L_x^{-1}(b)} = j|X_0 = b, J_0 = i\right)$$

for $i, j \in E_a$. This observation may be compared with equation (2.16) in [14]. Thus it suffices to determine the matrix-valued function

$$M(x, \gamma) := \mathbb{E}\left(e^{-\gamma L_x^{-1}(b)}; D > x | X_0 = b\right)$$

This is the content of our main result.

Theorem 1 *The distribution of the total dividends paid out under a barrier strategy at the level b, given that $X_0 = b$ and $J_0 \in E_a$, is matrix-exponential. Specifically,*

$$M(x, \gamma) = e^{G(b) \cdot x}$$

for $\gamma, x \geq 0$, where $G(b) = \left(U^+ e^{-U^+ b} + C^- e^{U^- b} U^- C^+\right) \cdot \left(e^{-U^+ b} - C^- e^{U^- b} C^+\right)^{-1}$.

Proof: We employ the following approximation. Assume that dividends are paid out in small batches of sizes $\varepsilon > 0$ rather than continuously. More exactly, we define a process $(\mathcal{X}^\varepsilon, \mathcal{J}^\varepsilon)$ as follows. The phase process $\mathcal{J}^\varepsilon$ shall equal $\mathcal{J}$ almost surely. The level process $\mathcal{X}^\varepsilon$ behaves like $\mathcal{X}$ in the interval $[0, b]$ but may go above the level b. Whenever $\mathcal{X}^\varepsilon$ reaches the level $b + \varepsilon$, we pay a dividend of amount ε whereupon $\mathcal{X}^\varepsilon$ jumps back to the level b. The phase process $\mathcal{J}^\varepsilon$ remains unchanged by this jump. The original process $(\mathcal{X}, \mathcal{J})$ is obtained if we let ε tend to 0.

Let D^ε denote the dividends obtained for $(\mathcal{X}^\varepsilon, \mathcal{J}^\varepsilon)$. Then D^ε has a matrix-geometric distribution, i.e.

$$M^\varepsilon(n, \gamma) := \mathbb{E}\left(e^{-\gamma T_n(\varepsilon)}; D^\varepsilon \geq n \cdot \varepsilon | X_0^\varepsilon = b\right) = \left(\Psi^+_{(a,a)}(b + \varepsilon | b)\right)^n$$

for $n \in \mathbb{N}$ and $\gamma \geq 0$, where

$$\Psi^+_{(a,a)}(b + \varepsilon | b) = \left(e^{U^+ \varepsilon} - C^- e^{U^- b} C^+ e^{U^+ \cdot (b+\varepsilon)}\right) \cdot \left(I - C^- e^{U^- \cdot (b+\varepsilon)} C^+ e^{U^+ \cdot (b+\varepsilon)}\right)^{-1}$$

according to (12), and $T_n(\varepsilon)$ denotes the time of the nth payment of an ε-batch of dividends.

Now letting ε tend to 0 we obtain that $M(x,\gamma)$ has a matrix-exponential distribution with parameter

$$\begin{aligned}
G(b) &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(\Psi^+_{(a,a)}(b+\varepsilon|b) - I\right)\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(e^{U^+\varepsilon} - C^- e^{U^- b}C^+ e^{U^+\cdot(b+\varepsilon)} - \left(I - C^- e^{U^-\cdot(b+\varepsilon)}C^+ e^{U^+\cdot(b+\varepsilon)}\right)\right)\\
&\qquad\cdot\left(I - C^- e^{U^-\cdot(b+\varepsilon)}C^+ e^{U^+\cdot(b+\varepsilon)}\right)^{-1}\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(e^{U^+\varepsilon} - I + C^- e^{U^- b}\left(e^{U^-\varepsilon} - I\right)C^+ e^{U^+\cdot(b+\varepsilon)}\right)\\
&\qquad\cdot\left(I - C^- e^{U^-\cdot(b+\varepsilon)}C^+ e^{U^+\cdot(b+\varepsilon)}\right)^{-1}\\
&= \left(U^+ + C^- e^{U^- b}U^- C^+ e^{U^+ b}\right)\cdot\left(I - C^- e^{U^- b}C^+ e^{U^+ b}\right)^{-1}\\
&= \left(U^+ e^{-U^+ b} + C^- e^{U^- b}U^- C^+\right)\cdot\left(e^{-U^+ b} - C^- e^{U^- b}C^+\right)^{-1}
\end{aligned}$$

which is the statement.
□

Remark 3 Defining an analogue of the γ-scale function by

$$W(x) := e^{-U^+ x} - C^- e^{U^- x}C^+$$

for $x > 0$, we see first that $G(b) = -W'(b)[W(b)]^{-1}$ where $W'(b)$ denotes the derivative of the function $W(x)$ at b. Further can the mean present value of the dividends before ruin be computed as

$$\begin{aligned}
V(b|u) &:= \mathbb{E}\left(e^{-\gamma L_D^{-1}(b)}|X_0 = u\right)\\
&= \Psi^+(b|u)\,\mathbb{E}\left(e^{-\gamma L_D^{-1}(b)}|X_0 = b\right)\\
&= \Psi^+(b|u)\,[-G(b)]^{-1}\\
&= \left(W^+ e^{-U^+ u} - W^- e^{U^- u}C^+\right)\cdot\left(-U^+ e^{-U^+ b} + C^- e^{U^- b}\left(-U^-\right)C^+\right)^{-1}
\end{aligned}$$

The mean total dividend payment $\mathbb{E}(D)$ before ruin is obtained by setting the discount factor γ to zero.

