Difference Forms

Elizabeth L. Mansfield

Institute of Mathematics, Statistics and Actuarial Science,

University of Kent, Canterbury, CT2 7NF, U.K.

E-mail: E.L.Mansfield@kent.ac.uk

Peter E. Hydon

Department of Mathematics and Statistics,

University of Surrey,

Guildford GU2 7XH, England.

E-mail: P.Hydon@surrey.ac.uk

November 8, 2006

Abstract

The development of geometric integrators, numerical models which contain analogues of geometric properties, is of great current interest. Our aim is to provide a means by which finite difference models can be seen to mirror *global* properties of physical systems. To this end, we introduce a cohomology theory for lattice varieties, on which finite difference systems are defined. These varieties are defined independently of any continuous space as a continuum limit, and indeed the equations and systems themselves need have no continuum limit to a differential system. This distinguishes these models from theories of "discrete differential forms" built on simplicial approximations, Whitney forms, and cohomology theories built on cubical complexes. We show that our cohomology can be calculated in terms of the pattern of a good cover, just as de Rham cohomology, and we postulate that the dimension of solution space of a globally defined linear recurrence relation equals the analogue of the Euler characteristic for the lattice variety. The fundamental property we use to prove our results is the natural ordering on the integer lattice. Different finite difference models yield slightly different theories.

1 Introduction

The new discipline of geometric integration focuses on the problem of numerically approximating differential equations whilst preserving important geometrical structures [8, 17] Integration schemes have been devised that preserve symplectic and multisymplectic structures [31, 14, 30], symmetries [7] first integrals [26], and conservation laws [16, 27]. In some instances, it is possible to construct approximations on a Lie group directly [22].

So far, most research in geometric integration has concentrated on the preservation of local structures. However, in many applications, it is also essential to preserve global structures. For smooth manifolds, the cohomology of the de Rham complex can be calculated by well-known techniques. There is much interest in the question of whether analogues of differential forms exist for spaces that are used in numerical methods, for such an analogue could produce a test for ensuring that cohomology is preserved by a numerical approximation.

Cohomology is important to numerical approximation because it encodes topological information, such as topological invariants, obstructions, and monodromy. For instance, the shallow water equations in computational meteorology have an invariant called potential vorticity. The evolution of the distribution of potential vorticity over a domain M largely determines the motion of large-scale weather systems. This is encoded in the potential vorticity 2-form

$$\Omega = \mathrm{d}u \wedge \mathrm{d}x + \mathrm{d}v \wedge \mathrm{d}y + f\mathrm{d}x \wedge \mathrm{d}y$$

where (u, v) is velocity at position (x, y) and f is the Coriolis parameter, which is nonzero away from the equator. The topology of the domain M constrains the total potential vorticity. For instance, if doubly-periodic boundary conditions are imposed, then

$$Q := \int_M \Omega \neq 0.$$

This is because M is topologically equivalent to a 2-torus; the area 2-form $\mathrm{d}x \wedge \mathrm{d}y$ is closed but not exact. For the same reason, the total potential vorticity Q on a sphere is nonzero [5], whereas Q can be zero on an annulus or a starshaped domain. This illustrates the necessity of preserving the cohomology of the original system in any numerical approximation.

Considerable effort has focused on analogues of differential forms for computational electromagnetism using the finite element method [1, 3, 4, 15, 18]; see [2] for a comprehensive discussion of recent results. These involve an analogue to the de Rham complex that is based on *Whitney forms* [36] rather than differential forms. A related approach to the general finite element method uses *discrete differential forms*, which arise naturally from the coboundary operator for the simplicial complex [33, 25, 10]. The finite element method uses a simplicial approximation to a smooth manifold; the space remains continuous even though differentiability is lost at boundaries of simplices.

It is important to realize that the term '*discrete* differential form' does not imply that the underlying space is discrete. By contrast, finite difference methods are defined in terms of mesh points, without there being a need for an underlying continuous space. The same is true for difference equations in general; where such equations are used to model an inherently discrete process, the imposition of a continuous structure can produce artefacts. Therefore, to deal with difference equations it is necessary to discard the continuous base space. At first sight this might seem to be disastrous, for most of the familiar and useful constructions are lost. These include the tangent bundle and the exterior derivative; indeed, difference operators are not derivations, that is, they have no Leibnitz or product rule. Moreover, results for constructions based on having an underlying continuum, such as one sees in cohomology theories based on cubical simplices, [23, 28], do not transfer. Despite the resemblance of the exterior difference operator to the cubical coboundary operator, constructions from cubical simplex theories can give some useful intuition but not theorems (see Remark 17 of §2.2). One major difference is the relative lack of maps taking the place of diffeomorphisms and homotopies on the base space.

Because there exist applications which are both inherently discrete and for which there are either multiple or no continuum limits, we prove results for the cohomology theory presented here using only those tools and constructions which pertain to difference systems themselves, such as the shift operator. We shall show that by exploiting a single property of difference equations it is possible to derive analogues of several major results concerning the de Rham complex. In finite difference methods, mesh points do not need to be evenly spaced [7], but they are ordered in each direction by an integer label, at least locally. Therefore, instead of dealing with mesh points directly, it is reasonable to regard the independent variables as being p-tuples of integers, where p is the dimension of the discretized problem. The usual ordering on $\mathbb{Z}$ provides sufficient structure, namely adjacency and orientation, to enable us to derive difference analogues of chains (§1), exterior algebra, a coboundary operator, the de Rham complex and Stokes' Theorem (§2); each analogue is adapted to a particular finite difference method. In §3, we construct a homotopy operator to prove that the difference complex is locally exact. The difference complex is combined with a Čech complex in §4, enabling the cohomology to be calculated. Several examples are presented in §5, and the paper concludes with some conjectures and open problems (§6). In particular, we give evidence for a conjecture that the dimension of solution space of a globally defined linear recurrence relation equals the analogue of the Euler characteristic for the lattice variety. Since this same quantity is that appearing in the Morse Index Theorem, this might be thought to be in some distant sense, an analogue of the Morse Index theorem for difference systems on lattice varieties. We point out, however, that our difference index is not a result of a discrete Morse theory, such as has been constructed by Forman [11].

2 Difference forms on lattice varieties

The simplest p-dimensional lattice is $\mathbb{Z}^p$, whose points are identified by the labels

$$\mathbf{n} = (n^1, \ldots, n^p), \qquad n^i \in \mathbb{Z}, \quad i = 1, \ldots, p.$$

These labels provide a coordinate system; points are ordered in the i^{th} direction by the coordinate n^i.

The natural ordering in $\mathbb{Z}$ yields the notions of adjacency and orientation, as well as, for example, the notions of both forward and backward difference. The most important of these in the construction of a lattice variety is *adjacency*.

Definition 1 *Two points with coordinates* $\mathbf{m} = (m^1, \ldots, m^p)$ *and* $\mathbf{n} = (n^1, \ldots, n^p)$ *in* $\mathbb{Z}^p$ *are said to be* adjacent *if and only if*

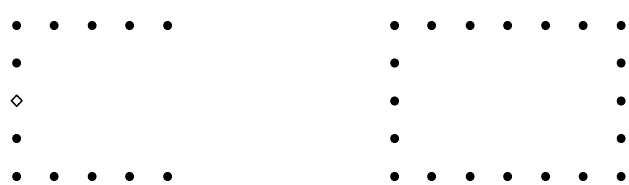
$$|\,\mathbf{m} - \mathbf{n}| := \sum_{i=1}^{p} \left|\, m^i - n^i \right| = 1.$$

Remark 2 *In the digital topology literature, adjacency is called '4-connectedness' when* $p = 2$*; the notion of '8-connectedness' is not needed here, see for example [24].*

The role of adjacency is similar to that of connectedness for manifolds. Consequently, the only coordinate changes that we will allow are those that preserve adjacency, namely translations and reflections.

Definition 3 *A* lattice L *is any subset of* $\mathbb{Z}^p$ *with the property the every point is adjacent to at least one other point. A lattice* L *is* projectable *if there exists a point* $\mathbf{n}_0 \in L$ *that can be reached from every point* $\mathbf{n} \in L$ *by projecting along each direction in a* fixed *order.*

In the diagram below, we show two examples of lattices in $\mathbb{Z}^2$. The one on the left is projectable. By projecting horizontally and then vertically, every point projects to the point marked by a diamond. The lattice on the right is not projectable since the order in which the projection happens needs to differ for the different points.



Definition 4 *For each* $k \in \{1, \ldots, p\}$*, the* k^{th} shift map *is defined by:*

$$S_k : \begin{cases} n^i \mapsto n^i + \delta^i_k; \\ \mathbf{n} \mapsto \mathbf{n} + \mathbf{1}_k. \end{cases} \tag{1}$$

where $\mathbf{1}_k$ *is the p-tuple whose only nonzero entry is the* k^{th}, *which is* 1.

The shift maps have the following properties:

1. $S_j S_k = S_k S_j$;
2. $S_k f(\mathbf{n}) = f(\mathbf{n}+\mathbf{1}_k)$, for all $f \in \mathcal{B}$;
3. $S_k(f(\mathbf{n})g(\mathbf{n})) = S_k(f(\mathbf{n}))S_k(g(\mathbf{n}))$ for all $f,\, g \in \mathcal{B}$.

The shift maps can only act in directions for which the shifted point belongs to L; otherwise, they are undefined.

2.1 Difference Forms and the Difference Map

Let $\textsf{Ex}(p)$ be the exterior algebra on p symbols, $\Delta_1, \ldots \Delta_p$, with real (complex) coefficients, so that

$$\Delta_i \wedge \Delta_i = 0, \qquad \Delta_i \wedge \Delta_j = -\Delta_i \wedge \Delta_j.$$

Let $\mathcal{B}$ denote the set of all real (complex) functions defined on $\mathbb{Z}^p$. Define the algebra of difference forms to be

$${}^{p}\textsf{Ex} = \bigcup_{\mathbf{n}\in\mathbb{Z}^p} \textsf{Ex}(p)\Big|_{\mathbf{n}},$$

with coefficients in $\mathcal{B}$. Difference r-forms on $\mathbb{Z}^p$ are written as $\omega \in {}^{p}\textsf{Ex}^r$. By analogy with differential forms,

$$\omega = \begin{cases} \sum\limits_{i_1<\cdots<i_p} P_{i_1\cdots i_p}(\mathbf{n})\Delta_{i_1}\wedge\Delta_{i_2}\wedge\cdots\wedge\Delta_{i_r}, & r \geq 1; \\ P_0(\mathbf{n}), & r = 0. \end{cases} \tag{2}$$

The action of the k^{th} shift map on difference forms is defined by

$$\begin{aligned} &S_k(\Delta_i) = \Delta_i; \\ &S_k(\eta\wedge\omega) = S_k(\eta)\wedge S_k(\omega), \qquad \eta,\omega \in {}^{p}\textsf{Ex}. \end{aligned} \tag{3}$$

Definition 5 *The* difference map $\boldsymbol{\Delta} : {}^{p}\textsf{Ex}^r \to {}^{p}\textsf{Ex}^{r+1}$ *is defined by*

$$\boldsymbol{\Delta}(\omega) = \sum_{k=1}^{p} \Delta_k \wedge (S_k - id)\omega. \tag{4}$$

Note that this map uses *forward* differences, whereas most finite difference methods do not. For simplicity, attention is restricted to forward differences until §6, where we describe minor changes that enable difference forms to be used far more widely, for example to backward difference and collocation methods.

