Autumn Term 2010 | University of Kent
Exercises on Stochastic Processes | L. Breuer

Exercise Sheet 2

NOTE: This exercise sheet will be assessed for students registered to MA836 only. Please return your solutions to the SMSAS general office before noon on Thursday, the 25th of November. Solutions will be presented on the same day at 1pm in RLT1.

Exercise 1: 10 Marks

Let $\mathcal{Y} = (Y_t : t \geq 0)$ denote a Markov process with state space $E = \{1, 2, 3\}$ and generator matrix

$$G = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 2 \\ 1 & 1 & -2 \end{pmatrix}$$

Write down the balance equations and determine the stationary distribution of $\mathcal{Y}$. Assuming that $Y_0 = 1$, what is the distribution of the first hitting time of state 3? Provide the distribution function and its density.

Exercise 2: 10 Marks

Let $\mathcal{Y} = (Y_t : t \geq 0)$ be a Poisson process with intensity $\lambda > 0$. Show that the transition probabilities are given by

$$P_{ij}(t) = e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} \qquad \text{for} \quad j \geq i \in \mathbb{N}_0$$

Hint: Use induction on $j \geq i$ with induction step starting at

$$P_{i,j+1}(t) = \int_0^t \lambda e^{-\lambda u} P_{i+1,j+1}(t-u)\, du$$

Further use $P_{i+1,j+1}(t-u) = P_{ij}(t-u)$.

Exercise 3: 10 Marks

Let $\mathcal{Y}^{(i)} = (Y_t^{(i)} : t \geq 0)$, $i = 1, 2$, denote two independent Poisson processes

with parameters $\lambda_i > 0$, $i = 1, 2$. Assume that $Y_0^{(1)} = 0$ and $Y_0^{(2)} = n - 1$ for some $n \in \mathbb{N}$. Determine the following probabilities:
(i) The process $\mathcal{Y}^{(1)}$ reaches state n before $\mathcal{Y}^{(2)}$ does.
(ii) The process $\mathcal{Y}^{(2)}$ reaches state n before $\mathcal{Y}^{(1)}$ does.

Exercise 4: 10 Marks

Consider the M/M/1 queue where new arrivals are discouraged by long queue lengths. This can be modelled by a Markov process $\mathcal{Y}$ with state space $E = \mathbb{N}_0$ and generator matrix $G = (g_{ij})_{i,j \in E}$ given by $g_{00} = -\lambda$ and

$$g_{ij} = \begin{cases} \frac{\lambda}{i+1}, & j = i + 1 \\ -\left(\frac{\lambda}{i+1} + \mu\right), & j = i \geq 1 \\ \mu, & 0 \leq j = i - 1 \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda, \mu > 0$.
(i) Show that $\mathcal{Y}$ is irreducible and determine its stationary distribution π.
(ii) Why does a stationary distribution exist regardless of the values for $\lambda, \mu > 0$?
(iii) Suppose now that $\lambda = \mu$. If the length of the queue at time 0 is n (including the user in service), calculate the probability that the first user to arrive after time 0 finds the system empty.