Two EPSRC-funded postgraduate scholarships

Available from Sept 25th, 2006, on the topic

Symmetric variational methods

To apply, send a CV, names and emails of three referees, and a letter saying why you are interested in this project, to Prof. E. Mansfield, IMSAS, University of Kent, Canterbury CT2 7NF UK *or* electronically to `E.L.Mansfield@kent.ac.uk`.

Applications close 31st July.

You should have a good first class degree in Mathematics, and be able to work in a small team. Experience with either Maple or Matlab would be helpful.

Summary in non-technical language: Suppose you are given a dirty photocopy and you need to touch it up. The human eye is very good at guessing how missing pieces of curves should be joined up. Now suppose you have hundreds of such copies. Can a computer do it? The problem is that we can do these things without knowing how we do it. A computer has to be told exactly what to do, in the computer's own language. This means we need to solve the problem of edge completion using mathematics. This project is a contribution to this and similar problems which can be described in the same mathematical terms. Basically, we try lots of different completions and ask, which looks best? Formulating what it means to "look best" using already solved mathematical problems is the key to success.

About the project

The projects concerns variational problems which have an inbuilt symmetry. The two main techniques used, in addition to the classical Calculus of Variations, are moving frames and discrete variational methods.

Motivational Example Consider the curve completion or "inpainting" problem. Suppose we are given a partially obscured curve in the plane, as in Figure 1,

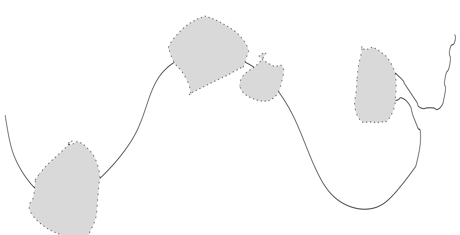

Fig. 1: A curve in the plane with occlusions

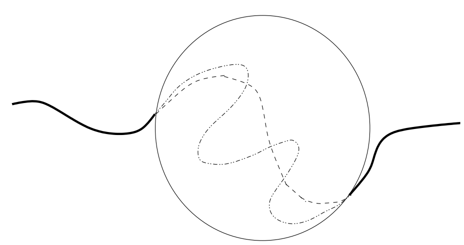

Fig. 2: Which infilling is best?

and we wish to fill in the parts of the curve that are missing. If the missing bit is small, then a straight line edge can be a cost effective solution, but this doesn't always give an aesthetically convincing look. Considering possible solutions to the curve completion problem, Figure 2, we arrive at three requirements on the resulting curve,

- it should look smooth to the human eye,
- if we rotate and translate the obscured curve and then fill it in, the result should be the same as filling it in and then rotating and translating,
- it should be the "simplest possible" in some sense.

The first requirement means that we have boundary conditions to satisfy as well as a function space in which we are working. The second means the formulation of the problem needs to be equivariant with respect to the standard action of the Euclidean group in the plane, as in Figure 3. This condition arises naturally: for example, if the image being repaired is a dirty photocopy, the result should not depend on the angle at which the original is fed into the photocopier. All three conditions can be satisfied if we require the resulting curve to be such as to minimise an integral which is invariant under the group action,

$$\int L(s, \kappa, \kappa_s, \ldots)\, \mathrm{d}s, \tag{1}$$

where s is arc length and κ the euclidean curvature, together with the relevant boundary conditions.

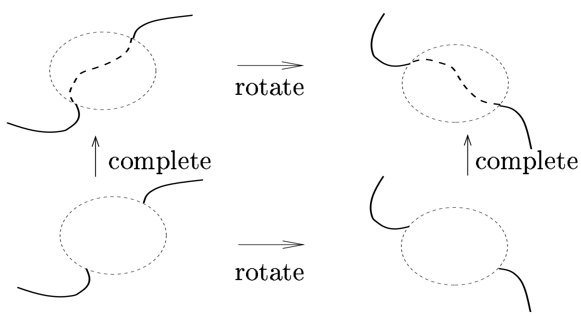

Fig. 3: The solution is equivariant.

One particular Lagrangian which is invariant under the euclidean group and which has been well studied is $\int \kappa^2\, \mathrm{d}s$. Solutions of the Euler-Lagrange equation,

$$\kappa_{ss} + \frac{1}{2}\kappa^3 = 0$$

are known as *Euler's elastica*. Modern applications include computer vision (the seminal paper is by Mumford [8]), the study of materials such as the draping problem [3], and numerical analysis ([2] is a seminal paper). This particular Lagrangian is not suited to the *smooth* curve completion problem (count the boundary conditions!) and $\int \kappa_2^2\, \mathrm{d}s$ (Euler spirals) has been proposed instead by some authors. However, smoothness is not necessary to fool the human eye, so other function spaces are considered.

Lagrangians that are invariant under the Euclidean or Lorentz groups abound in physics and engineering. Computer generated solutions of the curve minimising such Lagrangians are also of interest to computer animators, since it is well known that animations only look convincing if the objects involved obey the relevant physical laws.

While rooted in classical applied analysis, the project builds on work which has appeared only in the last 6 years. The seminal papers by Fels and Olver [4] on the moving frame method appeared in 1998-99, while the Kogan and Olver paper [5] on the invariantized variational complex appeared in 2004. The PI's simplification [7] is to appear, as is her paper on Noether's theorem for finite difference and finite element Lagrangians [6].

References

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