The difference map plays the role of the exterior derivative, and has the analogous properties

i) $\mathbf{\Delta}^2 = 0$;

ii) $\mathbf{\Delta}(n_i) = \Delta_i$.

However, unlike the exterior derivative, the difference map is not a derivation:

$$\mathbf{\Delta}(\eta \wedge \omega) \neq \mathbf{\Delta}(\eta) \wedge \omega \pm \eta \wedge \mathbf{\Delta}(\omega).$$

The analogue of the de Rham complex, the *difference complex*, will be defined in §2.3.

2.2 Lattice varieties

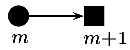
We introduce the idea of p-dimensional "lattice varieties", which are covered by pieces of projectable lattice in much the same way that manifolds are covered by open subsets of $\mathbb{R}^p$. There are two ways to build up lattice varieties. One is to use the cellular route, in which fundamental cubes (defined in this section) are joined together. This notion has a natural algebraic extension to that of *difference chains* which are formal sums of cubes. The second way is akin to covering a manifold by coordinate charts. Here, we glue together pieces of projectable lattice. These are equivalent since any projectable piece of lattice is a sum of fundamental cubes. Both methods have certain advantages and places in the overall theory.

2.2.1 Fundamental cubes as domains for difference forms

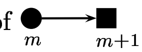
The fundamental p-cube is the unit cube in $\mathbb{Z}^p$, together with information concerning what kinds of forms are defined where. The fundamental cubes in dimensions 0, 1 and 2 are as follows. The suffix σ is an index that identifies a copy of the fundamental cube uniquely; indices m and n are merely provided as typical coordinates.

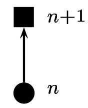
$p = 0$ $C^0_\sigma =$ 

A single point; only 0-forms can be defined here.

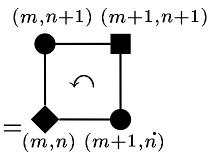
$p = 1$ $C^1_\sigma =$ 

The forward difference of a function is defined only at m, so 1-forms are definable there. The arrow indicates the orientation.

$p = 2$ Take an ordered product of 

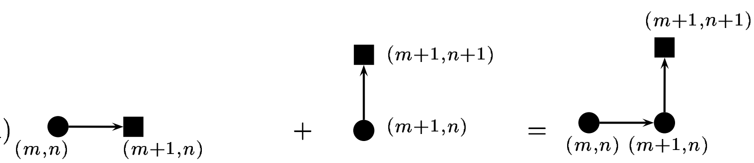
with 

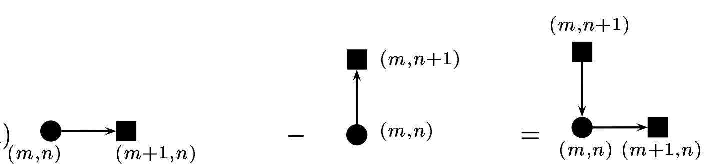
to get the 2-cube with anti-clockwise orientation,

$C^2_\sigma =$ 

The diamond at (m, n) indicates that a 2-form may be defined there.

If a difference form can be defined at a point, it remains defined there when other chains are added. For example,

(i) 

(ii) 

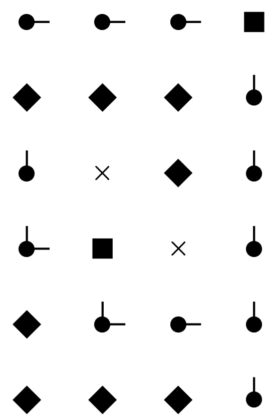
In example (ii) above, there is no diamond at (m, n) because the square is incomplete; even though the forward difference can be taken in each direction, 2-forms cannot properly be defined there and thus the set of 2-forms at such a point is taken to be $\{0\}$. Thus, $\boldsymbol{\Delta}$ acting on a 1-form at (m, n) is zero.

Definition 6 *The corner of a fundamental p-cube where the p-form is defined will be denoted the* south west corner. *The remaining points will be denoted collectively as the* TopRight *points.*

This definition will be generalised in Definition 16. The TopRight points become significant in the proofs in §4. In §6.2 we discuss, "What is so significant about the South West?".

Lattices can be viewed as sums of fundamental cubes, as in Figure 1.

Figure 1: **Lattice with two points removed**



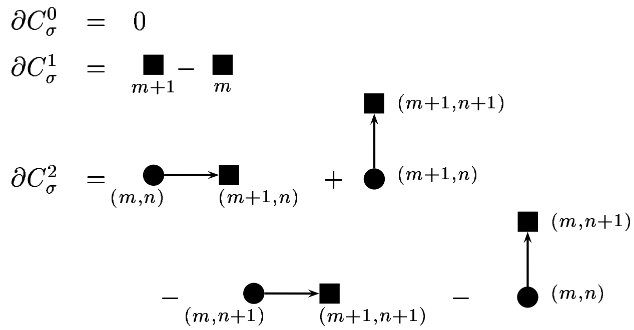
Key: ● one form defined in direction indicated
■ zero form ◆ two form defined
× point removed

2.2.2 Difference chains

Sums of fundamental cubes can be viewed algebraically as difference chains. These are the natural way to prove the analogue of Stokes' Theorem, which we state in this section. The proof is straightforward.

Definition 7 *A* p-chain *is a formal sum of fundamental* p*-cubes with integer coefficients.*

The *boundary* of a p-chain C^p is denoted by ∂C^p. For the fundamental cubes given above, the boundaries are listed below.



(5)

where the minus sign denotes the reversal of orientation; note that

$$\partial(\partial C^2_\sigma)) = 0.$$

The definition for an arbitrary dimensional cube is defined recursively as follows. If C^p is considered to be $C^{p-1} \times C^1$ then

$$\partial C^p = (\partial C^{p-1}) \times C^1 + (-1)^{p-1} C^{p-1} \times \partial C^1.$$

A non-recursive formula given by Fulton ([13], p. 333) is easily adapted to the present case.

Definition 8 *Given a p-chain C^p,*

$$C^p = \sum_\sigma a_\sigma C^p_\sigma, \qquad a_\sigma \in \mathbb{Z},$$

and a p-form

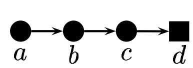
$$\omega = F(n^1, \ldots, n^p)\Delta_1 \wedge \cdots \wedge \Delta_p$$

the oriented sum of ω over C^p is defined to be

$$\sum_{C^p} \omega = \sum_{(n^1,\ldots,n^p) \in C^p_\sigma} a_\sigma F(n^1, \ldots, n^p)$$

provided that ω is defined at $(n^1, \ldots, n^p)$.

For example,

p	C^p	ω	$\sum_{C^p} \omega$
0		F	$F(a)$
1		$F\Delta_m$	$F(a) + F(b) + F(c)$

Theorem: 'Stokes' Theorem for difference forms' Let C^p be a finite p-chain and let ω be a $(p-1)$-form. Then

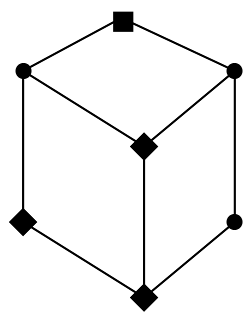
$$\sum_{C^p} \Delta\omega = \sum_{\partial C^p} \omega.$$

In particular, $\sum_{\partial C^p} \omega = 0$ if ω is closed.

2.2.3 Lattice varieties

The building blocks of lattice varieties are the fundamental cubes in $\mathbb{Z}^p$, which are glued together along boundaries (of dimension $p-1$). It is necessary to ensure that adjacency and ordering are respected. The boundary L is the sum of the boundaries of the cubes in L. For example, the cube corner shown below consists of three two-cells glued together; the diamonds show where two-forms are defined, the square where only a zero-form is defined.

A corner of a cube's surface



Rather than working solely with fundamental p-cubes, it is helpful to glue together larger pieces of lattice. We join p-dimensional *projectable* lattices together pairwise, allowing only intersections that respect adjacency and ordering. Each disjoint piece of the intersection is either an overlap (of dimension p) or part of a boundary (of dimension $p-1$).

If adjacency is respected but ordering is not, it is possible to construct non-orientable lattice varieties such as a discrete version of the Möbius strip. Except for a brief discussion in §6, we shall deal only with orientable lattice varieties henceforth. It is possible to respect orientation whilst violating adjacency (see §6.3.3); in the context of numerical methods, this situation occurs if a mesh is refined locally but not globally. For simplicity, we do not allow this.

The gluing process may be described formally as follows.

Definition 9 *A subset $L_\alpha \subset L$ and a map*

$$\phi_\alpha : supp(\phi) = L_\alpha \mapsto \mathbb{Z}^p$$

is said to be a co-ordinate chart *on L if ϕ is injective, and $\phi(L_\alpha)$ is a projectable lattice (Definition 4, §2).*

Definition 10 *Given two such "charts" on L, we say that the gluing map or coordinate change map*

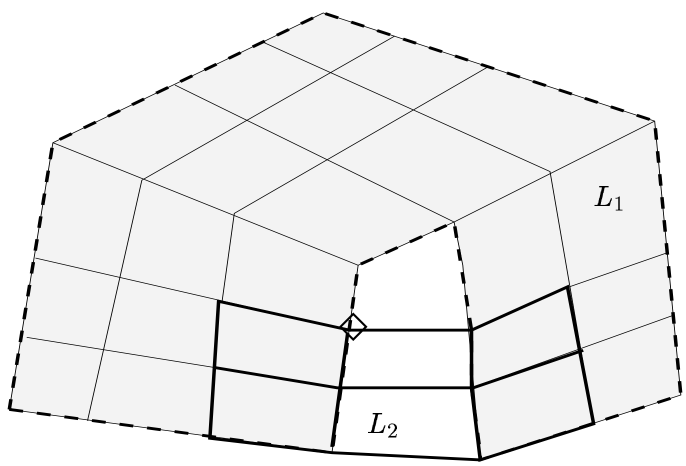
$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(supp(\phi_\alpha) \cap supp(\phi_\beta)) \to \mathbb{Z}^p$$

is admissible *if for any two points x_1 and x_2, $\phi_\alpha(x_1)$ is adjacent to $\phi_\alpha(x_2)$ if and only if $\phi_\beta(x_1)$ is adjacent to $\phi_\beta(x_2)$.*

In other words, admissible gluing maps maintain adjacency. However, we need more than a cover of L with admissible interchange maps.

If a variety is constructed out of charts, then for a forward difference to be defined at some point, it must be defined in some L_α which contains it. It is not enough that "the dots are all there".