Example 6 We continue example 4 of a Brownian motion fluid flow. Since there is only one phase, we get $W^+ = W^- = C^+ = C^- = 1$ and hence

$$V(b|u) = \frac{e^{ru} - e^{su}}{re^{rb} - se^{sb}}$$

which is equation (2.11) in [14]. Note that for $\gamma = 0$ we obtain

$$(s,r) = \begin{cases} \left(-2\frac{\mu}{\sigma^2}, 0\right), & \mu > 0 \\ \left(0, -2\frac{\mu}{\sigma^2}\right), & \mu < 0 \end{cases}$$

This implies

$$\mathbb{E}(D) = \begin{cases} \dfrac{\sigma^2}{2\mu}\left(e^{2\mu b/\sigma^2} - e^{2\mu(b-u)/\sigma^2}\right), & \mu > 0 \\ -\dfrac{\sigma^2}{2\mu}\left(e^{2\mu(b-u)/\sigma^2} - e^{2\mu b/\sigma^2}\right), & \mu < 0 \end{cases}$$

cf. equation (2.22) in [14] for the case $\mu > 0$.

Example 7 Another example is the compound Poisson model. Starting in the ascending phase (collecting premiums), we obtain

$$\begin{aligned} V(b|u) &= \left(e^{-U^+u} - A^- e^{U^-u} A^+\right) \cdot \left(-U^+ e^{-U^+b} + A^- e^{U^-b}\left(-U^-\right) A^+\right)^{-1} \\ &= \frac{e^{\rho u} - \frac{\beta - R}{\beta + \rho} e^{-Ru}}{\rho e^{\rho b} + R \frac{\beta - R}{\beta + \rho} e^{-Rb}} \\ &= \frac{(\beta + \rho) e^{\rho u} - (\beta - R) e^{-Ru}}{\rho \cdot (\beta + \rho) e^{\rho b} + R \cdot (\beta - R) e^{-Rb}} \end{aligned}$$

which is formula (7.8) in [13].

References

[1] H. Albrecher, M. Claramunt, and M. Marmol. On the distribution of dividend payments in a Sparre Andersen model with generalized Erlang(n) interclaim times. *Insur. Math. Econ.*, 37:324–334, 2005.

[2] H. Albrecher and R. Kainhofer. Risk theory with a nonlinear dividend barrier. *Computing*, 68:289–311, 2002.

[3] H. Albrecher, J.-F. Renaud, and X. Zhou. A Lévy insurance risk process with tax. *J. Appl. Prob.*, 45:363–375, 2008.

[4] S. Asmussen. *Applied Probability and Queues*. New York etc.: Springer, 2003.

[5] S. Asmussen, F. Avram, and M. Pistorius. Russian and American put options under exponential phase-type Lévy models. *Stochastic Processes and their Applications*, 109:79–111, 2004.

[6] S. Asmussen and G. Koole. Marked point processes as limits of Markovian arrival streams. *J. Appl. Probab.*, 30(2):365–372, 1993.

[7] S. Asmussen and M. Taksar. Controlled diffusion models for optimal dividend pay-out. *Insur. Math. Econ.*, 20:1–15, 1997.

[8] A. Badescu, S. Drekic, and D. Landriault. Analysis of a threshold dividend strategy for a MAP risk model. *Scand. Act. J.*, 4:227–247, 2007.

[9] N. Bean, M. O'Reilly, and P. G. Taylor. Hitting probabilities and hitting times for stochastic fluid flows. *Probab. Eng. Inf. Sci.*, 23:121–147, 2009.

[10] J. Bertoin. *Lévy Processes*. Cambridge University Press, Cambridge etc., 1996.

[11] L. Breuer. First passage times for Markov–additive processes with positive jumps of phase–type. *J. Appl. Prob.*, 45(3):779–799, 2008.

[12] E. Frostig. On risk model with dividends payments perturbed by a Brownian motion - an algorithmic approach. *Astin Bulletin*, 38(1):183–206, 2008.

[13] H. Gerber and E. Shiu. On the time value of ruin. *North American Actuarial Journal*, 2(1):48–78, 1998.

[14] H. Gerber and E. Shiu. Optimal Dividends: Analysis with Brownian motion. *North American Actuarial Journal*, 8(1):1–20, 2004.

[15] Z. Jiang and M. Pistorius. On perpetual American put valuation and first-passage in a regime-switching model with jumps. *Finance Stoch.*, 12:331–355, 2008.

[16] A. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, 2006.

[17] A. Kyprianou and Z. Palmowski. Distributional study of de Finetti's dividend problem for a general Lévy insurance risk process. *J. Appl. Prob.*, 44:428–443, 2007.

[18] S. Li. The distribution of the divident payments in the compound Poisson risk model perturbed by diffusion. *Scand. Act. J.*, 2:73–85, 2006.

[19] R. Loeffen. On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes. *Ann. Appl. Prob.*, 18(5):1669–1680, 2008.

[20] D. M. Lucantoni. New results on the single server queue with a batch Markovian arrival process. *Commun. Stat., Stochastic Models*, 7(1):1–46, 1991.

[21] M. F. Neuts. A versatile Markovian point process. *J. Appl. Probab.*, 16:764–774, 1979.

[22] J.-F. Renaud and X. Zhou. Distribution of the present value of dividend payments in a Lévy risk model. *J. Appl. Prob.*, 44:420–427, 2007.

[23] R. Schassberger. *Warteschlangen.* Wien-New York: Springer-Verlag, 1973.