Figure 2: **Cautionary example.**

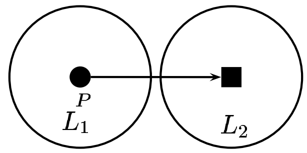


Key: $\diamond$ two-form undefined

In Figure 2, the lattice variety looks like a corner of a cube. But the 2-form at $\diamond$ is not defined in either L_1 or L_2. Indeed, $L_1 \cap L_2$ has two components. It will transpire that the cohomology of this lattice variety is *not* that of a disc, but rather a ring. As we will see in §4, *the pattern of the cover and their intersections matter.*

Definition 11 *We say a cover $\mathcal{L} = \{L_\alpha \,|\, \alpha \in A\}$ of a lattice variety L is* **valid** *if for every lattice point $P \in L$ there is an $\alpha = \alpha(P)$ such that any form definable at P is definable in L_α.*

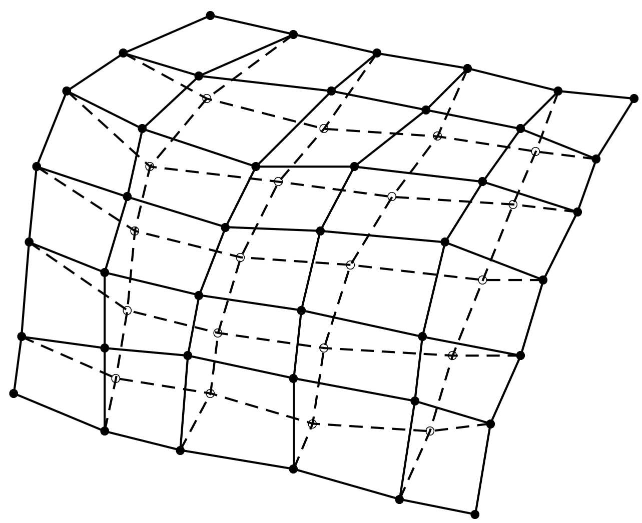
Example 12 *An invalid cover of a 1-cell. The 1-form at the point P is defined in neither L_1 nor L_2.*



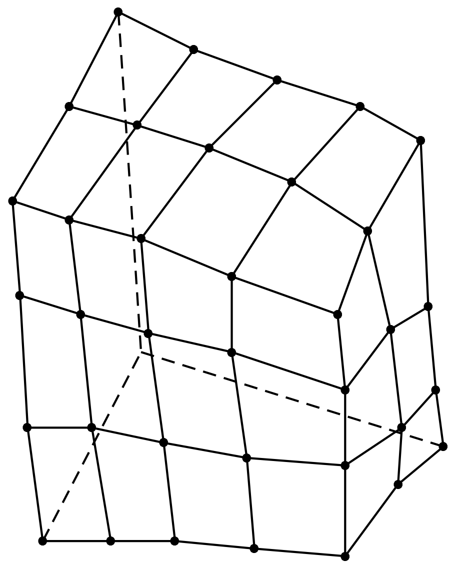
Definition 13 *A lattice variety L may be defined as a collection of charts $\{(L_\alpha, \phi_\alpha)\}$ such that $L = \cup_\alpha L_\alpha$, the cover $\mathcal{L} = \{L_\alpha\}$ is valid, and whenever $\phi_\alpha \circ \phi_\beta^{-1}$ is defined, then it is admissible.*

Two models of the lattice sphere are depicted below.

"Pillow" model of a 2-sphere



“Cube” model of a 2-sphere

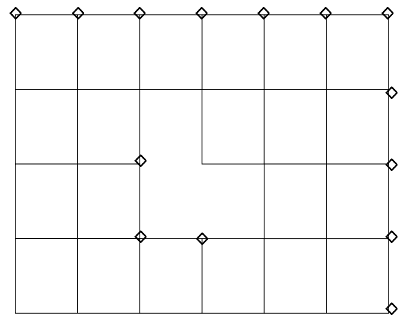


Back faces not indicated for simplicity

Those points in a lattice variety of dimension p where a p-form cannot be defined is an important set. Since domains add when we glue lattices together, we may generalise Definition 6 and define the TopRight of a lattice L as follows.

Definition 14 *The TopRight points of a lattice variety of dimension* p *are those where in no* L_α *containing that point is a* p*-form defined.*

Example 15 *For the two-dimensional lattice ring shown below, the TopRight points with respect to the standard chart (given by the obvious inclusion into* $\mathbb{Z}^2$*) are marked with a diamond. Changing the chart, for example by rotating the lattice ring before inclusion, will result in a different set of TopRight points.*

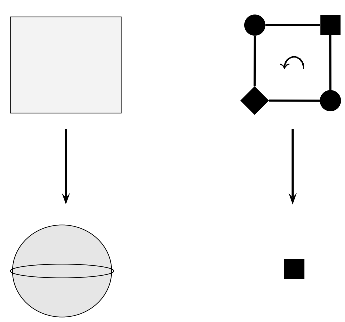


Definition 16 *At any point* $\mathbf{n} \in L$ *where forms of dimension higher than* q *(say) are not defined, we set*

$$^{p}\mathsf{Ex}(L)|_{\mathbf{n}} \equiv \{0\}, \qquad p \geq q.$$

Remark 17 *There is a strong visual analogy between lattice varieties defined as a sum of cells, and simplicial complexes. As the following example shows, the correspondence can only be taken so far. On the left is the projection of* $[0,1]^2$*, to* S^2*, which is achieved by identifying the boundary to a point. On the right, we have the same projection but this time of a discrete 2-cell. When emulating constructions with smooth manifolds, this example shows that problems occur when there are "not enough dots".*

Example 18 *A cautionary example.*



Projections of smooth (left) and discrete (right) 2-cells

2.3 The difference complex

Forms and the difference operator are defined on lattice varieties in the usual way, via the co-ordinate system provided by the charts. The de Rham complex for differential forms has a difference analogue, which arises from the property $\mathbf{\Delta}^2 = 0$. A difference

r-form ω is *closed* if $\mathbf{\Delta}(\omega) = 0$, and is *exact* if there exists a $(r-1)$-form η such that $\omega = \mathbf{\Delta}(\eta)$. Clearly, every exact r-form is closed; we shall prove that the converse is true (for $r \geq 1$) on projectable domains.

Definition 19 *The* difference complex *is*

$$0 \to \mathbb{R} \to {}^p\mathbf{Ex}^0 \xrightarrow{\mathbf{\Delta}} {}^p\mathbf{Ex}^1 \xrightarrow{\mathbf{\Delta}} \cdots \xrightarrow{\mathbf{\Delta}} {}^p\mathbf{Ex}^p \xrightarrow{\mathbf{\Delta}} 0. \tag{6}$$

For any lattice variety L, the cohomology groups are

$$H^r_\Delta(L) = \frac{\{\text{closed } r\text{-forms on } L\}}{\{\text{exact } r\text{-forms on } L\}} = \frac{\ker \mathbf{\Delta}\big|_{\mathbf{Ex}^r(L)}}{\mathrm{im}\mathbf{\Delta}\big|_{\mathbf{Ex}^{r-1}(L)}}.$$

For $r = 0$, the dimension of the group $H^0_\Delta(L)$ (regarded as a vector space) is the number of distinct connected pieces in L. In §3, we shall prove a generalization of the following result, which was proved in Hydon & Mansfield (2004).
Theorem For any $p \geq 1$, there exists a homotopy operator H on $\mathbb{Z}^p$ such that

$$H(\mathbf{\Delta}\omega) + \mathbf{\Delta}H(\omega) = \omega$$

for every r-form $(r \geq 1)$ ω defined on $\mathbb{Z}^p$. Consequently ω is closed $(\mathbf{\Delta}\omega = 0)$ if and only if it is exact. Therefore

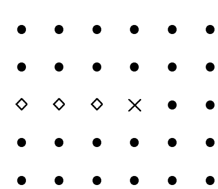
$$\begin{aligned} H^0_\Delta(\mathbb{Z}^p) &\cong \mathbb{R}, && \text{(constant functions on } \mathbb{Z}^p\text{)};\\ H^r_\Delta(\mathbb{Z}^p) &\cong 0, && r \geq 1. \end{aligned}$$

Our generalization of the homotopy operator provides one means of constructing the cohomology groups. In §5.1, we prove the following.

Example 20 The punctured planar lattice
For $L = \mathbb{Z}^2 \setminus \{(0,0)\}$, consider the 1-forms

$$\omega_c\big|_{(m,n)} = \begin{cases} c\Delta_2, & \text{if } n = 0 \text{ and } m \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

where c is constant. In the portion of the lattice L shown, the form equals $c\Delta_2$ on the diamonds and is zero on dots. The missing point is marked by $\times$.



Clearly, ω_c is closed, but if $c \neq 0$ then it is not exact. This can be seen by trying to find a pre-image by "integrating" around the hole. In fact,

$$\begin{aligned} H^1_\Delta(\mathbb{Z}^2 \setminus \{(0,0)\}) &= \{\omega_c \;:\; c \in \mathbb{R}\} \\ &\cong \mathbb{R} \end{aligned}$$

so the punctured lattice has cohomology which is isomorphic to the de Rham cohomology of $\mathbb{R}^2 \setminus \{(0,0)\}$.

3 Local exactness

For a manifold, the Poincaré Lemma states that every closed r-form is exact on a starshaped domain D if $r \geq 1$. The proof uses integration along the ray between a fixed point $\mathbf{x}_0 \in D$ and an arbitrary point $\mathbf{x} \in D$. For lattice varieties, a different approach is needed, because points on most rays are not adjacent. Instead, a homotopy operator is formed by projecting out each direction in turn, reducing the dimension of the problem by one at each stage. Consequently, the replacement for a starshaped domain is a projectable domain. The difference analogue of the Poincaré Lemma is as follows.

Lemma 21 *The difference complex (6) is exact on projectable domains; hence*

$$\ker\boldsymbol{\Delta}\big|_{{}^p\mathbf{Ex}^r} = \mathrm{im}\boldsymbol{\Delta}\big|_{{}^p\mathbf{Ex}^{r-1}}, \qquad r = 1, \ldots p-1,$$

and $\ker\boldsymbol{\Delta}\big|_{{}^p\mathbf{Ex}^0} = \mathbb{R}$.

3.1 Homotopy operators

We prove Lemma 21 by constructing a homotopy operator $H : {}^p\mathbf{E}^r \to {}^p\mathbf{E}^{r-1}$ that satisfies

$$H(\boldsymbol{\Delta}\omega) + \boldsymbol{\Delta}H(\omega) = \omega, \qquad \omega \in {}^p\mathbf{E}^r, \qquad r = 1, \ldots, p. \tag{7}$$

All 0-forms are mapped to zero by H. To begin with, consider the simplest case, in which the domain D is one-dimensional and connected (so that it is projectable); set $p = r = 1$ and $\omega = f(n^1)\Delta_1$. Clearly ω is closed for arbitrary $f(n^1)$, so we must find a homotopy operator $H : {}^1\mathbf{E}^1 \to {}^1\mathbf{E}^0$ such that

$$\boldsymbol{\Delta}\big(H(\omega)\big) = \omega, \tag{8}$$

for all functions $f(n^1)$. Let $H(\omega) = g(n^1)$; then (8) amounts to

$$\big(S - \mathrm{id}\big)g(n^1) = f(n^1),$$

whose solution is $g(n^1) = g(n_0^1) + h_1(\omega)$, where

$$h_1\big(f(n^1)\Delta_1\big) = \begin{cases} \sum_{k=n_0^1}^{n^1-1} f(k) & : n^1 > n_0^1, \\ 0 & : n^1 = n_0^1, \\ -\sum_{k=n^1}^{n_0^1-1} f(k) & : n^1 < n_0^1. \end{cases}$$

and $n_0^1 \in D$ is an arbitrary fixed point. The arbitrary constant $g(n_0^1)$ is annihilated by $\boldsymbol{\Delta}$, so we discard it. Hence a suitable homotopy operator for $p = 1$ is $H = h_1$. Now consider $\omega \in {}^1\mathbf{Ex}^0$, so that $\omega = f(n^1)$ for some function f. Then

$$\begin{aligned} H\boldsymbol{\Delta}(\omega) + \boldsymbol{\Delta}H(\omega) &= h_1(\{f(n^1+1) - f(n^1)\}\Delta_1) \\ &= f(n^1) - f(n_0^1) \\ &= \omega - \omega\big|_{n^1=n_0^1}. \end{aligned} \tag{9}$$

In particular, if $\omega \in \ker\boldsymbol{\Delta}$ then the left-hand side of (9) vanishes, so

$$\omega = \omega\big|_{n^1=n_0^1}.$$

Consequently Lemma 21 holds for $p = 1$. We now develop homotopy operators for higher-dimensional domains, by combining operators similar to h_1 with projection operators and using induction on p. Let D be a p-dimensional projectable domain, with every point projectable to $\mathbf{n}_0 = (n_0^1, \dots, n_0^p) \in D$. Let $\mathbf{n} = (n^1, \dots, n^p)$ be an arbitrary point in D. It is helpful to use a formal analogue of the interior product of a vector field and a differential form, by defining operators $\partial_{n^i} \lrcorner : {}^j\mathbf{Ex}^r \to {}^j\mathbf{Ex}^{r-1}$ by the relations $\partial_{n^i} \lrcorner\, \Delta_k = \delta_k^i$, where δ is the Kronecker symbol. This is extended to all difference forms by linearity and the product rule. Let $h_i : {}^j\mathbf{Ex}^r \to {}^j\mathbf{Ex}^{r-1}$ be defined by

$$h_i(\omega) = \begin{cases} \sum_{k=n_0^i}^{n^i-1} (\partial_{n^i} \lrcorner\, \omega)\big|_{n^i=k} & : n^i > n_0^i, \\ 0 & : n^i = n_0^i, \\ -\sum_{k=n^i}^{n_0^i-1} (\partial_{n^i} \lrcorner\, \omega)\big|_{n^i=k} & : n^i < n_0^i. \end{cases} \tag{10}$$

Roughly speaking, $h_i(\omega)$ sums the r-form ω along the 1-chain obtained by varying the i^{th} coordinate from n_0^i to n^i, whilst leaving the other coordinates unchanged. Therefore we call the operators h_i *summation operators.* Define the projection maps

$$\Pi_j : {}^j\mathbf{Ex}^r \to {}^{j-1}\mathbf{Ex}^r, \qquad \Pi_j(\omega) = \omega\big|_{n^j=n_0^j, \Delta_j=0} \tag{11}$$

and note that

$$\Pi_r \circ \Pi_{r+1} \circ \cdots \circ \Pi_p \omega = 0 \tag{12}$$

for all $\omega \in {}^p\mathbf{Ex}^r$, $r \geq 1$. For simplicity, we shall assume that there for every point $\mathbf{n} \in D$ one can first project in the n^p-direction, then in n^{p-1} and so on down to n^1 last. In other words, D must contain the points

$$\begin{array}{rl} (n^1, n^2, \ldots, n^{p-1}, k), & \text{for all } k \in \mathbb{Z} \text{ between } n_0^p \text{ and } n^p, \\ (n^1, n^2, \ldots, n^{p-2}, k, n_0^p), & \text{for all } k \in \mathbb{Z} \text{ between } n_0^{p-1} \text{ and } n^{p-1}, \\ \vdots & \vdots \\ (k, n_0^2, \ldots, n_0^{p-1}, n_0^p), & \text{for all } k \in \mathbb{Z} \text{ between } n_0^1 \text{ and } n^1. \end{array}$$

This allows us to construct the homotopy map by induction on the number of edges needed to get to $\mathbf{n}$ (that is, on the dimension of the lattice). There is no essential loss of generality in doing this, for the resulting homotopy operator can be adapted to other projectable lattices merely by using the projection maps and summation operators in the required order.

Theorem 22 *Under the above assumptions on the domain of definition D of $\omega \in {}^p\mathbf{Ex}^r/\mathcal{B}$, let*

$$h(\omega) = h_p(\omega) + \sum_{i=1}^{p-1} h_i \left(\Pi_{i+1} \circ \Pi_{i+2} \circ \cdots \circ \Pi_p \omega\right). \tag{13}$$

Then

$$H_{\mathcal{B}}(\omega) = \begin{cases} h(\omega) & \omega \in {}^p\mathbf{Ex}^r, \ r > 0 \\ \omega\big|_{\mathbf{n}=\mathbf{n}_0} & \omega \in {}^p\mathbf{Ex}^0 \end{cases} \tag{14}$$

is a homotopy operator for the complex ${}^p\mathbf{Ex}$ over $\mathcal{B}$.

Example 23 *If $p = 2$ then for 1-forms $\omega = \alpha(n^1, n^2)\Delta_1 + \beta(n^1, n^2)\Delta_2$ the homotopy map is*

$$h(\omega) = h_2(\omega) + h_1(\Pi_2(\omega)) = h_2(\omega) + h_1(\alpha(n^1, n_0^2)\Delta_1)$$

whereas for 2-forms $\omega = f(n^1, n^2)\Delta_1 \wedge \Delta_2$ the homotopy map is

$$h(\omega) = h_2(\omega) + h_1(\Pi_2(\omega)) = h_2(\omega).$$

Notes:

1. If $r \geq 1$ then, from (12), the sum in (13) need only be taken from $i = r$ to $i = p-1$.
2. If $\tilde{\omega} = \Pi_p \omega$ then

$$h(\omega) = h_p(\omega) + h(\tilde{\omega}) \tag{15}$$

Proof: It is sufficient to prove that

$$h(\boldsymbol{\Delta}\omega) + \boldsymbol{\Delta} h(\omega) = \omega - \Pi_1 \circ \cdots \circ \Pi_p \omega. \tag{16}$$

To see this, note that if $\omega \in {}^p\mathbf{Ex}^r$ and $r \geq 1$ then, by (12), $\Pi_1 \circ \cdots \circ \Pi_p \omega = 0$ and thus $H_{\mathcal{B}} = h$ is a homotopy map. To show exactness at ${}^p\mathbf{Ex}^0$, we need to show that

$$h(\mathbf{\Delta}\omega) + \omega\big|_{\mathbf{n}=\mathbf{n}_0} = \omega$$

for $\omega \in {}^p\mathbf{Ex}^0$. But this is precisely (16), since

$$\omega \in {}^p\mathbf{Ex}^0 \Longrightarrow h(\omega) = 0, \qquad \Pi_1 \circ \cdots \circ \Pi_p \omega = \omega\big|_{\mathbf{n}=\mathbf{n}_0}.$$

The proof of (16) is by induction on p. First note that if $\omega \in {}^1\mathbf{Ex}^0$ then $\omega = f(n^1)$ for some function f, and therefore

$$\begin{aligned} h(\mathbf{\Delta}\omega) + \mathbf{\Delta}h(\omega) &= h(\{f(n^1+1) - f(n^1)\}\Delta_1) \\ &= f(n^1) - f(n_0^1) \\ &= \omega - \omega\big|_{n^1=n_0^1} \\ &= \omega - \Pi_1\omega. \end{aligned} \tag{17}$$

Also if $\omega \in {}^p\mathbf{Ex}^p$ then ω is a multiple of the p-form $\Delta_1 \wedge \Delta_2 \wedge \cdots \wedge \Delta_p$ and so $\Pi_p\omega = 0$ and $\mathbf{\Delta}\omega = 0$. Hence

$$\begin{aligned} h(\mathbf{\Delta}\omega) + \mathbf{\Delta}h(\omega) &= \mathbf{\Delta}h_p(\omega) \\ &= \omega \\ &= \omega - \Pi_1 \circ \cdots \circ \Pi_p\omega. \end{aligned} \tag{18}$$

Now fix $r < p$ and suppose that $H_{\mathcal{B}}$ is a homotopy operator for all $p' < p$. We set $\tilde{\omega} = \Pi_p(\omega)$ and observe that $\tilde{\omega} \in {}^{p-1}\mathbf{Ex}^r$; the induction hypothesis implies that

$$h(\mathbf{\Delta}\tilde{\omega}) + \mathbf{\Delta}h(\tilde{\omega}) = \tilde{\omega} - \Pi_1 \circ \cdots \circ \Pi_{p-1}\tilde{\omega} \tag{19}$$

The last term is nonzero only if $r = 0$. Note that

$$\begin{aligned} \Pi_p(\mathbf{\Delta}\omega) &= \Pi_p(\textstyle\sum_{j=1}^p \Delta_j \wedge (S_j - \mathrm{id})\omega) \\ &= \textstyle\sum_{j=1}^{p-1} \Delta_j \wedge (S_j - \mathrm{id})(\Pi_p\omega) \\ &= \mathbf{\Delta}\tilde{\omega} \end{aligned}$$

and so, from (15),

$$\begin{aligned} h(\mathbf{\Delta}\omega) &= h_p(\mathbf{\Delta}\omega) + h(\Pi_p(\mathbf{\Delta}\omega)) \\ &= h_p(\mathbf{\Delta}\omega) + h(\mathbf{\Delta}\tilde{\omega}). \end{aligned}$$

Also from (15),

$$\mathbf{\Delta}h(\omega) = \mathbf{\Delta}h_p(\omega) + \mathbf{\Delta}h(\tilde{\omega}),$$

and therefore, using (19),

$$\begin{aligned} h(\mathbf{\Delta}\omega) + \mathbf{\Delta}h(\omega) &= h_p(\mathbf{\Delta}\omega) + \mathbf{\Delta}h_p(\omega) + \tilde{\omega} - \Pi_1 \circ \Pi_2 \circ \cdots \circ \Pi_{p-1}\tilde{\omega} \\ &= h_p(\mathbf{\Delta}\omega) + \mathbf{\Delta}h_p(\omega) + \Pi_p(\omega) - \Pi_1 \circ \Pi_2 \circ \cdots \circ \Pi_p\omega. \end{aligned}$$

So to prove the correctness of the homotopy formula, we need only show that

$$h_p(\mathbf{\Delta}\omega) + \mathbf{\Delta}h_p(\omega) = \omega - \Pi_p\omega.$$

This can be verified by direct calculation, as follows.

$$\begin{aligned} h_p(\mathbf{\Delta}\omega) \quad &+ \mathbf{\Delta}h_p(\omega) \\ &= \partial_{n^p} \lrcorner (\Delta_p \wedge \omega) - (\partial_{n^p} \lrcorner (\Delta_p \wedge \omega))\big|_{n^p=n_0^p} + \Delta_p \wedge (\partial_{n^p} \lrcorner\, \omega) \\ &= \omega - \omega\big|_{n^p=n_0^p, \Delta_p=0} \\ &= \omega - \Pi_p\omega \end{aligned}$$

as required, where we have used the identity

$$\partial_{n^p} \lrcorner (\Delta_j \wedge \eta) + \Delta_j \wedge (\partial_{n^p} \lrcorner \eta) = \delta_j^p \eta.$$

Equations (17) and (18) show that (16) holds for $p = r$ if $r \geq 1$ and for $p = 1$ if $r = 0$. By induction, (16) holds for all p, r, as required. □

4 From local to global

As for de Rham cohomology of a manifold, the difference cohomology of a lattice variety may be calculated in terms of the pattern of intersections of a "good cover". We follow the line of argument in Weil's celebrated proof [35], as expounded in [6], pointing out the relevant technical differences.

As a corollary, we show that if the cover of a space and its lattice approximation has the same pattern, then the $\mathbf{\Delta}$-cohomology will match the smooth de Rham cohomology in a well-defined way. There are, however, limits to the analogy with the smooth case, which the examples show. In the next section, we calculate examples using the techniques developed in this section.

4.1 The Čech difference operator

The purpose of the Čech difference operator is to measure when forms defined locally can be "glued" or pieced together to make forms with extended domains.

Let L_α, $\alpha \in \mathcal{A}$, be a finite collection of lattices composed of sums of cells. In our application, these will be the pieces that comprise a lattice variety. Denote

$$\begin{aligned} L_{\alpha_0} \cap L_{\alpha_1} \qquad &\text{by} \qquad L_{\alpha_0\alpha_1} \\ L_{\alpha_0} \cap L_{\alpha_1} \cap L_{\alpha_2} \qquad &\text{by} \qquad L_{\alpha_0\alpha_1\alpha_2} \end{aligned}$$

and so forth, and let $\coprod$ denote disjoint union.

The inclusions $\partial_i : L_{\alpha_0 \cdots \alpha_i \cdots \alpha_j} \to L_{\alpha_0 \cdots \widehat{\alpha_i} \cdots \alpha_j}$ give rise to the sequence

$$\coprod L_\alpha \;\underset{\overset{\partial_1}{\leftarrow}}{\overset{\partial_0}{\leftarrow}}\; \coprod L_{\alpha_0\alpha_1} \;\begin{matrix}\overset{\partial_0}{\leftarrow}\\ \overset{\partial_1}{\leftarrow}\\ \overset{\partial_2}{\leftarrow}\end{matrix}\; \coprod L_{\alpha_0\alpha_1\alpha_2} \;\begin{matrix}\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow\end{matrix}\; \cdots$$

These in turn give rise to a sequence of restrictions of difference forms

$$\bigoplus \mathbf{Ex}^*(L_\alpha) \;\begin{matrix}\overset{\delta_0}{\to}\\ \overset{\delta_1}{\to}\end{matrix}\; \bigoplus \mathbf{Ex}^*(L_{\alpha_0\alpha_1}) \;\begin{matrix}\overset{\delta_0}{\to}\\ \overset{\delta_1}{\to}\\ \overset{\delta_2}{\to}\end{matrix}\; \cdots$$

where, for example,

$$\delta_0 : \mathbf{Ex}^*(L_{\alpha_1\alpha_2}) \to \mathbf{Ex}^*(L_{\alpha_0\alpha_1\alpha_2})$$

Definition 24 *The Čech difference operator*

$$\delta : \bigoplus \mathbf{Ex}^*(L_{\alpha_0 \cdots \alpha_p}) \to \bigoplus \mathbf{Ex}^*(L_{\alpha_0 \cdots \alpha_{p+1}})$$

is defined for each integer $p \geq 0$. If $\omega \in \bigoplus \mathbf{Ex}^*(L_{\alpha_0 \cdots \alpha_p})$ has components

$$\omega_{\alpha_0 \cdots \alpha_p} \in \mathbf{Ex}^q(L_{\alpha_0 \cdots \alpha_p})$$

then $\delta\omega$ has components

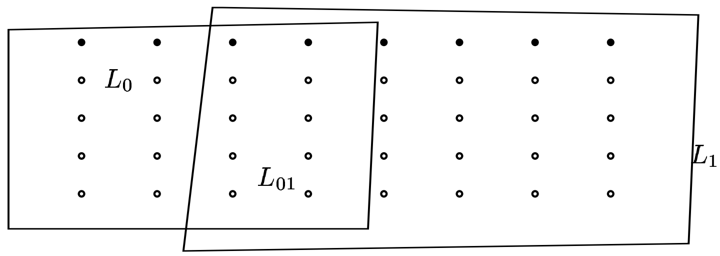
$$(\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \cdots \widetilde{\alpha_i} \cdots \alpha_{p+1}} \big|_{L_{\alpha_0 \cdots \alpha_{p+1}}}$$

Important Remark 25 *The key difference to the smooth case is that here, the restriction sets to zero any inappropriate forms at points which are TopRight points in the range space.*

Theorem 26 *The Čech difference operator δ satisfies $\delta^2 = 0$*

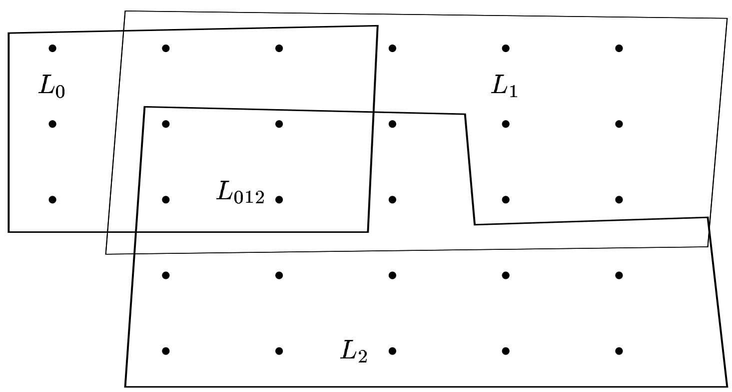
Example 27 *For the lattice covers shown, we give the Čech difference operator.*

(i)



$$\omega = (\omega_0, \omega_1), \qquad (\delta\omega)_{01} = (\omega_1 - \omega_0)|_{L_{01}}$$

(ii)



$$\omega = (\omega_{01}, \omega_{12}, \omega_{02}), \qquad (\delta\omega)_{012} = (\omega_{12} - \omega_{02} + \omega_{01})|_{L_{012}}$$

Theorem 28 *The sequence*

$$\bigoplus \mathsf{Ex}^i(L_\alpha) \xrightarrow{\delta} \bigoplus \mathsf{Ex}^i(L_{\alpha\beta}) \xrightarrow{\delta} \bigoplus \mathsf{Ex}^i(L_{\alpha\beta\gamma}) \cdots \tag{20}$$

is exact for each i.

Proof of exactness of the Čech sequence requires a partition of unity, $\{\rho_\alpha\}$,

$$\sum \rho_\alpha = 1, \quad \text{support}\rho_\alpha \subseteq L_\alpha$$

For $\omega \in \bigoplus \mathsf{Ex}^i(L_{\alpha_0 \cdots \alpha_p})$, define

$$(K\omega)_{\alpha_0 \cdots \alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0 \cdots \alpha_{p-1}}.$$

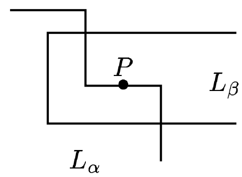
Then

$$\delta K + K\delta = \mathrm{id} \tag{21}$$

except perhaps at TopRight points in the range spaces where the restriction maps send some forms to zero. To ensure (21) is valid for all points, so that K is a homotopy operator for the sequence (20), the functions ρ_α must take the value 1 at any point P such that

$$\omega|_{L_\alpha}(P) \neq \omega|_{L_{\alpha\beta}}(P).$$

Such a point is shown in the next Figure, where P is a TopRight point in L_α but not $L_\alpha \cup L_\beta$.



For example, if $\omega(P) = a\Delta_1 + b\Delta_2$ in L_β, then $\omega|_{L_{\alpha\beta}} = a\Delta_1$, since Δ_2 is not defined at P in $L_{\alpha\beta}$. This constraint on the ρ poses no difficulties as unlike the smooth case, the partition functions need not be continuous.

Example 29 Example 11(ii) cont. *Suppose*

$$\delta(\omega_{12}, \omega_{02}, \omega_{01}) = (\omega_{12} - \omega_{02} + \omega_{01})|_{L_{012}} = 0,$$

then there exist $\omega_0 \in \mathbf{Ex}^i(L_0)$, $\omega_1 \in \mathbf{Ex}^i(L_1)$ *and* $\omega_2 \in \mathbf{Ex}^i(L_2)$ *such that*

$$\omega_{12} = \omega_1 - \omega_2$$
$$\omega_{01} = \omega_1 - \omega_0$$
$$\omega_{02} = \omega_0 - \omega_2$$

Theorem 30 *The Čech difference operator δ and the difference operator $\mathbf{\Delta}$ commute, that is,*

$$\mathbf{\Delta}\delta = \delta\mathbf{\Delta}.$$

This follows as the smooth case, except at TopRight points, where the restrictions ensure commutativity.

4.2 Čech cochains

For a lattice variety L with cover $\{L_\alpha \,|\, \alpha \in A\}$, where A is the index set for the cover, let $\mathcal{L}$ denote the set $\{L_{\alpha_0\alpha_1\cdots\alpha_r} \,|\, r \in \mathbb{N}, \alpha_i \in A\}$. Further, let $\mathcal{L}^i = \{L_{\alpha_0\alpha_1\cdots\alpha_i}\}$.

For a set S, let $\langle S \rangle_{\mathbb{F}}$ denote the set

$$\langle S \rangle_{\mathbb{F}} = \{\sum a_s s \,|\, s \in S, a_s \in \mathbb{F}\}$$

of formal sums of the elements of S with coefficients in $\mathbb{F}$.

Definition 31 *Čech cochains are defined as*

$$\check{\mathcal{C}}^i(\mathcal{L}) = \{f : \langle \mathcal{L}^i \rangle \to \mathbb{R}\}, \qquad i \in \mathbb{N}$$

The Čech p-cochains can be viewed as constant valued degree zero $\mathbf{\Delta}$-forms on the $p+1$ intersections. Let s denote the injection map,

$$s : \check{\mathcal{C}}^i(\mathcal{L}) \to \oplus \mathbf{Ex}^0(L_{\alpha_0 \alpha_1 \cdots \alpha_i}).$$

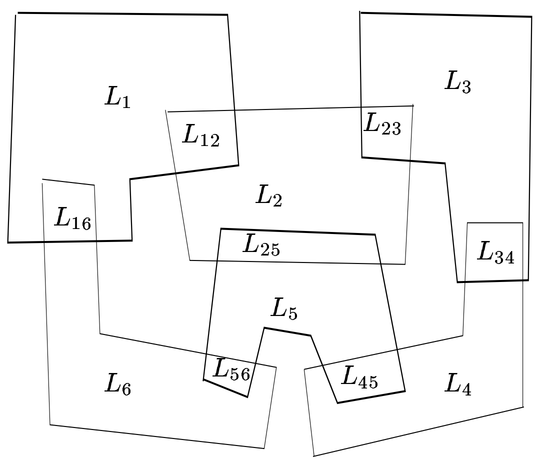
The maps $\delta : \check{\mathcal{C}}^i \to \check{\mathcal{C}}^{i+1}$ are then restrictions of the Čech difference operator given in Definition 24 above to the $\mathbf{\Delta}$ forms with which they are identified via the map s, so that $s\delta = \delta s$. Note also that $\mathbf{\Delta} \circ s = 0$, since the image of s are locally constant.

Definition 32 *The Čech cohomology groups of $\mathcal{L}$ are defined to be*

$$\check{H}^i(\mathcal{L}) = \frac{\ker \delta|_{\check{\mathcal{C}}^i(\mathcal{L})}}{\mathrm{im}\delta|_{\check{\mathcal{C}}^{i-1}(\mathcal{L})}}$$

Once the pattern of the L_α and their intersections is known, the Čech cohomology of $\mathcal{L}$ is easily worked out. This first example is a simple expository one for the sake of completeness.

Example 33 *Consider the following diagram of lattices and their intersections,*



In this example, $\check{\mathcal{C}}^0 \cong \mathbb{R}^6$ and $\check{\mathcal{C}}^1 \cong \mathbb{R}^7$. The Čech sequence is

$$0 \longrightarrow \check{\mathcal{C}}^0 \xrightarrow{\delta} \check{\mathcal{C}}^1 \longrightarrow 0,$$

where

$$\delta = \begin{array}{c} \\ L_{12} \\ L_{23} \\ L_{34} \\ L_{45} \\ L_{25} \\ L_{56} \\ L_{16} \end{array} \begin{array}{c} \begin{array}{cccccc} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Let χ_α be the characteristic function for L_α, that is, takes the value 1 on L_α and is zero elsewhere. Then

$$\check{H}^0(\mathcal{L}) = \ker \delta = \sum_{j=1}^{6} \chi_j$$

and

$$\begin{aligned} \check{H}^1(\mathcal{L}) &= \check{\mathcal{C}}^1/\mathrm{im}\ \delta \\ &\cong \langle \chi_{12} + \chi_{25} + \chi_{56} + \chi_{16}, \chi_{12} + \chi_{23} + \chi_{34} + \chi_{45} + \chi_{56} + \chi_{16} \rangle_{\mathbb{R}} \end{aligned}$$

These may be calculated in a symbolic computing environment using commands for the kernel, transpose and column space on the matrix form of δ.

In §2, we stressed the need for a valid cover of the lattice variety. Consider the variety shown in Figure 2 in §2.3.3. Since $L_1 \cap L_2$ has two components, the Čech cohomology is the same as a two-dimensional ring, which is *not* isomorphic to that of a corner. Thus the Čech cohomology captures the fact that there is a central point at which 2-forms are not defined.

4.3 The Čech-$\mathbf{\Delta}$ double complex

The double complex we describe in this section is the "Divide and Conquer" method for calculating cohomology groups, pioneered by Weil [35].

Let L be a lattice variety with cover $\{L_\alpha\}$.

Definition 34 Good cover *If the cover $\mathcal{L}$ of L is not only valid but has the property that each $L_{\alpha_0 \cdots \alpha_r}$ has trivial $\mathbf{\Delta}$-cohomology, we say that $\mathcal{L}$ is a* good cover *of L.*

Example 35 *The cover of the pillow model of the cube consisting of the front and back faces, is not a good cover. The intersection is a one dimensional ring which has nontrivial $\mathbf{\Delta}$ cohomology.*

Good covers of the punctured plane, the lattice ring and the cube model of the sphere will be given in §5.

Define $r : \mathbf{Ex}^i(L) \to \bigoplus \mathbf{Ex}^i(L_\alpha)$, by

$$r(\omega) = (\omega|_{L_0}, \omega|_{L_1}, \dots).$$

Then $r\mathbf{\Delta} = \mathbf{\Delta} r$. Combining results from the previous discussions, we have that the following diagram is commutative, where $\mathbf{\Delta}$ on a direct sum is assumed to act componentwise.

$$
\begin{array}{cccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & \uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & \\
0 & \to & \mathsf{Ex}^2(L) & \xrightarrow{r} & \bigoplus \mathsf{Ex}^2(L_\alpha) & \xrightarrow{\delta} & \bigoplus \mathsf{Ex}^2(L_{\alpha\beta}) & \xrightarrow{\delta} & \bigoplus \mathsf{Ex}^2(L_{\alpha\beta\gamma}) & \xrightarrow{\delta} \\
 & & \uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & \\
0 & \to & \mathsf{Ex}^1(L) & \xrightarrow{r} & \bigoplus \mathsf{Ex}^1(L_\alpha) & \xrightarrow{\delta} & \bigoplus \mathsf{Ex}^1(L_{\alpha\beta}) & \xrightarrow{\delta} & \bigoplus \mathsf{Ex}^1(L_{\alpha\beta\gamma}) & \xrightarrow{\delta} \\
 & & \uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & \\
0 & \to & \mathsf{Ex}^0(L) & \xrightarrow{r} & \bigoplus \mathsf{Ex}^0(L_\alpha) & \xrightarrow{\delta} & \bigoplus \mathsf{Ex}^0(L_{\alpha\beta}) & \xrightarrow{\delta} & \bigoplus \mathsf{Ex}^0(L_{\alpha\beta\gamma}) & \xrightarrow{\delta} \\
\cline{5-10}
 & & \uparrow & & \uparrow s & & \uparrow s & & \uparrow s & \\
 & & 0 & \to & \check{\mathcal{C}}^0 & \xrightarrow{\delta} & \check{\mathcal{C}}^1 & \xrightarrow{\delta} & \check{\mathcal{C}}^2 & \xrightarrow{\delta} \\
 & & & & \uparrow & & \uparrow & & \uparrow & \\
 & & & & 0 & & 0 & & 0 &
\end{array}
$$

The first important point is that those parts of the columns above the long horizontal line are exact by definition of a good cover. Those parts of the rows above the line are exact as they are the Čech sequences. The second important point is that the kernels of the $\mathbf{\Delta}$ in the leftmost column are, by definition, the *global* $\mathbf{\Delta}$-forms.

The complex is used to construct a global $\mathbf{\Delta}$-form from a Čech form, and vice versa. As an example, consider now the *descent diagram* for $p = 2$ beginning on the left with $\omega \in \mathsf{Ex}^2(L)$:

$$
\begin{array}{ccccccccc}
0 = \mathbf{\Delta}(\omega) & & 0 & & & & & & \\
\uparrow \mathbf{\Delta} & & \uparrow \mathbf{\Delta} & & & & & & \\
\omega \in \mathsf{Ex}^2(L) & \xrightarrow{r} & (\omega|_{L_0}, \dots) & \longrightarrow & 0 & & & & \\
 & & \uparrow \mathbf{\Delta} & & \uparrow & & & & \\
 & & (\eta_0, \eta_1, \dots) & \overset{\delta}{\to} & (\xi_{01}, \dots) & \overset{\delta}{\to} & 0 & & \\
 & & & & \uparrow \mathbf{\Delta} & & \uparrow & & \\
 & & & & (\chi_{01}, \dots) & \overset{\delta}{\to} & (\phi_{012}, \dots) & \overset{\delta}{\to} & 0 \\
\cline{3-9}
 & & & & & & \uparrow s & & \\
 & & & & & & \phi \in \check{\mathcal{C}}^2 & \overset{\delta}{\to} & 0
\end{array}
$$

Assume that $\mathbf{\Delta}\omega = 0$, that is, ω is a closed $\mathbf{\Delta}$-form. Since each $\omega|_{L_\alpha}$ is closed on L_α which has trivial $\mathbf{\Delta}$-cohomology, we use the relevant homotopy operator to obtain η_α

on L_α such that $\mathbf{\Delta}(\eta_\alpha) = \omega|_{L_\alpha}$. Applying δ to $(\eta_0, \eta_1, \dots)$ yields $\xi = (\xi_{01}, \dots)$, say. By commutativity of the diagram, each component of ξ is closed on its domain. Thus, there exists for each $\xi_{\alpha\beta}$, a $\chi_{\alpha\beta}$, such that on $L_{\alpha\beta}$, $\mathbf{\Delta}(\chi_{\alpha\beta}) = \xi_{\alpha\beta}$. Taking $\delta(\chi_{01}, \dots)$ yields

$$\phi = (\phi_{012}, \dots) \in \bigoplus \mathsf{Ex}^0(L_{\alpha_0\alpha_1\alpha_2})$$

such that $\mathbf{\Delta}(\phi) = 0$ and $\delta(\phi) = 0$.

Since zero-forms in $\ker \mathbf{\Delta}$ are just constants, we have that ϕ can be identified with a Čech cocycle,

$$\phi : \langle \{L_{\alpha\beta\gamma}\} \rangle_{\mathbb{R}} \to \mathbb{R}, \qquad \phi(L_{\alpha\beta\gamma}) = \phi_{\alpha\beta\gamma}.$$

Standard algebraic arguments are used to prove the following theorem.

Theorem 36 *A $\mathbf{\Delta}$ p-cocycle produces a Čech p-cocycle via the descent diagram. Moreover, exact $\mathbf{\Delta}$ cocycles are mapped to exact Čech cocycles.*

Since the Čech cohomology is easy to compute, we would like to use it to compute the $\mathbf{\Delta}$-cohomology of the lattice-like surface. To this end, we reverse the descent procedure to obtain an *ascent* procedure. Instead of using the homotopy maps for exact $\mathbf{\Delta}$ sequences (the columns), we now use those for the exact Čech sequences (the rows). Then a standard argument yields the following theorem.

Theorem 37 *If $\mathcal{L}$ is a good cover for the lattice surface L, then*

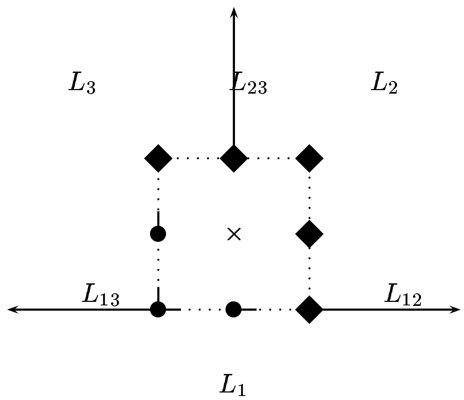
$$H^p_{\mathbf{\Delta}}(L) \cong \check{H}^p(\mathcal{L}).$$

5 Examples

We now look at three examples to demonstrate the process of finding the $\mathbf{\Delta}$-cohomology from the Čech cohomology.

5.1 Punctured Plane

Consider the cover of the punctured plane given below.



The diagram shows a good cover $\mathcal{L}$ for the "punctured plane"; $L = \mathbb{Z}^2 \setminus \{(0,0)\}$ with the origin, marked by a cross, removed. The double intersections are along the lines, as marked, and there are no triple intersections. It should be noted that no 2-form at $(0,-1)$ can be defined. As in §2, points where 2-forms are definable are marked by diamonds, points where only 1-forms are definable are marked by discs.

The Čech cohomology of $\mathcal{L}$ is

$$\begin{array}{rcll} \check{H}^i(\mathcal{L}) & = & 0, & i \geq 2 \\ \check{H}^1(\mathcal{L}) & \cong & \langle \eta \rangle_{\mathbb{R}} & \\ \check{H}^0(\mathcal{L}) & \cong & \mathbb{R} & \text{(constant 0-forms)} \end{array} \tag{22}$$

where $\eta|_{L_{13}} = 1$, and $\eta|_{L_{23}} = \eta|_{L_{12}} = 0$.

The ascent diagram is

$$\begin{array}{ccccccc} 0 & & 0 & & & & \\ \uparrow & & \uparrow & & & & \\ \mathbf{Ex}^1 \ni \omega & \longrightarrow & \boldsymbol{\Delta}(\chi) & \to & 0 & & \\ & & \boldsymbol{\Delta}\uparrow & & \uparrow & & \\ & & \chi & \xrightarrow{\delta} & s(\eta) & \to & 0 \\ \hline & & & & s\uparrow & & \\ & & & & \eta \in \check{C}^1 & \to & 0 \end{array} \tag{23}$$

The first step is to find a pre-image χ, under δ, of $s(\eta)$, which is the constant 0-form $(0,1,0) \in \mathbf{Ex}^0(L_{12}) \oplus \mathbf{Ex}^0(L_{13}) \oplus \mathbf{Ex}^0(L_{23})$. We may take

$$\chi = (\chi_1, \chi_2, \chi_3) \in \mathbf{Ex}^0(L_1) \oplus \mathbf{Ex}^0(L_2) \oplus \mathbf{Ex}^0(L_3),$$

to be

$$\chi_1 \equiv 0, \qquad \chi_2 \equiv 0$$

and

$$\chi_3(n_1, n_2) = \begin{cases} 0 & (n_1, n_2) \in L_3 \setminus L_{13} \\ 1 & (n_1, n_2) \in L_{13} \end{cases}$$

The next step is to calculate $\boldsymbol{\Delta}\chi$, which is done component-wise. We have $\boldsymbol{\Delta}\chi_1 = \boldsymbol{\Delta}\chi_2 \equiv 0$, while

$$\boldsymbol{\Delta}\chi_3(n_1, n_2) = \begin{cases} 0 & (n_1, n_2) \in L_1 \setminus L_{13} \\ -\Delta_2 & (n_1, n_2) \in L_{13} \end{cases}$$

We now observe that $\boldsymbol{\Delta}\chi$ is the component wise restriction of a global closed 2-form ω, given by

$$\omega(n_1, n_2) = \begin{cases} 0 & (n_1, n_2) \in L \setminus L_{13} \\ -\Delta_2 & (n_1, n_2) \in L_{13} \end{cases}$$

To see this, first note that $\omega|_{L_1} \equiv 0$, since 1-forms on the top boundary of L_1 may involve Δ_1 only, the restriction sets any Δ_2 terms to zero. Secondly, $\boldsymbol{\Delta}\omega \equiv 0$ since 2-forms are not defined at $(0, -1)$.

We remark that taking χ_3, which is defined on L_3, to be

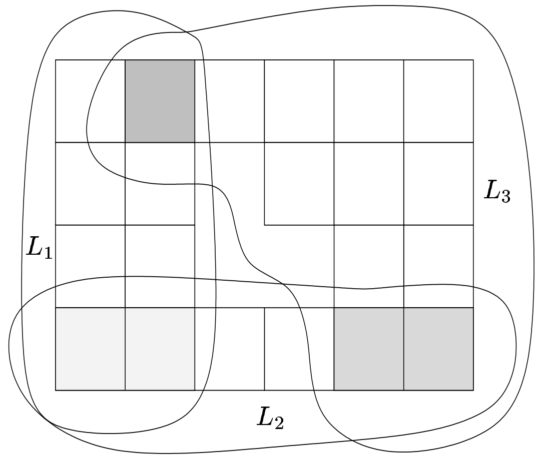
$$\chi_3(n_1, n_2) = \begin{cases} 0 & n_1, n_2 > 0 \\ 1 & n_2 = 0, -1 \end{cases}$$

yields a representative for $H^1_{\boldsymbol{\Delta}}$ which is more obviously both closed and the restriction of a global form, such as that shown in Example 19 in §2.4. Our choice of a good cover and pre-image highlights some of the subtleties of calculations with difference forms which are not present in the smooth case.

Different covers, different representatives η of $\check{H}^1$ and different pre-images of η all yield different representatives of $H^1(L)$. However, the difference of any two representatives of $H^1(L)$ is an exact form.

5.2 Lattice Ring

A good cover of a lattice ring L consists of three lattice pieces, L_i, $i = 1, 2, 3$, as in the diagram, with no triple intersections. The Čech cohomology of $\mathcal{L}$ is the same as that for the punctured plane, given in equation (22).



The ascent diagram is the same as given in (23) in §4.1. For variety, we take a different representative of $\check{H}^1$, which takes the value 1 on L_{23} and is zero elsewhere.

The first step is to find a pre-image χ, under δ, of $s(\eta)$, which is the constant 0-form $(0,0,1) \in \mathbf{Ex}^0(L_{12}) \oplus \mathbf{Ex}^0(L_{13}) \oplus \mathbf{Ex}^0(L_{23})$. We may take

$$\chi = (\chi_1, \chi_2, \chi_3) \in \mathbf{Ex}^0(L_{12}) \oplus \mathbf{Ex}^0(L_{13}) \oplus \mathbf{Ex}^0(L_{23}),$$

to be

$$\chi_2 \equiv 1, \qquad \chi_3 \equiv 0$$

and χ_1, which is defined on L_1, is given diagrammatically by

$$\chi_1 = \begin{array}{ccc} 0^{\bullet} & 0^{\bullet} & 0^{\bullet} \\ 0^{\bullet} & 0^{\bullet} & 0^{\bullet} \\ 1^{\bullet} & 1^{\bullet} & 1^{\bullet} \\ 1^{\bullet} & 1^{\bullet} & 1^{\bullet} \\ 1^{\bullet} & 1^{\bullet} & 1^{\bullet} \end{array}$$

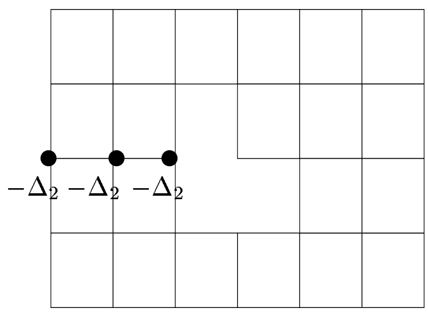
The next step is to calculate $\mathbf{\Delta}\chi$ component wise. This yields,

$$\mathbf{\Delta}\chi_2 \equiv 0, \qquad \mathbf{\Delta}\chi_3 \equiv 0$$

and

$$\boldsymbol{\Delta}\chi_1 = \begin{array}{ccc} 0^\bullet & 0^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet \\ -\Delta_2{}^\bullet & \text{-}\Delta_2{}^\bullet & \text{-}\Delta_2{}^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet \end{array}$$

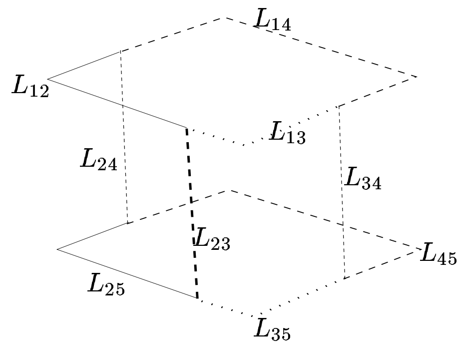
We can now see that $\boldsymbol{\Delta}\chi$ is the restriction onto components of a global form ω, since on the intersections L_{ij}, the values of the $\boldsymbol{\Delta}\chi_i$ are identical. This form is closed but not exact 1-form, since we began with a closed but not exact Čech 1-form.

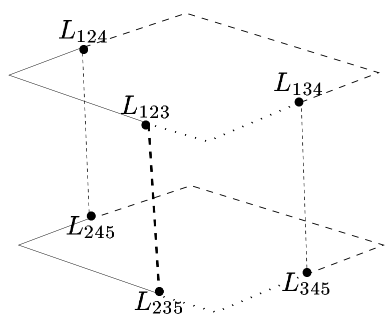


The global 2-form ω; the value at unmarked points is zero.

5.3 The lattice sphere

The cube model of the two-dimensional lattice sphere was pictured in §2.3. Suppose the cover is given as: L_1 is the top face, L_5 is the bottom face, while L_2, L_3 and L_4 wrap around the "belly". The intersections L_{ij} are given below, followed by a diagram of the triple intersections. This example shows another significant difference from the smooth case, which is that the intersections do not need to have the same dimension as the original variety.





The triple intersections, labelled on the double intersection skeleton.

From the pattern of the intersections of the cover, it is simple to calculate the Čech cohomology of $\mathcal{L}$,

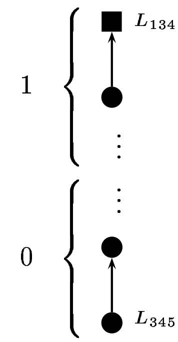
$$\check{H}^i = \begin{cases} 0, & i \neq 0, 2 \\ \mathbb{R} & i = 0, 2 \end{cases} .$$

We now construct a closed but not exact difference 2-form via the ascent procedure. The ascent diagram for $p = 2$ is given in §3.3.

Step 1 A representative of $\check{H}^2$ is given by $\phi_{134} = 1$ on L_{134} and zero else.

Step 2 Construct a preimage $\chi \in \oplus \mathbf{Ex}^0(L_{\alpha\beta})$ under δ of η. This may be taken to be zero on all $L_{\alpha\beta}$ except L_{34}, where it takes the form,

$$\chi_{34} =$$



Indeed,

$$\begin{aligned} (\delta\chi)_{134} &= (\chi_{34} - \chi_{14} - \chi_{13})|_{134} \\ &= 1 \\ (\delta\chi)_{235} &= (\chi_{35} - \chi_{25} - \chi_{23})|_{235} \\ &= 0 \end{aligned}$$

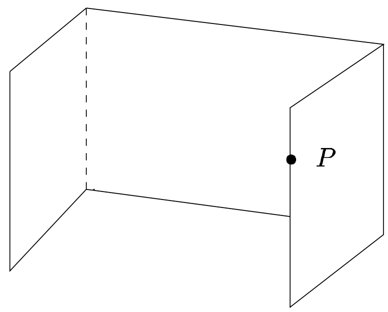
and so forth.

Step 3 Calculate the components of $\xi = \boldsymbol{\Delta}\chi$:

$$\boldsymbol{\Delta}\chi_{34} = \begin{array}{cl} 0 & \left\{\begin{array}{l} \bullet(L_{134}) \\ \vdots \end{array}\right. \\ \Delta_2 & \bullet \\ 0 & \left\{\begin{array}{l} \vdots \\ \bullet(L_{345}) \end{array}\right. \end{array}$$

and zero elsewhere.

Step 4 Calculate the components of a preimage, η, under δ of $\boldsymbol{\Delta}\chi$ in $\oplus\mathbf{Ex}^1(L_i)$. This can be taken to be zero on all L_i for $i \neq 4$, and on L_4 is given in the figure following;



$\eta(P) = -\Delta_2$, and is zero elsewhere

Step 5 Calculate $\mathbf{\Delta}\eta \in \oplus \mathsf{Ex}^2(L_i)$. This is zero on all L_i for $i \neq 4$, and equals $\Delta_1 \wedge \Delta_2$ at the point P in L_4 (see figure above).

- We can now see that $\mathbf{\Delta}\eta$ is a restriction of a global 2-form,

$$\omega = \begin{cases} \Delta_1 \wedge \Delta_2 & \text{at P} \\ 0 & \text{elsewhere} \end{cases}$$

(Recall that restrictions on TopRight points of components are zero if forms of the appropriate dimension are not otherwise definable).

The overall result is,

$$\begin{aligned} H^2_{\mathbf{\Delta}}(LS^2) &\cong \langle \omega \rangle_{\mathbb{R}} \\ H^1_{\mathbf{\Delta}}(LS^2) &= 0 \\ H^0_{\mathbf{\Delta}}(LS^2) &\cong \mathbb{R} \end{aligned}$$

where the zero-forms are the constant forms.

6 Open problems, Discussion and Conjectures

6.1 Conjectures concerning Discrete Index Theorems

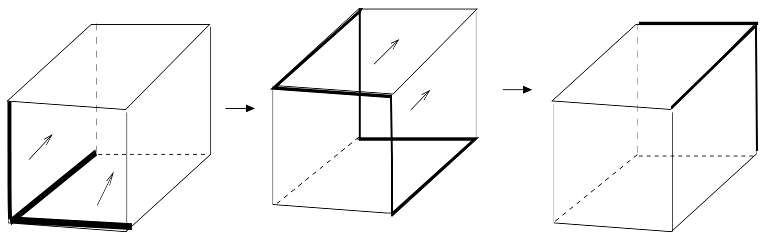
The Morse Index theorem relates the Euler characteristic of a manifold to features of the solution of a generic vector field on it, specifically to the indices of the vector field about its critical points (see for example [12], §14.3e). In this section we conjecture a result, for generic linear equations on lattice varieties, relating the Euler characteristic of a lattice variety to an index, which we define, of the generic solution of the equation.

Suppose you want to solve, globally, a 4-point scheme Ω defined locally as

$$\Omega: \qquad 0 = au_{n,m} + bu_{n+1,m} + cu_{n,m+1} + du_{n+1,m+1} \tag{24}$$

with $abcd \neq 0$, on a 2-dimensional lattice variety. For simplicity, we consider only linear equations.

We demonstrate the kinds of calculations involved by considering solutions of (24) on the cube model of a sphere. In the diagram below, we have omitted the lattice points on the faces of the model, for simplicity.



Begin with generic initial data defined on the points in the 1-dimensional sublattice variety shown as darkened lines in the left most sphere in the diagram. Then the equation can be solved at points in the bottom, left and front faces by sequentially solving for u in the direction of the arrows on the left most sphere. This yields values for $u_{m,n}$ on the 1-dimensional sublattice (shown as darkened lines) of the middle sphere model in the figure. Continuing solving for u (in the direction of the arrows drawn on the middle sphere) on the top, right and back faces, we obtain compatibility conditions on the initial data coming from the need to have u well-defined on that sublattice of the third lattice model shown as darkened lines in the diagram.

A careful count reveals that regardless of the number of 2-cells that make up the lattice sphere, and regardless of the position of the initial data, we have that

$$\sharp\text{initial conditions} - \sharp\text{compatibility conditions} \equiv 2.$$

Conjecture 38 *Let L be a lattice model for a boundary-free manifold, and $\Omega = 0$ be a generic linear equation defined on L. Define the* index $\mathcal{S}(\Omega)$ *to be*

$$\mathcal{S}(\Omega) = \sharp\textit{initial conditions} - \sharp\textit{compatibility conditions}$$

for the generic solution, that is, arbitrary initial conditions. Then

$$\mathcal{S}(\Omega) = \sum(-1)^i \dim H^i_{\mathbf{\Delta}}(L). \tag{25}$$

We may take the right hand side of equation (25) to define the *Euler Characteristic* of L by direct analogy of the result for smooth manifolds,

$$\chi(M) = \sum(-1)^i \dim H^i_{\text{de Rham}}(M).$$

We remark that the conjecture is not true for lattice models with a boundary, such as a lattice ring.

One can look at the Čech cohomology with local coefficients in the same way as for smooth manifolds, since the same constructions will hold (see [6], §10). If we take local coefficients on $L_{\boldsymbol{\alpha}} = L_{\alpha_0\cdots\alpha_p}$ to be the vector space $S_{\boldsymbol{\alpha}}$ of solutions of the linear equation $\Omega = 0$ on $L_{\boldsymbol{\alpha}}$, then we have that the $S_{\boldsymbol{\alpha}}$ form a pre-sheaf $\Sigma = \Sigma(\Omega)$ on L. Moreover,

$$\begin{aligned}\chi(L,\Sigma) &:= \sum(-1)^i \check{H}^i(L,\Sigma)\\ &= \sum(-1)^i \check{C}^i(L,\Sigma)\\ &= \sum(-1)^i \left(\sum_{|\boldsymbol{\alpha}|=i} \dim S_{\boldsymbol{\alpha}}\right)\end{aligned}$$

follows from standard arguments.

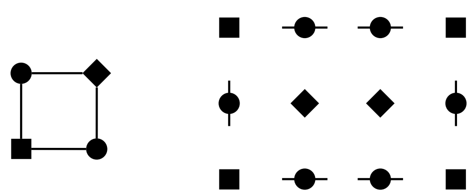
Conjecture 39 *Let Ω be a generic linear equation on a lattice model L of a manifold perhaps with boundary. Then*

$$\chi(L,\Sigma(\Omega)) = \mathcal{S}(\Omega).$$

6.2 What is special about the South-West?

So far, the south west point of the fundamental p-cube is distinguished as the only point where a p-form is definable.

If instead of forward difference we want to use a backward difference or a consistent collocation, the entire theory above can be developed for different models of the basic fundamental cube. Simply, one fixes points at which the various forms are defined. In the diagram below, on the left is shown a fundamental 2-cube for a backward difference, while on the right an example is shown of a fundamental 2-cell for a collocation scheme.



The right hand cell has no TopRight points, so the proofs for such a model are simpler than those we have presented. Gauss-Legendre, Marker and Cell, and Preissmann schemes all have a fundamental model for what is defined where in a fundamental cube, and thus these schemes can be studied using the methods described.

6.3 Open problems

6.3.1 Independence from the good cover

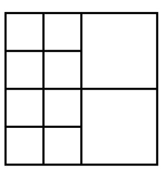
In the smooth theory, one proves that the Čech forms are independent of the particular cover of the manifold used to calculate them by proving stability under refinement of the cover. Then since any two covers have a common refinement, independence follows. We do not have any such theory yet for lattice-like surfaces. One conjecture would be that any two lattice models of the same manifold have the same cohomology, provided there are sufficient numbers of points.

6.3.2 Non-orientable lattice varieties

We leave open the problem of generalising the methods used here to allow for lattice approximations of non-orientable manifolds. It is not hard to imagain a lattice approximation of a Möbius band, for example. They are excluded from the theory described here, as their construction necessarily violates the condition to respect the ordering inherited from $\mathbb{Z}^p$ when pieces of lattice are glued together to form a lattice variety. One interesting possibility is to find the analogue of a "twisted differential form" or pseudoform, see for example ([12], §2.8).

6.3.3 Localised refinements

A common numerical technique is that of the *localised refinement* of a mesh. This is used to obtain more accurate information where the functions being calculated are rapidly changing. An open problem is how to adapt the constructions given in this article to cover meshes such as in the diagram. They are not covered by the theory described here as they voliate the adjacency condition stipulated in the lattice variety construction.



Using different scale meshes violates the adjacency condition.

6.3.4 Functorial properties

An open problem is to complete the difference form constructions described in this article to a fully functorial theory. This requires a discussion of mappings between lattice varieties, and the associated maps they generate between cohomologies of the underlying spaces.

7 Conclusions

A strong analogy between global difference ($\mathbf{\Delta}$) forms of a lattice model of a manifold M, and smooth de Rham forms for M, exists by virtue of their relationship to a cover or decoposition of the underlying spaces. Moreover, we show how to compute $\mathbf{\Delta}$ forms effectively. However, our definitions do not rely on any underlying smooth model existing, and therefore our definitions make sense in the absence of any such model.

References

[1] D. Arnold, *Differential complexes and numerical stability*, Plenary address at ICM Beijing, 2002. `http://www.ima.umn.edu/~arnold/papers/icm2002.pdf`

[2] D. Arnold, R.S. Falk and R. Winthur, Finite element exterior calculus, homological techniques and applications, *Acta Numerica* **15** 1–155 (2006).

[3] A. Bossavit, Differential forms and the computation of fields and forces in electromagnetism, *Eur. J. Mech B-Fluid* **10** 474–488 (1991).

[4] A. Bossavit, Whitney forms - a class of finite-elements for 3-dimensional computations in electromagnetism, *IEE Proc-A* **135** 493–500 (1988).

[5] A.R. Mohebalhojeh, On shallow water potential vorticity inversion by Rossby-number expansions, *Q. J. Roy. Met. Soc.*, **128** 679–694 (2002).

[6] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics **82**, Springer Verlag, New York, 1982.

[7] C.J. Budd and G.J. Collins, An invariant moving mesh scheme for the nonlinear diffusion equation, *App. Num. Math*, **26** 23–29 (1998).

[8] C.J. Budd and M.D. Piggott, *Geometric Integration and its Applications*, Department of Mathematical Sciences Preprint 0024, University of Bath, 2000.

[9] J.A. Chard and V. Shapiro, A Multivector Data Structure for Differential Forms and equations, *Math. Comp. Sim.* **54** 33–64 (2000).

[10] M. Desbrun, M. Leok and J.E. Marsden, Discrete Poincaré Lemma, *Appl. Num. Math.* **53** 231–248 (2005).

[11] R. Forman, Morse theory for cell complexes, *Topology* **37** 945–979 (1998).

[12] T. Frankel, *The Geometry of Physics*, Cambridge University Press, Cambridge, 1997.

[13] W. Fulton, *Algebraic Topology, A First Course*, Graduate Texts in Mathematics **153**, Springer Verlag, New York, 1995.

[14] Ge Zhong and J.E. Marsden, Lie-Poisson Hamilton-Jacobi theory and Lie-Poisson integrators, *Phys. Lett. A* **133** 134–139 (1988).

[15] P.W. Gross and P.R. Kotiuga, *Electromagetic Theory and Computation: A Topological Approach*, MSRI Publications 48, Cambridge Universitry Press, Cambridge, 2004.

[16] Gu Chaohao (Ed.), *Soliton Theory and its Applications*, Springer Verlag, Berlin, (1995).

[17] E. Hairer, C. Lubich and G. Wanner, *Geometric Numerical Integration*, Springer Series in Computational Mathematics **31**, Springer Verlag, Berlin, 2002.

[18] R. Hiptmair, Finite elements in computational electromagnetism, *Acta Numerica* **11** 237–340 (2002).

[19] P.E. Hydon, Symmetries and first integrals of ordinary difference equations *Proc. Roy. Soc. A* **456** 2835–2855 (2000).

[20] P.E. Hydon, Conservation laws of partial difference equations with two independent variables, *J. Phys. A: Math. Gen.* **34** 10347–10355 (2001).

[21] P.E. Hydon and E.L. Mansfield, A variational complex for difference equations, *J. Foundations of Computational Mathematics* **4** 187–217 (2004).

[22] A. Iserles, H. Munthe-Kaas, S. Norsett, and A. Zanna, Lie Group Methods, *Acta Numerica* 215-365 (2000).

[23] T. Kaczynski, K. Mischaikow and M. Mrozek, Computing Homology, *Homology, Homotopy and Applications* **5** 233–256 (2003).

[24] T.Y. Kong, R.D. Kopperman, and P.R. Meyer, *A topological approach to digital topology*, American Mathematical Monthly **98** 901-917 (1991).

[25] M. Leok, *Foundations of Computational Geometric Mechanics*, PhD Thesis, California Institute of Technology, 2004.

[26] R.I. McLachlan and G.R.W. Quispel, Six lectures on the geometric integration of ODEs, in *Foundations of Computational Mathematics*, Editors RA DeVore et al, Cambridge University Press, Cambridge, pp155-210 (2001).

[27] E.L. Mansfield and P.E. Hydon, Towards approximations which preserve integrals, in Proc. ISSAC, Ed. B. Mourrain, ACM Publishing, pp. 217–222, (2001).

[28] W.S. Massey, *Singular Homology Theory*, Graduate Texts in Mathematics 70, Springer Verlag, Berlin, 1980.

[29] C. Mattiussi, An analysis of finite volume, finite element, and finite difference methods using some concepts from algebraic topology, *J. Comp. Phys* **133** 289–309 (1997).

[30] T.J. Bridges and S. Reich, Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that preserve symplecticity, *Phys. Lett. A* **284** 184–193 (2001).

[31] J.M. Sanz-Serna and M.P. Calvo, *Numerical Hamiltonian Problems*, Chapman and Hall, London, 1994.

[32] W. Schwalm, B. Moritz, M. Giona and M. Schwalm, Vector difference calculus for physical lattice models, *Phys Rev E* **59** 1217-1233 (1999).

[33] Y.Y. Tong, S. Lombeya, A.N. Hirani and M. Desbrun, Discrete multiscale vector field decomposition, *ACM Trans. Graphics*, **22** 445–452 (2003).

[34] E. Tonti, *On the formal structure of physical theories*, Istituto di Matematica del Politecnico di Milano, Milan, 1975.
`http://www.dic.univ.trieste.it/perspage/tonti/papers.htm`

[35] A. Weil, Sur les théorèmes de De Rham, *Comment. Math. Hel.*, **26** (1952) 119–145.

[36] Hassler Whitney, *Geometric Integration Theory*, Princeton University Press, Princeton, 1957.